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The Mass of the Lightest MSSM Higgs Boson: A Compact Analytical Expression at the Two-Loop Level

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Abstract

A compact approximation formula for the mass of the lightest neutral $\mathcal{CP}$-even Higgs boson, m_h, in the Minimal Supersymmetric Standard Model (MSSM) is derived from the diagrammatic two-loop result for m_h up to $\mathcal{O}(\alpha\alpha_s)$. By analytically expanding the diagrammatic result the leading logarithmic and non-logarithmic as well as the dominant subleading contributions are obtained. The approximation formula is valid for general mixing in the scalar top sector and arbitrary choices of the parameters in the Higgs sector of the model. Its quality is analyzed by comparing it with the full diagrammatic result. We find agreement with the full result better than 2 GeV for most parts of the MSSM parameter space.

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1 Introduction

One of the most striking phenomenological implications of Supersymmetry (SUSY) is the prediction of a relatively light Higgs boson, which is common to all supersymmetric models whose couplings remain in the perturbative regime up to a very high energy scale [1]. The search for the lightest Higgs boson thus allows a crucial test of SUSY, and is one of the main goals at the present and the next generation of colliders. A precise knowledge of the dependence of the mass m_h of the lightest Higgs boson on the relevant SUSY parameters is necessary for a detailed analysis of SUSY phenomenology at LEP2, the upgraded Tevatron, and also for the LHC and a future linear e^+e^- collider, where a high-precision measurement of m_h might become possible.

In the Minimal Supersymmetric Standard Model (MSSM) [2], at the tree level the mass of the lightest Higgs boson is restricted to be smaller than the Z-boson mass M_Z. This bound, however, is strongly affected by radiative corrections, resulting in an upper bound of about 135 GeV [3–15]. Results beyond one-loop order have been obtained using several different approaches: a Feynman-diagrammatic calculation of the leading QCD corrections has been performed [8–10]; renormalization group (RG) methods have been applied in order to obtain leading logarithmic higher-order contributions [11–13]; the leading two-loop QCD corrections have been calculated in the effective potential method [14, 15]. Until recently phenomenological analyses have been based either on RG improved one-loop calculations [11–13] or on the complete Feynman-diagrammatic one-loop on-shell result [5–7]. Their numerical results, however, differ by up to 20 GeV. Recently the Feynman-diagrammatic result for the dominant contributions in $\mathcal{O}(\alpha\alpha_s)$ to the masses of the neutral $\mathcal{CP}$-even Higgs bosons has become available [8]. By combining these contributions with the complete one-loop on-shell result [6], the presently most precise result for m_h based on diagrammatic calculations is obtained [9, 10]. In comparison with the results obtained by RG methods good agreement is found in the case of vanishing mixing in the scalar top sector, while sizeable deviations which can exceed 5 GeV occur when mixing in the $\tilde{t}$-sector is taken into account [9, 10].

The Feynman-diagrammatic two-loop result for m_h, however, is very lengthy, making the evaluation of the Higgs-boson masses in this approach relatively slow. This could limit the applicability of this result e.g. in Monte Carlo simulations. In the present paper we derive, by means of a Taylor expansion, a short analytical approximation formula from the diagrammatic two-loop result up to $\mathcal{O}(\alpha\alpha_s)$ [8, 9]. The purpose of this is not only to provide a compact analytical expression for m_h suitable for a very fast numerical evaluation without losing too much of accuracy, but also to isolate the most important contributions, thus allowing a better qualitative understanding of the source of the dominant corrections. The compact approximation formula contains, besides the relevant parts of the one-loop contributions, partly taken over from Ref. [13], the leading logarithmic and non-logarithmic two-loop corrections for general mixing in the $\tilde{t}$-sector, the leading Yukawa corrections [12,16] and leading QCD contributions beyond $\mathcal{O}(\alpha\alpha_s)$.

The values for m_h obtained from the approximation formula are compared with the full result [9, 10]. The dependence on the various MSSM parameters from the stop sector, the Higgs sector and the chargino-neutralino sector is analyzed. We find that the approximation formula agrees with the full result better than 2 GeV for most parts of the MSSM parameter space.

The paper is organized as follows: In Sect. 2 a Taylor expansion of the diagrammatic two-loop result up to $\mathcal{O}(\alpha\alpha_s)$ is performed and a compact approximation formula is derived. The accuracy of the approximation formula is discussed in Sect. 3, where also the location of the maximal values of m_h, depending on the mixing in the $\tilde{t}$-sector, is analyzed analytically. In Sect. 4 we give our conclusions.

2 The compact analytical formula

2.1 The scalar top sector of the MSSM

In order to fix our notations and to explain the approximations employed in this sector, we first list the conventions for the MSSM scalar top sector: the mass matrix in the basis of the current eigenstates $\tilde{t}_L$ and $\tilde{t}_R$ is given by

$$\mathcal{M}^2_{\tilde{t}} = \begin{pmatrix} M^2_{\tilde{t}_L} + m_t^2 + \cos 2\beta\,(\frac{1}{2} - \frac{2}{3} s_W^2) M_Z^2 & m_t M_t^{LR} \\ m_t M_t^{LR} & M^2_{\tilde{t}_R} + m_t^2 + \frac{2}{3}\cos 2\beta\; s_W^2 M_Z^2 \end{pmatrix}, \tag{1}$$

where

$$m_t M_t^{LR} = m_t (A_t - \mu \cot\beta\,)\ . \tag{2}$$

Neglecting the numerically small contributions proportional to M_Z^2 and setting[1]

$$M_{\tilde{t}_L} = M_{\tilde{t}_R} := m_{\tilde{q}}, \;\; M_S^2 := m_{\tilde{q}}^2 + m_t^2 \tag{3}$$

one arives at

$$\mathcal{M}^2_{\tilde{t}} = \begin{pmatrix} M_S^2 & m_t M_t^{LR} \\ m_t M_t^{LR} & M_S^2 \end{pmatrix}\ . \tag{4}$$

Diagonalizing the $\tilde{t}$-mass matrix (4) yields the mass eigenvalues $m_{\tilde{t}_1}, m_{\tilde{t}_2}$ and the $\tilde{t}$-mixing angle $\theta_{\tilde{t}}$, which relates the current eigenstates to the mass eigenstates:

$$\begin{pmatrix} \tilde{t}_1 \\ \tilde{t}_2 \end{pmatrix} = \begin{pmatrix} \cos\theta_{\tilde{t}} & \sin\theta_{\tilde{t}} \\ -\sin\theta_{\tilde{t}} & \cos\theta_{\tilde{t}} \end{pmatrix} \begin{pmatrix} \tilde{t}_L \\ \tilde{t}_R \end{pmatrix}\ . \tag{5}$$

In the above approximation, which we will use throughout the rest of the paper, the $\tilde{t}$-masses and the mixing angle are given by

$$\begin{aligned} m^2_{\tilde{t}_1} &= M_S^2 - |m_t M_t^{LR}| = M_S^2\,(1 - \Delta_{\tilde{t}})\,, \\ m^2_{\tilde{t}_2} &= M_S^2 + |m_t M_t^{LR}| = M_S^2\,(1 + \Delta_{\tilde{t}})\,, \end{aligned} \tag{6}$$

$$\theta_{\tilde{t}} = \begin{cases} \frac{\pi}{4} & \text{for } M_t^{LR} < 0 \\ 0 & \text{for } M_t^{LR} = 0 \\ -\frac{\pi}{4} & \text{for } M_t^{LR} > 0 \end{cases}\ , \tag{7}$$

[1]Later we will also discuss the case $M_{\tilde{t}_L} \neq M_{\tilde{t}_R}$, for which the same formalism applies, but with a different definition for M_S, see eq. (19).

with

$$\Delta_{\tilde{t}} = \frac{|m_t M_t^{LR}|}{M_S^2} = \frac{m_{\tilde{t}_2}^2 - m_{\tilde{t}_1}^2}{m_{\tilde{t}_2}^2 + m_{\tilde{t}_1}^2} \; . \tag{8}$$

From the above definition it follows that $0 \leq \Delta_{\tilde{t}} \leq 1$, otherwise the $\tilde{t}$-mass matrix eq. (4) would have a negative eigenvalue.

2.2 Calculation of the mass of the lightest Higgs boson

Here we only give a very brief outline of the calculation of the mass of the lightest neutral $\mathcal{CP}$-even Higgs boson in the MSSM; for notations and a detailed description see Refs. [8–10]. We focus on the different steps of approximations made in order to derive a compact analytical expression from the full diagrammatic result of $\mathcal{O}(\alpha\alpha_s)$.

At the tree level, the mass matrix of the neutral $\mathcal{CP}$-even Higgs bosons in the basis of the weak eigenstates ϕ_1, ϕ_2 can be expressed in terms of M_Z, M_A (the mass of the $\mathcal{CP}$-odd Higgs boson) and $\tan\beta = v_2/v_1$ (the ratio of the vacuum expectation values of the two Higgs doublets) as follows:

$$\begin{aligned} M_{\text{Higgs}}^{2,\text{tree}} &= \begin{pmatrix} m_{\phi_1}^2 & m_{\phi_1\phi_2}^2 \\ m_{\phi_1\phi_2}^2 & m_{\phi_2}^2 \end{pmatrix} \\ &= \begin{pmatrix} M_A^2 \sin^2\beta + M_Z^2 \cos^2\beta & -(M_A^2 + M_Z^2)\sin\beta\,\cos\beta \\ -(M_A^2 + M_Z^2)\sin\beta\,\cos\beta & M_A^2 \cos^2\beta + M_Z^2 \sin^2\beta \end{pmatrix}. \end{aligned} \tag{9}$$

Taking into account higher-order corrections, the Higgs-boson masses can to a good approximation be obtained by diagonalizing the matrix

$$M_{\text{Higgs}}^2 = \begin{pmatrix} m_{\phi_1}^2 - \hat{\Sigma}_{\phi_1}(0) & m_{\phi_1\phi_2}^2 - \hat{\Sigma}_{\phi_1\phi_2}(0) \\ m_{\phi_1\phi_2}^2 - \hat{\Sigma}_{\phi_1\phi_2}(0) & m_{\phi_2}^2 - \hat{\Sigma}_{\phi_2}(0) \end{pmatrix} \; , \tag{10}$$

where the $\hat{\Sigma}_s(0)$ $(s = \phi_1, \phi_1\phi_2, \phi_2)$ denote the renormalized Higgs-boson self-energies (in the ϕ_1, ϕ_2 basis). As a first step of approximation, the momentum dependence, which is numerically rather small, has been neglected in the $\hat{\Sigma}_s(p^2)$.

The mass of the lightest Higgs boson receives contributions from all sectors of the MSSM, but not all are numerically of equal relevance. In order to derive a compact analytical expression, we make the following further approximations:

The contributions from the $t, \tilde{t}$-sector up to the two-loop level: The contributions arising from the $t, \tilde{t}$-sector can be written as

$$\hat{\Sigma}_s(0) = \hat{\Sigma}_s^{(1)}(0) + \hat{\Sigma}_s^{(2)}(0), \;\; s = \phi_1, \phi_1\phi_2, \phi_2. \tag{11}$$

Here the $\hat{\Sigma}_s^{(1)}(0)$ denote the one-loop contributions of the $t, \tilde{t}$-sector to the renormalized Higgs-boson self-energies. Their explicit form (including also the momentum dependence) can be found e.g. in Ref. [6]. The $\hat{\Sigma}_s^{(2)}(0)$ denote the two-loop contributions from the $t, \tilde{t}$-sector at zero external momentum from the Yukawa part of the theory (neglecting the gauge couplings) [8]. We first consider the dependence of the mass shift Δm_h^2 ($m_h^2 = m_h^{2,\text{tree}} + \Delta m_h^2$) on $\tan\beta$ and μ. For the leading contributions to Δm_h^2 from the Yukawa part of the theory the dependence on $\tan\beta$ and μ drops out in the limit $M_A \gg M_Z$.[2] This holds both for the contributions in $\mathcal{O}(\alpha)$ and $\mathcal{O}(\alpha\alpha_s)$. Since $\tan\beta$ and μ enter only via non-leading corrections in Δm_h^2, the dependence on them is relatively mild. We therefore use the approximation $\mu = 0$ in the $\hat{\Sigma}_s(0)$. We furthermore extract a common prefactor $(1/\sin^2\beta)$ and set otherwise $\sin\beta = 1$ in the non-logarithmic one-loop contributions, while the full dependence on $\sin\beta$ is kept in the logarithmic one-loop and the two-loop contributions. Since the variation of m_h with $m_{\tilde{g}}$ is ± 2 GeV at most, see Ref. [10], we have eliminated the dependence of the $\hat{\Sigma}_s^{(2)}(0)$ on the gluino mass by setting

$$m_{\tilde{g}} = M_{SUSY} \equiv \sqrt{M_S^2 - m_t^2}, \tag{12}$$

where in the case $M_{\tilde{t}_L} = M_{\tilde{t}_R} = m_{\tilde{q}}$ the SUSY scale is given by $M_{SUSY} = m_{\tilde{q}}$.

As the main step of our approximations, we have performed a Taylor expansion in $\Delta_{\tilde{t}}$ of the $\hat{\Sigma}_s(0)$ by inserting eq. (6) for the $\tilde{t}$-masses and eq. (7) for the $\tilde{t}$-mixing angle. For the one-loop correction we have expanded up to $\mathcal{O}(\Delta_{\tilde{t}}^8)$; all three renormalized one-loop Higgs-boson self-energies give a contribution. In the one-loop self-energies we have kept terms up to $\mathcal{O}(M_Z^4/m_t^4)$ while terms of $\mathcal{O}(M_Z^2/M_S^2)$ have been neglected. We have checked that the numerical effect of the latter terms is insignificant. Concerning the two-loop self-energies, the expansion has been carried out up to $\mathcal{O}(\Delta_{\tilde{t}}^4)$. With the above described approximations only $\hat{\Sigma}_{\phi_2}^{(2)}$ gives a non-zero contribution. The Taylor expanded self-energies have then been inserted into the Higgs-boson mass matrix eq. (10).

The one-loop contributions from the other sectors: For the one-loop corrections from the other sectors of the MSSM we use the logarithmic approximation given in Ref. [4]. In this approximation the scale of the soft SUSY-breaking parameter in the gaugino sector, M, is chosen as $M = M_{SUSY}$, where M_{SUSY} is defined as in eq. (12). Besides M_{SUSY}, this contribution is parameterized in terms of M_A, the mixing angle β, and the SM parameters, see eq. (25). A higher accuracy of the non-leading one-loop contributions in the approximation formula can be achieved by including further terms of the one-loop logarithmic approximation given in Ref. [13].

Corrections beyond $\mathcal{O}(\alpha\alpha_s)$: Leading contributions beyond $\mathcal{O}(\alpha\alpha_s)$ have been taken into account by incorporating the leading two-loop Yukawa correction of $\mathcal{O}(G_F^2 m_t^6)$ [12,16] and by expressing the $t, \tilde{t}$-contributions through the $\overline{\text{MS}}$ top-quark mass

$$\overline{m}_t = \overline{m}_t(m_t) \approx \frac{m_t}{1 + \frac{4}{3\pi}\alpha_s(m_t)} \tag{13}$$

instead of the pole mass m_t. This leads to an additional contribution in $\mathcal{O}(\alpha\alpha_s^2)$.

[2]Note that M_t^{LR} is treated as a free parameter in eq. (2) and therefore does not depend on μ.

2.3 The analytical approximation formula for m_h

2.3.1 The case for general M_A

With the approximations described above we obtain the following contributions to the renormalized Higgs-boson self-energies from the $t, \tilde{t}$-sector (expressed in terms of the top-quark pole mass) at one-loop order:

$$\hat{\Sigma}^{(1)}_{\phi_1}(0) = \frac{G_F\sqrt{2}}{\pi^2} M_Z^4 \Lambda \cos^2\beta \, \log\left(\frac{m_t^2}{M_S^2}\right) , \tag{14}$$

$$\hat{\Sigma}^{(1)}_{\phi_1\phi_2}(0) = -\frac{G_F\sqrt{2}}{\pi^2} M_Z^2 \cot\beta \left[-\frac{3}{8} m_t^2 + M_Z^2 \Lambda \sin^2\beta\right] \log\left(\frac{m_t^2}{M_S^2}\right) , \tag{15}$$

$$\begin{aligned}
\hat{\Sigma}^{(1)}_{\phi_2}(0) = {} & \frac{G_F\sqrt{2}}{\pi^2} \frac{m_t^4}{8\sin^2\beta} \Bigg\{ -2\frac{M_Z^2}{m_t^2} + \frac{11}{10}\frac{M_Z^4}{m_t^4} \\
& + \left[12 - 6\frac{M_Z^2}{m_t^2}\sin^2\beta + 8\frac{M_Z^4}{m_t^4}\Lambda\sin^4\beta\right] \log\left(\frac{m_t^2}{M_S^2}\right) \\
& + \frac{\left(M_t^{LR}\right)^2}{M_S^2}\left[-12 + 4\frac{M_Z^2}{m_t^2} + 6\frac{m_t^2}{M_S^2}\right] + \frac{\left(M_t^{LR}\right)^4}{M_S^4}\left[1 - 4\frac{m_t^2}{M_S^2} + 3\frac{m_t^4}{M_S^4}\right] \\
& + \frac{\left(M_t^{LR}\right)^6}{M_S^6}\left[\frac{3}{5}\frac{m_t^2}{M_S^2} - \frac{12}{5}\frac{m_t^4}{M_S^4} + 2\frac{m_t^6}{M_S^6}\right] \\
& + \frac{\left(M_t^{LR}\right)^8}{M_S^8}\left[\frac{3}{7}\frac{m_t^4}{M_S^4} - \frac{12}{7}\frac{m_t^6}{M_S^6} + \frac{3}{2}\frac{m_t^8}{M_S^8}\right]\Bigg\} ,
\end{aligned} \tag{16}$$

with

$$\Lambda = \left(\frac{1}{8} - \frac{1}{3}s_W^2 + \frac{4}{9}s_W^4\right), \qquad s_W^2 = 1 - \frac{M_W^2}{M_Z^2}. \tag{17}$$

We have verified that the logarithmic terms in eqs. (14)–(16) agree with the ones given in Ref. [4].

The two-loop contributions read:

$$\begin{aligned}
\hat{\Sigma}^{(2)}_{\phi_1}(0) &= 0 , \\
\hat{\Sigma}^{(2)}_{\phi_1\phi_2}(0) &= 0 , \\
\hat{\Sigma}^{(2)}_{\phi_2}(0) &= \frac{G_F\sqrt{2}}{\pi^2}\frac{\alpha_s}{\pi}\frac{m_t^4}{\sin^2\beta}\Bigg[3\log^2\left(\frac{m_t^2}{M_S^2}\right) - 6\log\left(\frac{m_t^2}{M_S^2}\right) - 6\frac{M_t^{LR}}{M_S} \\
&\qquad - 3\frac{\left(M_t^{LR}\right)^2}{M_S^2}\log\left(\frac{m_t^2}{M_S^2}\right) + \frac{3}{4}\frac{\left(M_t^{LR}\right)^4}{M_S^4}\Bigg] .
\end{aligned} \tag{18}$$

M_S has to be chosen according to

$$M_S = \begin{cases} \sqrt{m_{\tilde{q}}^2 + m_t^2} & : \quad M_{\tilde{t}_L} = M_{\tilde{t}_R} = m_{\tilde{q}} \\ \left[M_{\tilde{t}_L}^2 M_{\tilde{t}_R}^2 + m_t^2(M_{\tilde{t}_L}^2 + M_{\tilde{t}_R}^2) + m_t^4\right]^{\frac{1}{4}} & : \quad M_{\tilde{t}_L} \neq M_{\tilde{t}_R} \end{cases} \tag{19}$$

The last formula requires some explanation: when performing the expansion in $\Delta_{\tilde{t}}$, it was assumed that $M_{\tilde{t}_L} = M_{\tilde{t}_R}$. Thus our result contains only one soft SUSY-breaking scale M_S. For the case $M_{\tilde{t}_L} \neq M_{\tilde{t}_R}$, M_S is chosen to reproduce the argument of the leading log correctly. According to eq. (19) the log yields in both cases

$$\log\left(\frac{m_t^2}{M_S^2}\right) = \log\left(\frac{m_t^2}{m_{\tilde{t}_1}\, m_{\tilde{t}_2}}\right) + \mathcal{O}\left(\Delta_{\tilde{t}}\right) \tag{20}$$

in agreement with eq. (6).

The one-loop contributions from the other sectors of the MSSM are not listed here but can be found in Refs. [4,13]. The combined self-energies have to be inserted into eq. (10). In order to incorporate the leading QCD corrections beyond $\mathcal{O}(\alpha\alpha_s)$, the top-quark pole mass should be replaced by the $\overline{\rm MS}$ top-quark mass eq. (13) in the two-loop contribution, as described above. Diagonalization of eq. (10) yields the square of the masses of the neutral $\mathcal{CP}$-even Higgs bosons. In order to incorporate leading contributions in $\mathcal{O}(\alpha^2)$, the leading Yukawa correction, see eq. (28) below, can be added.

2.3.2 The case $M_A \gg M_Z$

The diagonalization of the mass matrix eq. (10) in the evaluation of m_h^2 incorporates contributions that are formally of higher order but are non-negligible in general. For large M_A these higher-order contributions are suppressed by inverse powers of M_A. Therefore it is possible in this case to perform an expansion in the loop order, leading to a very compact formula for m_h^2 of the form

$$m_h^2 = m_h^{2,\rm tree} + \Delta m_h^{2,\alpha,t/\tilde{t}} + \Delta m_h^{2,\alpha,\rm rest} + \Delta m_h^{2,\alpha\alpha_s} + \Delta m_h^{2,\alpha^2}. \tag{21}$$

At the two-loop level, only the ϕ_2 self-energy contributes in our approximation, yielding the term

$$\Delta m_h^{2,\alpha\alpha_s} = -\sin^2\beta\ \hat{\Sigma}^{(2)}_{\phi_2}(0) \tag{22}$$

with $\hat{\Sigma}^{(2)}_{\phi_2}(0)$ from eq. (18). In contrast to eqs. (14)–(18) we give here an expression for m_h in which the $\overline{\rm MS}$ top-quark mass is used everywhere. Inserting eq. (13) into $\hat{\Sigma}^{(1)}_{\phi_2}(0)$ yields additional contributions to $\Delta m_h^{2,\alpha\alpha_s}$, see eq. (27). From eq. (22) it becomes also clear that the β dependence of the leading two-loop contribution drops out, as mentioned above.
The tree-level and one-loop contributions of eq. (21) are given by

$$\begin{aligned}
m_h^{2,\rm tree} &= \frac{1}{2}\left[M_A^2 + M_Z^2 - \sqrt{(M_A^2+M_Z^2)^2 - 4M_Z^2 M_A^2\cos^2 2\beta}\,\right], \qquad (23)\\
\Delta m_h^{2,\alpha,t/\tilde{t}} &= \frac{G_F\sqrt{2}}{\pi^2}\,\overline{m}_t^4\Bigg[\log\left(\frac{\overline{m}_t^2}{M_S^2}\right)\Bigg\{-\frac{3}{2} - \frac{3}{4}\frac{M_Z^2}{\overline{m}_t^2}\cos 2\beta\ - \frac{M_Z^4}{\overline{m}_t^4}\Lambda\cos^2 2\beta \\
&\quad - \frac{M_Z^2}{M_A^2}\cos^2\beta\,\cos 2\beta\,\left(6 + \frac{3}{2}\frac{M_Z^2}{\overline{m}_t^2}(1-4\sin^2\beta\,) - \frac{M_Z^4}{\overline{m}_t^4}8\Lambda\cos 2\beta\,\sin^2\beta\right)\Bigg\}
\end{aligned}$$

$$
+\left\{\frac{1}{4}\frac{M_Z^2}{\overline{m}_t^2}-\frac{11}{80}\frac{M_Z^4}{\overline{m}_t^4}+\frac{\left(M_t^{LR}\right)^2}{M_S^2}\left(\frac{3}{2}-\frac{1}{2}\frac{M_Z^2}{\overline{m}_t^2}-\frac{3}{4}\frac{\overline{m}_t^2}{M_S^2}\right)\right.
$$

$$
+\frac{\left(M_t^{LR}\right)^4}{M_S^4}\left(-\frac{1}{8}+\frac{1}{2}\frac{\overline{m}_t^2}{M_S^2}-\frac{3}{8}\frac{\overline{m}_t^4}{M_S^4}\right)+\frac{\left(M_t^{LR}\right)^6}{M_S^6}\left(-\frac{3}{40}\frac{\overline{m}_t^2}{M_S^2}+\frac{3}{10}\frac{\overline{m}_t^4}{M_S^4}-\frac{1}{4}\frac{\overline{m}_t^6}{M_S^6}\right)
$$

$$
\left.+\frac{\left(M_t^{LR}\right)^8}{M_S^8}\left(-\frac{3}{56}\frac{\overline{m}_t^4}{M_S^4}+\frac{3}{14}\frac{\overline{m}_t^6}{M_S^6}-\frac{3}{16}\frac{\overline{m}_t^8}{M_S^8}\right)\right\}\left(1+4\frac{M_Z^2}{M_A^2}\cos^2\beta\,\cos 2\beta\right)\Bigg], \quad (24)
$$

$$
\Delta m_h^{2,\alpha,\mathrm{rest}} = \frac{G_F\sqrt{2}}{\pi^2}\frac{M_Z^4}{24}\Bigg[\log\left(\frac{M_{SUSY}^2}{M_Z^2}\right)\Bigg\{12N_c\frac{m_b^4}{M_Z^4}-6N_c\cos 2\beta\,\frac{m_b^2}{M_Z^2}+\cos^2 2\beta\,(P_b+P_f)
$$

$$
\left.+(P_g+P_{2H})(\sin^4\beta+\cos^4\beta)-2\sin^2\beta\,\cos^2\beta\,(P'_g+P'_{2H})\right\}+\frac{6N_c m_b^4}{M_Z^4}\times
$$

$$
\left[2\frac{\left(M_b^{LR}\right)^2}{M_{SUSY}^2}\left(1-\frac{\left(M_b^{LR}\right)^2}{12\,M_{SUSY}^2}\right)-\frac{M_Z^2}{2\,m_b^2}\cos 2\beta\,\left(\frac{\left(M_b^{LR}\right)^2}{M_{SUSY}^2}+\frac{A_b^2-\mu^2\tan^2\beta}{3\,M_{SUSY}^2}\right)\right]
$$

$$
-\log\left(\frac{M_A^2}{M_Z^2}\right)\Bigg\{(\cos^4\beta+\sin^4\beta)P_{2H}-2\cos^2\beta\,\sin^2\beta\,P'_{2H}-P_{1H}\Bigg\}\Bigg], \quad (25)
$$

where Λ is defined as in eq. (17), and

$$
\begin{aligned}
M_b^{LR} &= A_b-\mu\tan\beta \\
P_b &= N_c(1+4Q_b s_W^2+8Q_b^2 s_W^4) \\
P_f &= N_c(N_g-1)(2-4s_W^2+8(Q_t^2+Q_b^2)s_W^4)+N_g(2-4s_W^2+8s_W^4) \\
P_g &= -44+106s_W^2-62s_W^4,\quad P'_g = 10+34s_W^2-26s_W^4 \\
P_{1H} &= -9\cos^4 2\beta+(1-2s_W^2+2s_W^4)\cos^2 2\beta \\
P_{2H} &= -10+2s_W^2-2s_W^4,\quad P'_{2H} = 8-22s_W^2+10s_W^4 \\
& Q_t=\frac{2}{3},\ Q_b=-\frac{1}{3},\ N_c=3,\ N_g=3\ .
\end{aligned} \quad (26)
$$

The dominant two-loop contribution of $\mathcal{O}(\alpha\alpha_s)$ to m_h^2 reads:

$$
\begin{aligned}
\Delta m_h^{2,\alpha\alpha_s} &= -\frac{G_F\sqrt{2}}{\pi^2}\frac{\alpha_s}{\pi}\,\overline{m}_t^4\Bigg[4+3\log^2\left(\frac{\overline{m}_t^2}{M_S^2}\right)+2\log\left(\frac{\overline{m}_t^2}{M_S^2}\right)-6\frac{M_t^{LR}}{M_S} \\
&\quad -\frac{\left(M_t^{LR}\right)^2}{M_S^2}\left\{3\log\left(\frac{\overline{m}_t^2}{M_S^2}\right)+8\right\} \\
&\quad +\frac{17}{12}\frac{\left(M_t^{LR}\right)^4}{M_S^4}\Bigg]\left(1+4\frac{M_Z^2}{M_A^2}\cos^2\beta\,\cos 2\beta\right).
\end{aligned} \quad (27)
$$

For the leading two-loop Yukawa correction we use the result given in [12,16]:

$$
\Delta m_h^{2,\alpha^2} = \frac{9}{16\pi^4}G_F^2\overline{m}_t^6\left[\tilde{X}t+t^2\right]\ , \quad (28)
$$

$$
\begin{aligned}
\tilde{X} &= \Bigg[\left(\frac{m_{\tilde{t}_2}^2 - m_{\tilde{t}_1}^2}{4\overline{m}_t^2} \sin^2 2\theta_{\tilde{t}} \right)^2 \left(2 - \frac{m_{\tilde{t}_2}^2 + m_{\tilde{t}_1}^2}{m_{\tilde{t}_2}^2 - m_{\tilde{t}_1}^2} \log\left(\frac{m_{\tilde{t}_2}^2}{m_{\tilde{t}_1}^2} \right) \right) \\
&\quad + \frac{m_{\tilde{t}_2}^2 - m_{\tilde{t}_1}^2}{2\overline{m}_t^2} \sin^2 2\theta_{\tilde{t}} \log\left(\frac{m_{\tilde{t}_2}^2}{m_{\tilde{t}_1}^2} \right) \Bigg], \quad t = \frac{1}{2} \log\left(\frac{m_{\tilde{t}_1}^2 m_{\tilde{t}_2}^2}{\overline{m}_t^4} \right).
\end{aligned} \tag{29}
$$

Here M_S is chosen in analogy with eq. (19), where the top-quark pole mass m_t has to be replaced by the $\overline{\rm MS}$ top-quark mass $\overline{m}_t$.

In the contributions of the $t, \tilde{t}$-sector at one-loop and two-loop order, eqs. (24) and (27), we have included correction factors of $\mathcal{O}(M_Z^2/M_A^2)$. In this way the compact formula eq. (21) gives a reliable approximation for M_A values down to at least $M_A = 200$ GeV.

The contribution of $\mathcal{O}(\alpha\alpha_s)$ given in eq. (27) can be compared with analytical formulas derived via the two-loop effective potential approach for the case of no mixing in the $\tilde{t}$-sector [14] and via RG methods [12, 13]. The leading term $\sim \log^2(\overline{m}_t^2/M_S^2)$ agrees with the results in Refs. [12–14]. The subleading term for vanishing $\tilde{t}$-mixing $\sim \log(\overline{m}_t^2/M_S^2)$ agrees with the result of the two-loop effective potential approach [14] and the result of the two-loop RG calculation [13, 14], but differs from the RG improved one-loop result [12, 13]. The term $\sim \log(\overline{m}_t^2/M_S^2)(M_t^{LR}/M_S)^2$ for non-vanishing $\tilde{t}$-mixing differs from the result given in Ref. [12, 13]. All other terms of $\mathcal{O}(\alpha\alpha_s)$ are new. The term $\sim M_t^{LR}/M_S$ shows that the result for m_h is not symmetric in $\pm M_t^{LR}$. The good numerical agreement with the RG results in the case of no mixing in the $\tilde{t}$-sector can qualitatively be understood by noting that in the no-mixing case the leading term in both approaches agrees, while for the corrections proportional to powers of M_t^{LR}/M_S deviations occur already in the leading contribution.

When comparing the results of the diagrammatic on-shell calculation with the RG results in terms of the parameters M_S and M_t^{LR} it should be noted that, due to the different renormalization schemes employed, the meaning of these (non-observable) parameters is not precisely the same in the two approaches starting from two-loop order (see the discussion in Ref. [9]). A more detailed comparison of our results with those obtained via RG methods will be performed in a forthcoming publication.

2.4 Implementation into *FeynHiggs*

The formulas given in the previous section have been implemented into the Fortran code *FeynHiggs* [17], thus allowing a direct comparison between the full result described in Refs. [9, 10] and the approximation formula (21).

We also provide the Fortran code *FeynHiggsFast* in which only the formula for the compact approximation of m_h is implemented, thus allowing an approximate but much faster evaluation of the Higgs-boson mass m_h. This program is shorter by a factor of 50 and faster by a factor of 3×10^4 with respect to *FeynHiggs*; it needs only 2×10^{-5} seconds on a Sigma station (Alpha CPU, 600 MHz) for the evaluation of m_h for one set of parameters. The quality of the prediction of m_h is reasonably good for a large part of the MSSM parameter space, as will be discussed in the next section. Into *FeynHiggsFast* we have also implemented the calculation of the MSSM contributions to $\Delta\rho$ [18]. Here the corrections arising from

$\tilde{t}/\tilde{b}$-loops up to $\mathcal{O}(\alpha\alpha_s)$ have been taken into account, neglegting only the gluino-exchange contribution which is very lengthy and vanishes for large $m_{\tilde{g}}$. The ρ-parameter can be used as an additional constraint (besides the experimental bounds) on the squark masses. A value of $\Delta\rho$ outside the experimentally preferred region of $\Delta\rho^{\rm SUSY} \lesssim 1 \times 10^{-3}$ [19] indicates experimentally disfavored $\tilde{t}$- and $\tilde{b}$-masses.

Both Fortran codes can be obtained via the WWW page
`http://www-itp.physik.uni-karlsruhe.de/feynhiggs` .

3 Discussion of the compact expression for m_h

In this section we analyze the quality of our compact approximation formula for m_h with respect to the full calculation, which contains the full diagrammatic one-loop contribution [6], the complete leading two-loop corrections in $\mathcal{O}(\alpha\alpha_s)$ [8–10], and the two contributions beyond $\mathcal{O}(\alpha\alpha_s)$, see eq. (13) and eq. (28). We use here the approximation formula for general M_A, as discussed in Sec. 2.3.1. As mentioned above, in the region $M_A \gtrsim 200$ GeV the compact formula given in Sec. 2.3.2 yields equally good results.

For $\tan\beta$ we restrict ourselves to two typical values which are favored by SUSY-GUT scenarios [20]: $\tan\beta = 1.6$ for the $SU(5)$ scenario and $\tan\beta = 40$ for the $SO(10)$ scenario. Other parameters are $M_Z = 91.187$ GeV, $M_W = 80.39$ GeV, $G_F = 1.16639 \times 10^{-5}$ GeV^{-2}, $\alpha_s(m_t) = 0.1095$, $m_t = 175$ GeV, and $m_b = 4.5$ GeV. In the numerical evaluation we have furthermore chosen $A_t = A_b$. The parameter M appearing in the plots is the $SU(2)$ gaugino mass parameter, it enters in the full result only, see the discussion of the one-loop contributions in Sect. 2.2. μ is the Higgs-mixing parameter. If not indicated differently, we have chosen $M = m_{\tilde{q}}$ and $\mu = -m_{\tilde{q}}$.

In Fig. 1 we show m_h as a function of $M_t^{LR}/m_{\tilde{q}}$ for $m_{\tilde{q}} = 200, 500, 1000$ GeV and $M_A = 500$ GeV. A maximum for m_h, evaluated with the full formula as well as with the approximation formula, is reached for about $M_t^{LR}/m_{\tilde{q}} \approx \pm 1.9$ in the $\tan\beta = 1.6$ scenario and in the $\tan\beta = 40$ scenario. This case we refer to as 'maximal mixing'. A minimum is reached around $M_t^{LR}/m_{\tilde{q}} \approx 0$ which we refer to as 'no mixing'. In general the approximation differs from the full result by less than 2 GeV up to $|M_t^{LR}/m_{\tilde{q}}| \lesssim 2$. For larger $|M_t^{LR}|$ sizeable deviations occur, which become very large for $|M_t^{LR}/m_{\tilde{q}}| \gtrsim 2.5$. The reason is that the expansion parameter $\Delta_{\tilde{t}}$ becomes rather large and approaches 1 in this region. The effect of the non-logarithmic terms in the two-loop contribution in $\mathcal{O}(\alpha\alpha_s)$ reaches up to about 5 GeV for maximal mixing.

The location of the extrema can be understood analytically as follows: taking into account only the leading one-loop corrections from the $t, \tilde{t}$-sector for the calculation of m_h, one can easily compute the values of $M_t^{LR}/m_{\tilde{q}}$ for which m_h reaches a maximum or a minimum. The well known results are

$$\frac{M_t^{LR}}{m_{\tilde{q}}} = \begin{cases} \sqrt{6} & \text{(maximum)} \\ 0 & \text{(minimum)} \\ -\sqrt{6} & \text{(maximum)} \end{cases} . \tag{30}$$

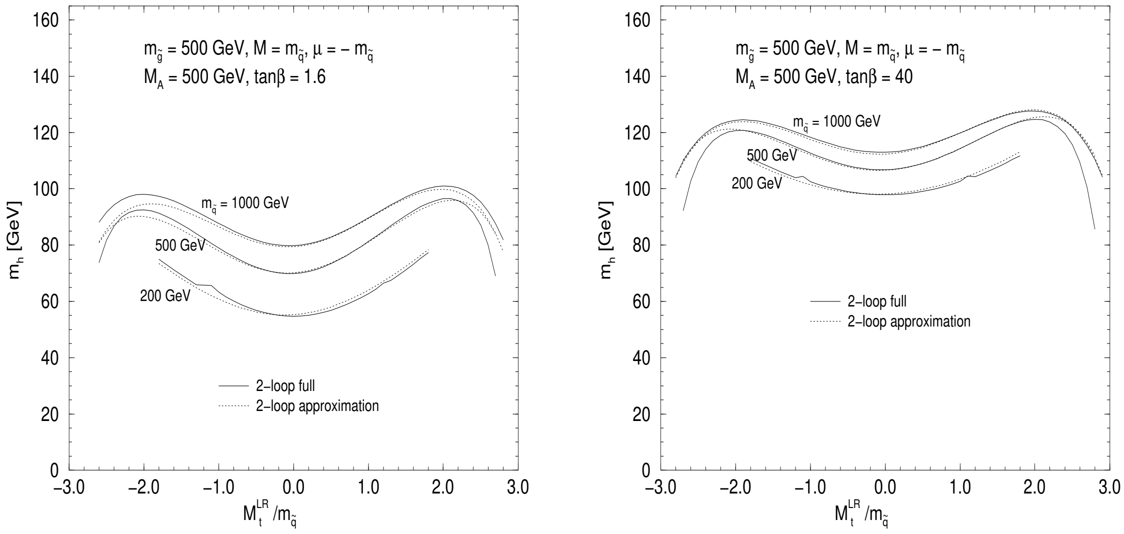

Figure 1: m_h as a function of $M_t^{LR}/m_{\tilde{q}}$, calculated from the full formula and from the approximation formula for $M_A = 500$ GeV, $m_{\tilde{g}} = 500$ GeV and $\tan\beta = 1.6$ or 40.

Taking into account the new two-loop corrections $\Delta m_h^{2,\alpha\alpha_s}$, given in eq. (27), the positions of the extrema in eq. (30) receive a shift, yielding up to $\mathcal{O}(\alpha\alpha_s)$:

$$\frac{M_t^{LR}}{m_{\tilde{q}}} = \begin{cases} \sqrt{6} - \frac{\alpha_s}{\pi}\left[-1 + 3\sqrt{6} - \sqrt{6}\log\left(\frac{\overline{m}_t^2}{M_S^2}\right)\right] & (\approx +1.92 \text{ for } M_S = 1000 \text{ GeV; max.}) \\ -2\frac{\alpha_s}{\pi} & (\approx -0.07; \text{ minimum}) \\ -\sqrt{6} + \frac{\alpha_s}{\pi}\left[1 + 3\sqrt{6} - \sqrt{6}\log\left(\frac{\overline{m}_t^2}{M_S^2}\right)\right] & (\approx -1.85 \text{ for } M_S = 1000 \text{ GeV; max.}) \end{cases} \tag{31}$$

The maxima are shifted to smaller absolute values of $M_t^{LR}/m_{\tilde{q}}$, the minimum is shifted to a slightly negative value (see the discussion in Refs. [8–10]).

Fig. 2 shows the dependence of m_h on $m_{\tilde{q}}$ for the cases of no mixing and maximal mixing, and we have set $M_A = 500$ GeV. Very good agreement is found in the no-mixing scenario as well as in the maximal-mixing scenario, the deviation lies below 2 GeV.

The dependence on M_A is shown in Fig. 3. The quality of the approximation is typically better than 1 GeV for the no-mixing case and better than 2 GeV for the maximal-mixing case. Only for very small (and experimentally already excluded) values of M_A a deviation of 5 GeV occurs. The peaks in the plot for $\tan\beta = 1.6$ in the full result are due to the threshold $M_A = 2\,m_t$ in the one-loop contribution, originating from the top-loop diagram in the A self-energy. This peak does not occur in the approximation formula and can thus lead to a larger deviation around the threshold.

In deriving the leading and subleading corrections we have set $m_{\tilde{g}} = \sqrt{M_S^2 - m_t^2}$, thus eliminating this additional scale. A variation of $m_{\tilde{g}}$ in the full result directly corresponds to a shift relative to the approximation formula. As it was shown in Ref. [10], this deviation is negligible for the no-mixing case and lies within ± 2 GeV in the maximal-mixing case. It was

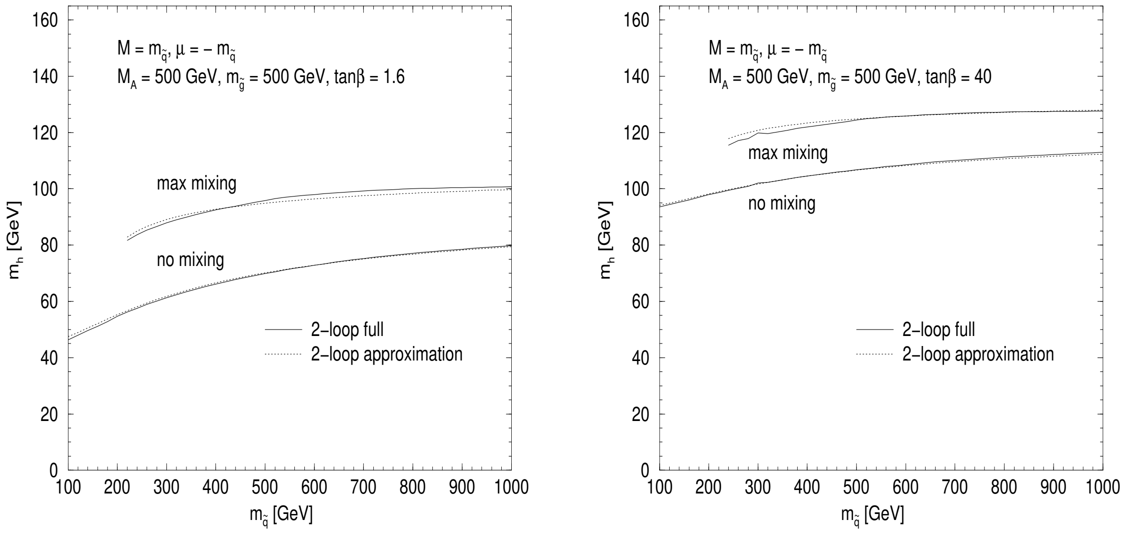

Figure 2: m_h as a function of $m_{\tilde{q}}$, calculated from the full formula and from the approximation formula for $M_A = 500$ GeV, $m_{\tilde{g}} = 500$ GeV and $\tan\beta = 1.6$ or 40.

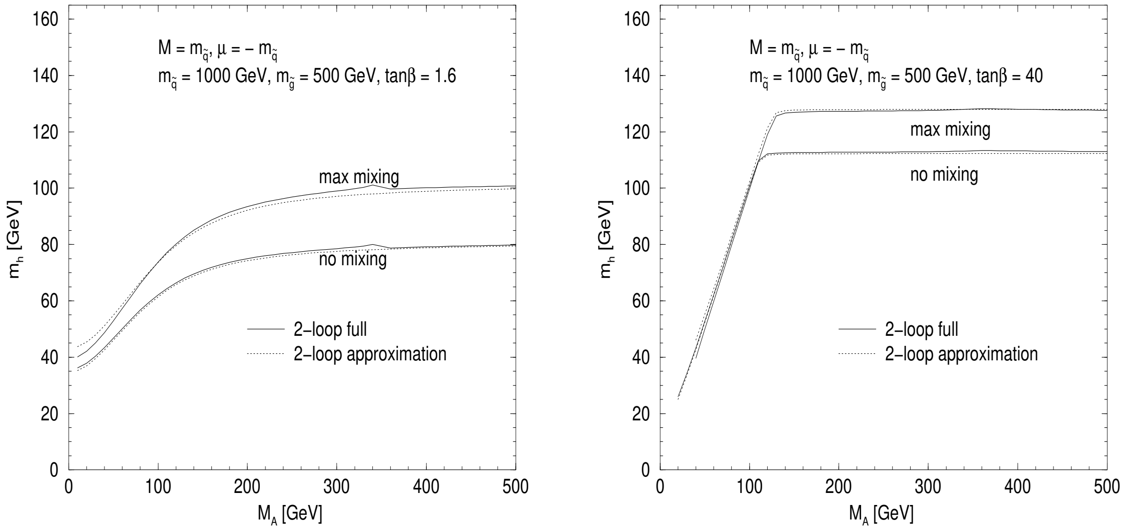

Figure 3: m_h as a function of M_A, calculated from the full formula and from the approximation formula for $m_{\tilde{q}} = 1000$ GeV, $m_{\tilde{g}} = 500$ GeV and $\tan\beta = 1.6$ or 40.

also analyzed in Ref. [10] that the variation of m_h with the $SU(2)$ gaugino mass parameter $M (= M_2)$ and with the Higgs-mixing parameter μ is relatively weak. We have found only small deviations of the approximation formula with respect to the full result, not more than 2 GeV for small values of M, and 1.5 GeV for small values of μ.

Finally we consider the case when $M_{\tilde{t}_L} \neq M_{\tilde{t}_R}$, which is shown in Fig. 4. Here also the case $M_t^{LR} = M_{\tilde{t}_L}$ is depicted, which we refer to as 'moderate mixing'. 'No mixing' here corresponds to $M_t^{LR} = 0$, 'maximal mixing' corresponds to the choice $M_t^{LR} = 2\, M_{\tilde{t}_L}$. For fixed $M_{\tilde{t}_R}$ this leads to a $\tilde{t}$-mass below the experimental lower bound for nearly the whole

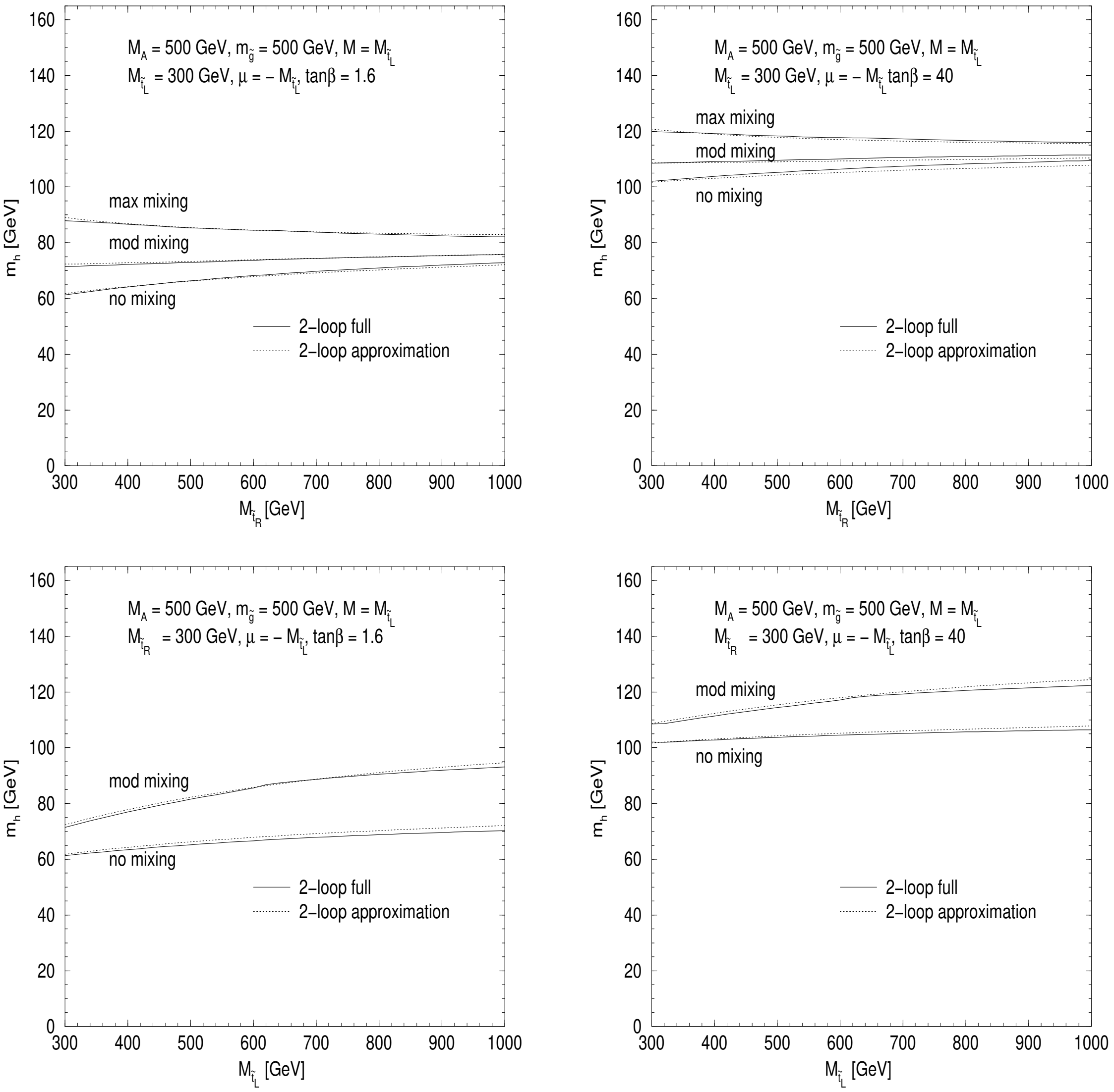

Figure 4: m_h as a function of $M_{\tilde{t}_R}$ or $M_{\tilde{t}_L}$, calculated from the full formula and from the approximation formula for varied $M_{\tilde{t}_R}$ ($M_{\tilde{t}_L}$), $M_{\tilde{t}_L}$ ($M_{\tilde{t}_R}$) $= 300$ GeV, $M_A = 500$ GeV, $m_{\tilde{g}} = 500$ GeV and $\tan\beta = 1.6$ or 40.

parameter space; thus we have omitted this case in this scenario (bottom of Fig. 4). Here we have made use of eq. (19). Agreement better than 2 GeV is found for all scenarios.

4 Conclusions

By means of a Taylor expansion of the diagrammatic two-loop result we have derived the leading logarithmic and non-logarithmic as well as subleading contributions to the lightest

Higgs-boson mass m_h up to $\mathcal{O}(\alpha\alpha_s)$. This result has been incorporated into a compact analytical formula for m_h^2, which can easily be implemented for numerical evaluation. It contains the tree-level expression, leading and subleading one-loop contributions, the leading logarithmic and non-logarithmic two-loop contributions, and two further corrections beyond $\mathcal{O}(\alpha\alpha_s)$. This formula is valid for general mixing in the scalar top sector and arbitrary choices of the parameters in the Higgs sector of the model. It has been included into the Fortran codes *FeynHiggs* and *FeynHiggsFast*, thus allowing a very fast and reasonably accurate evaluation of m_h. The codes are available via the WWW page
`http://www-itp.physik.uni-karlsruhe.de/feynhiggs` .

The approximation formula has been compared with the full diagrammatic result, in which the complete one-loop and the dominant two-loop corrections of $\mathcal{O}(\alpha\alpha_s)$ are taken into account without Taylor expansion. Good agreement, better than 2 GeV, is found for most parts of the MSSM parameter space. Larger deviations, exceeding a few GeV, can occur for large mixing in the $\tilde{t}$-sector, $|M_t^{LR}/m_{\tilde{q}}| \gtrsim 2$. The effect of the corrections of $\mathcal{O}(\alpha\alpha_s)$ of shifting the maxima for m_h towards smaller values of $|M_t^{LR}/m_{\tilde{q}}|$ can easily be read off from the compact formula by analytically determining the extrema of m_h.

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References

[1] G. Kane, C. Kolda and J. Wells, *Phys. Rev. Lett.* **70** (1993) 2686, hep-ph/9210242;
J. Espinosa and M. Quirós, *Phys. Lett.* **B 302** (1993) 51, hep-ph/9212305; *Phys. Rev. Lett.* **81** (1998) 516, hep-ph/9804235.

[2] H. Haber and G. Kane, *Phys. Rep.* **117** (1985) 75.
H.P. Nilles, *Phys. Rep.* **110** (1984) 1.

[3] H. Haber and R. Hempfling, *Phys. Rev. Lett.* **66** (1991) 1815;
Y. Okada, M. Yamaguchi and T. Yanagida, *Prog. Theor. Phys.* **85** (1991) 1;
J. Ellis, G. Ridolfi and F. Zwirner, *Phys. Lett.* **B 257** (1991) 83; *Phys. Lett.* **B 262** (1991) 477;
R. Barbieri and M. Frigeni, *Phys. Lett.* **B 258** (1991) 395.

[4] H. Haber and R. Hempfling, *Phys. Rev.* **D 48** (1993) 4280, hep-ph/9307201.

[5] P. Chankowski, S. Pokorski and J. Rosiek, *Nucl. Phys.* **B 423** (1994) 437.

[6] A. Dabelstein, *Nucl. Phys.* **B 456** (1995) 25, hep-ph/9503443; *Z. Phys.* **C 67** (1995) 495, hep-ph/9409375.

[7] J. Bagger, K. Matchev, D. Pierce and R. Zhang, *Nucl. Phys.* **B 491** (1997) 3, hep-ph/9606211.

[8] S. Heinemeyer, W. Hollik and G. Weiglein, *Phys. Rev.* **D 58** (1998) 091701, hep-ph/9803277.

[9] S. Heinemeyer, W. Hollik and G. Weiglein, *Phys. Lett.* **B 440** (1998) 296, hep-ph/9807423.

[10] S. Heinemeyer, W. Hollik and G. Weiglein, CERN-TH/98-405, hep-ph/9812472, to appear in *Eur. Phys. Jour.* **C**.

[11] J. Casas, J. Espinosa, M. Quirós and A. Riotto, *Nucl. Phys.* **B 436** (1995) 3, E: *ibid.* **B 439** (1995) 466, hep-ph/9407389.

[12] M. Carena, J. Espinosa, M. Quirós and C. Wagner, *Phys. Lett.* **B 355** (1995) 209, hep-ph/9504316;
M. Carena, M. Quirós and C. Wagner, *Nucl. Phys.* **B 461** (1996) 407, hep-ph/9508343.

[13] H. Haber, R. Hempfling and A. Hoang, *Z. Phys.* **C 75** (1997) 539, hep-ph/9609331.

[14] R. Hempfling and A. Hoang, *Phys. Lett.* **B 331** (1994) 99, hep-ph/9401219.

[15] R.-J. Zhang, *Phys. Lett.* **B 447** (1999) 89, hep-ph/9808299.

[16] M. Carena, P. Chankowski, S. Pokorski and C. Wagner, *Phys. Lett.* **B 441** (1998) 205, hep-ph/9805349.

[17] S. Heinemeyer, W. Hollik and G. Weiglein, CERN-TH/98-389, hep-ph/9812320.

[18] A. Djouadi, P. Gambino, S. Heinemeyer, W. Hollik, C. Jünger and G. Weiglein, *Phys. Rev. Lett.* **78** (1997) 3626, hep-ph/9612363; *Phys. Rev.* **D 57** (1998) 4179, hep-ph/9710438.

[19] G. Altarelli, hep-ph/9811456.

[20] M. Carena, S. Pokorski and C. Wagner, *Nucl. Phys.* **B 406** (1993) 59, hep-ph/9303202; W. de Boer et al., *Z. Phys.* **C 71** (1996) 415, hep-ph/9603350.