CPT anomaly in two-dimensional chiral $U(1)$ gauge theories

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(preprint KA–TP–11–2000, NBI–HE–00–27, hep-th/0006154)

The CPT anomaly, which was first seen in perturbation theory for certain four-dimensional chiral gauge theories, is also present in the exact result for a class of two-dimensional chiral $U(1)$ gauge theories on the torus. Specifically, the chiral determinant for periodic fermion fields changes sign under a CPT transformation of the background gauge field. There is, in fact, an anomaly of Lorentz invariance, which allows for the CPT theorem to be circumvented.

PACS numbers 11.15.-q; 11.10.Kk; 11.30.Cp; 11.30.Er

I. INTRODUCTION

Recently, a CPT anomaly has been found in certain four-dimensional chiral gauge theories, with the topology and spin structure of the spacetime manifold playing a crucial role [1]. The well-known CPT theorem [2] is circumvented by the breakdown of Lorentz invariance at the quantum level [1,3]. The calculation of Ref. [1] was done perturbatively and more or less the same type of anomaly was expected to occur in appropriate higher- and lower-dimensional chiral gauge theories. Here, we consider the two-dimensional chiral $U(1)$ gauge theory over the torus, for which the chiral determinant is known *exactly* [4–6]. The aim of this paper is to determine whether or not the exact result contains the CPT anomaly and perhaps to learn more about the anomaly itself [7].

II. CHIRAL DETERMINANT

We consider in this paper two-dimensional Euclidean chiral $U(1)$ gauge theory, defined over the torus T^2. For simplicity, we take a particular torus (modulus $\tau = i$), with Cartesian coordinates $x^\mu \in [0, L]$, $\mu = 1, 2$, and Euclidean metric $g_{\mu\nu} = \delta_{\mu\nu}$. The theory has the fermionic action

$$S[A,\bar\psi,\psi] = -\int_0^L \mathrm{d}x^1 \int_0^L \mathrm{d}x^2 \; \bar\psi\, \sigma^\mu (\partial_\mu + iA_\mu)\, \psi \; , \quad (1)$$

with $\sigma^1 = 1$ and $\sigma^2 = i$. The boundary conditions for the real gauge potential $A(x) \equiv A_\mu(x)\, \mathrm{d}x^\mu$ and the 1-component Weyl field $\psi(x)$ are *both* taken to be periodic,

$$\begin{aligned} A(x^1 + mL, x^2 + nL) &= A(x^1, x^2) \; , \\ \psi(x^1 + mL, x^2 + nL) &= \psi(x^1, x^2) \; , \end{aligned} \quad (2)$$

for arbitrary integers m and n.

The two-dimensional gauge potential in the trivial topological sector can be decomposed as follows [4]

$$A_\mu(x) = \epsilon_{\mu\nu}\, g^{\nu\rho}\, \partial_\rho \phi(x) + 2\pi h_\mu / L + \partial_\mu \chi(x) \; , \quad (3)$$

with $\phi(x)$ and $\chi(x)$ real periodic functions and h_μ real constants (the harmonic pieces of the gauge potential). Here, $\chi(x)$ corresponds to the gauge degree of freedom. Furthermore, the gauge potential $A_\mu(x)$ is taken to be smooth, i.e. without delta-function singularities.

The chiral determinant (the exponential of minus the Euclidean effective action) is then given by the following functional integral:

$$\begin{aligned} D^{\mathrm{PP}}[A] &\equiv \exp\left(-\Gamma^{\mathrm{PP}}[A]\right) \\ &= \int_{\mathrm{PP}} \mathcal{D}\psi\, \mathcal{D}\bar\psi \; \exp\left(-S[A,\bar\psi,\psi]\right) \; , \end{aligned} \quad (4)$$

where PP indicates the doubly-periodic boundary condition (2) on the fermion field. This chiral determinant has been calculated using various regularization methods. See Refs. [4–6] and references therein. Ref. [6], in particular, introduces a local counterterm to restore translation invariance and obtains the following result [9]:

$$\begin{aligned} D^{\mathrm{PP}}[A] &= \hat\vartheta(h_1 + \tfrac{1}{2}, h_2 + \tfrac{1}{2}) \; \exp\left(\frac{i\pi}{2}(h_1 - h_2)\right) \\ &\quad \times \exp\left(\frac{1}{4\pi} \int \mathrm{d}^2x \left(\phi\, \partial^2 \phi + i\phi\, \partial^2 \chi\right)\right) , \end{aligned} \quad (5)$$

with, for real variables k_1 and k_2, the definition [5,6]

$$\begin{aligned} \hat\vartheta(k_1, k_2) &\equiv \exp\left(-\pi (k_2)^2 + i\pi k_1 k_2\right) \\ &\quad \times \vartheta(k_1 + ik_2; i)/\eta(i) \; , \end{aligned} \quad (6)$$

in terms of the Riemann theta function and Dedekind eta function

$$\begin{aligned} \vartheta(z;\tau) &\equiv \sum_{n=-\infty}^{\infty} \exp\left(\pi i n^2 \tau + 2\pi i n z\right) , \\ \eta(\tau) &\equiv \exp\left(\pi i \tau / 12\right) \prod_{m=1}^{\infty} \left(1 - \exp\left(2\pi i m \tau\right)\right) \; . \end{aligned} \quad (7)$$

The result (5) holds for the chiral determinant of a single positive chirality (right-moving) Weyl fermion of unit charge, cf. Eq. (1). If the charge is q_{R1} instead, then the variables h_μ, $\phi(x)$, and $\chi(x)$ in Eq. (5) each need to be multiplied by a factor q_{R1}. For a negative chirality (left-moving) Weyl fermion of charge q_{L1}, one also has to take the complex conjugate of the whole expression (5).

For the 345-model (three chiral fermions with charges $q_{R1}=3$, $q_{R2}=4$, and $q_{L3}=5$), one obtains the following chiral determinant [6]:

$$D_{345}^{\mathrm{PP}}[A]=D^{\mathrm{PP}}[3A]\,D^{\mathrm{PP}}[4A]\,\left(D^{\mathrm{PP}}[5A]\right)^{*}\ . \tag{8}$$

The chiral determinant (8) of the 345-model is gauge invariant. Indeed, it is straightforward to verify both the χ-independence and the invariance under large gauge transformations $h_\mu \to h_\mu + n_\mu$ for arbitrary integers n_μ [10]. We will first focus on this particular chiral model. Other chiral models will be discussed later.

III. CPT NONINVARIANCE

The question, now, is how the gauge-invariant chiral determinant (8) of the 345-model behaves under a CPT transformation of the background gauge field,

$$A_\mu(x)\ \to\ A_\mu^{\mathrm{CPT}}(x)\equiv -A_\mu(-x)\ . \tag{9}$$

Using the elementary properties of the theta function [11], one finds

$$D_{345}^{\mathrm{PP}}[A^{\mathrm{CPT}}]=-D_{345}^{\mathrm{PP}}[A]\ , \tag{10}$$

with each of the three chiral fermions contributing a multiplicative factor -1 on the right-hand side. Hence, the effective action of the chiral $U(1)$ gauge theory with PP spin structure over the torus *changes* under a CPT transformation (9) of the background gauge field, provided the total number (N_F) of charged chiral fermions of the theory is *odd* (e.g. $N_F=3$ for the 345-model). The result (10) thus provides conclusive evidence for a CPT anomaly of the chiral model considered.

The asymmetry (10) implies the vanishing of the chiral determinant (8) for $A_\mu(x)=0$. For gauge fields (3) with $\phi(x)=\chi(x)=0$ and infinitesimal harmonic pieces h_μ, one has, in fact,

$$D_{345}^{\mathrm{PP}}[h_1,h_2]=c\,(h_1+ih_2)\,(h_1^2+h_2^2)+\mathrm{O}(h^5)\ , \tag{11}$$

with a nonvanishing complex constant c. This result follows from the observation that the analytic function $\vartheta(z;i)$ appearing in Eq. (5) has a simple zero at $z=(1+i)/2$. More directly, the holomorphic factor (h_1+ih_2) in Eq. (11) corresponds to one of the eigenvalues of the Weyl operator $\sigma^\mu\,(\partial_\mu+iA_\mu)$ with doubly-periodic boundary condition and constant gauge potential, as do the holomorphic and antiholomorphic factors contained in $(h_1^2+h_2^2)$. Eq. (11) agrees, of course, with the general result (10) on CPT violation. But the real importance of Eq. (11) is that, for this special case, the *origin* of the two-dimensional CPT anomaly can be identified explicitly, namely one particular eigenvalue of the Weyl operator. (See [12] for further details.)

The chiral determinant of the 345-model over the torus [6] is CPT invariant for the other spin structures AA, PA, and AP, where (A)P stands for (anti-)periodic boundary conditions on the fermion fields (the three fermion species being treated equally). This appears to be related to the observation that the CPT anomaly is not expected for the AA spin structure [1,3] and the fact that the chiral determinants for the AA, PA, and AP spin structures [6] transform into each other under modular transformations (global diffeomorphisms, cf. Ref. [4]), whereas the chiral determinant of the PP spin structure is invariant up to a phase. It is important to realize that this extra requirement of modular invariance for the AA, PA, and AP spin structures restricts the type of theories considered, and also possible regularization methods [13]. For the general question of how to sum over the different spin structures, see, for example, the discussion in Refs. [15,16]. In our case, the two-dimensional CPT anomaly would be present as long as the PP spin structure appears in the sum.

IV. LORENTZ NONINVARIANCE

Given that CPT invariance no longer holds for the 345-model with doubly-periodic spin structure over the torus, $SO(1,1)$ Lorentz invariance, or rather $SO(2)$ invariance for the Euclidean theory, is expected to be broken as well [1,3]. Concretely, this can be tested by comparing the (translation-invariant) chiral determinant (8) for two different, *localized* gauge fields which are related by a Lorentz transformation [17].

Consider, for example, a gauge potential $\tilde{A}_\mu(x)$ which, up to periodicity, is allowed to be nonzero only for $|x^\mu - L/2| < \ell$, with a fixed length $\ell << L/2$, and which has infinitesimal, but nonvanishing, harmonic pieces $\tilde{h}_\mu \equiv (2\pi L)^{-1}\int \mathrm{d}^2x\,\tilde{A}_\mu(x)$. In other words, the gauge potential $\tilde{A}_\mu(x)$ has local support (set by ℓ) and produces small, but nonzero, averages $\tilde{h}_\mu$ (typically of order ℓ/L). According to Eq. (11), the chiral determinant for this gauge field is then proportional to $(\tilde{h}_1+i\tilde{h}_2)=\sigma^\mu\,\tilde{h}_\mu$. Similarly, the chiral determinant for the $SO(2)$ Lorentz transformed ("boosted") gauge potential

$$\begin{pmatrix}\tilde{A}_1'(x)\\ \tilde{A}_2'(x)\end{pmatrix}=\Lambda\cdot\begin{pmatrix}\tilde{A}_1(\Lambda\,x)\\ \tilde{A}_2(\Lambda\,x)\end{pmatrix},\quad \Lambda\equiv\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}, \tag{12}$$

is proportional to $\sigma^\mu\,\tilde{h}_\mu'$. But these two particular factors differ by a phase factor $\exp(i\alpha)$, as can be readily verified. All other factors of the two chiral determinants being equal, this then implies

$$D_{345}^{\mathrm{PP}}[\tilde{A}']=\exp(i\alpha)\,D_{345}^{\mathrm{PP}}[\tilde{A}]\ . \tag{13}$$

Note that Eq. (13), for $\alpha=\pi$, agrees with the previous result (10). Also remark that the noninvariance of the factor $\sigma^\mu\,\tilde{h}_\mu$ in the chiral determinant directly carries over to the theory with Minkowskian metric $g_{\mu\nu}=$ diag

$(+1,-1)$. In short, the Lorentz invariance of the chiral determinant (8) for the localized gauge field $\tilde{A}_\mu(x)$ is broken through its $\tilde{h}_\mu$ dependence. (The term $\int \mathrm{d}^2x\, \tilde{\phi}\, \partial^2 \tilde{\phi}$ from Eq. (5) is, of course, Lorentz invariant.)

As far as the gauge potential is concerned, this localized configuration $\tilde{A}_\mu(x)$ could also have been embedded in the Euclidean plane $\mathbb{R}^2$. The Lorentz noninvariance of the effective gauge field action comes from the chiral fermions which are sensitive to the topology of the torus T^2. More physically, the periodic boundary conditions *predispose* the chiral fermions of the 345-model to select specific $\tilde{h}_\mu$–dependent terms from the local interaction with the gauge field. These special terms in the effective action then make the local dynamics of the (classical) gauge field $\tilde{A}_\mu(x)$ Lorentz noninvariant.

V. OTHER CHIRAL MODELS

Up till now, we have focused on the 345-model, which has an odd number of charged chiral fermions ($N_F = 3$). A chiral model with even N_F does not have the CPT anomaly discussed above, but can still be Lorentz noninvariant. An example for $N_F = 10$ would be the $1^9 3$-model, which has ten chiral fermions with charges $q_{Ri} = 1$, for $i = 1, \ldots, 9$, and $q_{L10} = 3$. For this model, the chiral determinant (11) becomes

$$D^{\mathrm{PP}}_{1^9 3}[h_1, h_2] = c' \, (h_1 + i h_2)^8 \, (h_1^2 + h_2^2) + \mathrm{O}(h^{12}) \, , \quad (14)$$

which is invariant under the CPT transformation (9), but changes under the $SO(2)$ Lorentz transformation (12) by a phase factor $\exp(i 8 \alpha)$. On the other hand, a chiral model with even N_F can also be Lorentz invariant, in the sense discussed above. An example would be the chiral model with $N_F = 6$ chiral fermions of charges $\{q_R\} = \{3, 4, 13\}$ and $\{q_L\} = \{5, 5, 12\}$, for which the chiral determinant is $c'' \, (h_1^2 + h_2^2)^3$ to lowest order. (Vector-like models, which have $\{q_R\} = \{q_L\}$, are always Lorentz invariant.) Clearly, a deeper understanding of what distinguishes these gauge-invariant chiral models remains to be desired.

VI. DISCUSSION

For the two-dimensional chiral $U(1)$ gauge theory with an odd number N_F of charged chiral fermions defined over the torus, we have thus seen that the CPT non-invariance of the effective gauge field action $\Gamma^{\mathrm{PP}}[A]$ is carried by the harmonic pieces h_μ of the gauge fields $A_\mu(x)$. These h_μ are of the same type as the local Chern–Simons-like terms encountered previously in four dimensions [1,3]. Indeed, the Chern–Simons one-form for an one-dimensional Abelian $U(1)$ gauge field $a(x)$ is given by

$$\omega_{\mathrm{CS}}[a] \equiv (2\pi)^{-1} \, a(x) \, \mathrm{d}x \, . \quad (15)$$

One possible two-dimensional Chern–Simons-like term is then the average over the x^2-coordinate of $2\pi i$ times the genuine Chern–Simons term for the x^1-space S^1, namely

$$\Gamma^{S^1 \times S^1}_{\mathrm{CS-like},1}[A] \equiv \int_0^L \frac{\mathrm{d}x^2}{L} \left(2\pi i \int_{S^1} \omega_{\mathrm{CS}}[A_1] \right)$$
$$= i \int_0^L \mathrm{d}x^1 \int_0^L \mathrm{d}x^2 \, A_1(x^1, x^2)/L = 2\pi i h_1 \, , \quad (16)$$

where h_1 is defined by Eq. (3). The other two-dimensional Chern–Simons-like term (based on the genuine Chern–Simons term for the x^2-space) equals $2\pi i h_2$. Hence, Chern–Simons-like terms play a role for the CPT anomaly in both two and four dimensions. There is, however, a difference, in that the four-dimensional Chern–Simons-like term immediately affects the gauge field propagation, with the vacuum becoming optically active [1,18].

In closing, we remark that the CPT noninvariance found here appears to be not directly related to the purely gravitational anomaly which afflicts Weyl fermions in two dimensions ($4\,k{+}2$ dimensions in general) [8]. The gravitational anomaly (breakdown of general coordinate invariance) of the two-dimensional 345-model, say, shows up for deviations from the Euclidean metric $g_{\mu\nu} = \delta_{\mu\nu}$, but in our case the metric is perfectly Euclidean and, still, the effective gauge field action $\Gamma^{\mathrm{PP}}[A]$ is CPT noninvariant. Instead of local spacetime fluctuations, it is the spacetime topology (and spin structure) that is relevant to the CPT anomaly. The CPT anomaly resembles in this respect the so-called topological Casimir effect [19].

ACKNOWLEDGMENTS

This work was started during the sabbatical leave of one of us (F.R.K.) and the hospitality of the High-Energy Theory group at the Niels Bohr Institute is gratefully acknowledged. J.N. is supported by the Japan Society for the Promotion of Science as a Postdoctoral Fellow for Research Abroad.

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[9] To arrive at Eq. (5) we have chosen factors $P[b_1] = +\frac{1}{2}$ and $P[b_2] = +\frac{1}{2}$ in Eq. (2.18) of Ref. [6] but other signs are also possible, as long as they are fixed once and for all.

[10] For the 345-model with PP spin structure, the chiral determinant from Ref. [5] would not be invariant under arbitrary large gauge transformations. This shows that the extra phase factor $\exp(i\pi(h_1 - h_2)/2)$ in Eq. (5), which traces back to the local counterterm of Ref. [6], also plays a role in maintaining the Abelian gauge invariance of the full chiral determinant (8).

[11] The relevant properties of $\vartheta(z;\tau)$ are its periodicity under $z \to z+1$ and quasi-periodicity under $z \to z+\tau$, together with the symmetry $\vartheta(-z;\tau) = \vartheta(z;\tau)$. We take the opportunity to correct Eq. (4.13) of Ref. [6], which must have an extra factor $\exp(i\pi n_1 n_2)$ on the right-hand side.

[12] For constant gauge potentials $A_\mu(x) = 2\pi h_\mu/L$, the single chiral determinant $D^{\mathrm{PP}}[h_1,h_2]$ from Eq. (4) is, formally, proportional to $\prod(n_1 + h_1 + in_2 + ih_2)$, with the product running over the integers n_1 and n_2. The CPT-odd factor $(h_1 + ih_2)$ is manifest for $n_1 = n_2 = 0$ (which corresponds to the eigenvector $\psi(x) = \mathrm{const.}$ of the Weyl operator), whereas the other contributions combine into CPT-even factors whose product still needs to be regularized appropriately. The full chiral determinant (8) of the 345-model then has factors $3\,(h_1+ih_2)$, $4\,(h_1+ih_2)$, and $5\,(h_1-ih_2)$, which combine to give the result (11).

[13] For the two-dimensional chiral $U(1)$ gauge theory on a torus, we have also calculated the chiral determinant for constant gauge potentials $A_1(x) = 2\pi h_1/L$ and $A_2(x) = 0$, using the partial regularization method adopted in Ref. [1]. For this setup, the Fourier modes of the x^2-direction decouple and each of them can be regarded as a one-dimensional Dirac fermion in x^1-space. The regularization method then amounts to introducing one-dimensional Pauli-Villars fields over x^1-space for each Fourier mode from the x^2-direction. The result for the chiral determinant differs in general from the one obtained in Ref. [6] and behaves differently under modular transformations. In particular, CPT violation is observed if the boundary conditions on the fermion fields are taken to be periodic in the x^2-direction and either periodic or antiperiodic in the x^1-direction, whereas the result from Ref. [6] is CPT violating only for the doubly-periodic (PP) spin structure, as shown by Eq. (10). The apparent regularization dependence of the theory deserves further study, cf. Ref [14]. Still, the existence of a CPT anomaly for the PP spin structure is essentially independent of the ultraviolet regularization, as discussed below Eq. (11) and in [12].

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