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Complete fermionic two-loop results for the $M_{\rm W}$–$M_{\rm Z}$ interdependence

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Abstract

The complete fermionic two-loop contributions to the prediction for the W-boson mass from muon decay in the electroweak Standard Model are evaluated exactly, i.e. no expansion in the top-quark and the Higgs-boson mass is made. The result for the W-boson mass is compared with the previous result of an expansion up to next-to-leading order in the top-quark mass. The predictions are found to agree with each other within about 4 MeV. A simple parameterization of the new result is presented, approximating the full result better than 0.4 MeV for $M_{\rm H} \leq 1$ TeV.

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The prediction of the W-boson mass, $M_{\rm W}$, in terms of the Z-boson mass, $M_{\rm Z}$, the Fermi constant, G_μ, and the fine structure constant, α, is one of the most important relations for testing the electroweak Standard Model (SM) and its extensions with high precision. This relation is derived from muon decay, as the Fermi constant is defined in terms of the muon lifetime, τ_μ, according to

$$\tau_\mu^{-1} = \frac{G_\mu^2\, m_\mu^5}{192\pi^3}\, F\left(\frac{m_{\rm e}^2}{m_\mu^2}\right)\left(1+\frac{3}{5}\frac{m_\mu^2}{M_{\rm W}^2}\right)(1+\Delta q)\,, \tag{1}$$

with $F(x) = 1 - 8x - 12x^2 \ln x + 8x^3 - x^4$. By convention, the QED corrections within the Fermi Model, Δq, are included in this defining equation for G_μ. The one-loop result for Δq [1], which has been known already for several decades, has recently been supplemented by the two-loop correction [2]. The tree-level W propagator effects giving rise to the (numerically insignificant) term $3m_\mu^2/(5M_{\rm W}^2)$ in eq. (1) are conventionally also included in the definition of G_μ although they do not belong to the Fermi Model prediction.

Comparing the prediction for the muon lifetime within the SM with eq. (1) yields the relation

$$M_{\rm W}^2\left(1-\frac{M_{\rm W}^2}{M_{\rm Z}^2}\right) = \frac{\pi\alpha}{\sqrt{2}G_\mu}\,(1+\Delta r)\,, \tag{2}$$

where the radiative corrections are summarized in the quantity Δr [3]. This relation can be used for deriving the prediction of $M_{\rm W}$ within the SM or extensions of it, to be confronted with the experimental result for $M_{\rm W}$. At present, the W-boson mass is measured with an accuracy of 5×10^{-4}, $M_{\rm W}^{\rm exp} = 80.419 \pm 0.038$ GeV [4]. The experimental precision on $M_{\rm W}$ will be further improved with the data taken at LEP2 in its final year of running, and at the upgraded Tevatron and the LHC, where an error of $\delta M_{\rm W} = 15$ MeV can be expected [5]. At a high-luminosity linear collider running in a low-energy mode at the $\rm W^+W^-$ threshold, a reduction of the experimental error down to $\delta M_{\rm W} = 6$ MeV can be envisaged [6]. This offers the prospect for highly sensitive tests of the electroweak theory [7], provided that the accuracy of the theoretical prediction matches the experimental precision.

The one-loop result for Δr within the SM [3] can be decomposed as (with $s_{\rm W}^2 = 1 - M_{\rm W}^2/M_{\rm Z}^2$)

$$\Delta r^{(\alpha)} = \Delta\alpha - \frac{c_{\rm W}^2}{s_{\rm W}^2}\Delta\rho + \Delta r_{\rm rem}(M_{\rm H}), \tag{3}$$

where the leading fermion-loop contributions $\Delta\alpha$ and $\Delta\rho$, arising from the charge and mixing-angle renormalization, are separated out, while the remainder part $\Delta r_{\rm rem}$ contains in particular the dependence on the Higgs-boson mass, $M_{\rm H}$. The QED-induced shift in the fine structure constant, $\Delta\alpha$, contains large logarithms of light-fermion masses. The leading contribution to the ρ parameter from the top/bottom weak isospin doublet, $\Delta\rho$, gives rise to a term with a quadratic dependence on the top-quark mass, $m_{\rm t}$ [8].

Beyond one-loop order, resummations of the leading one-loop contributions $\Delta\alpha$ and $\Delta\rho$ are known [9]. They correctly take into account the terms of the form $(\Delta\alpha)^2$, $(\Delta\rho)^2$, $(\Delta\alpha\Delta\rho)$, and $(\Delta\alpha\Delta r_{\rm rem})$ at the two-loop level and the leading powers in $\Delta\alpha$ to all orders.

While the QCD corrections to Δr are known at $\mathcal{O}(\alpha\alpha_s)$ [10] and $\mathcal{O}(\alpha\alpha_s^2)$ [11], only partial results are available up to now for the electroweak two-loop contributions. They have been obtained using expansions for asymptotically large values of m_t [12, 13] and M_H [14]. The terms derived by expanding in the top-quark mass of $\mathcal{O}(G_\mu^2 m_t^4)$ [12] and $\mathcal{O}(G_\mu^2 m_t^2 M_Z^2)$ [13] were found to be numerically sizable. The $\mathcal{O}(G_\mu^2 m_t^2 M_Z^2)$ term, involving three different mass scales, has been obtained by two separate expansions in the regions $M_W, M_Z, M_H \ll m_t$ and $M_W, M_Z \ll m_t, M_H$ and an interpolation between the two expansions. This formally next-to-leading order term turned out to be of similar magnitude as the formally leading term of $\mathcal{O}(G_\mu^2 m_t^4)$, entering with the same sign. Its inclusion (both for M_W and the effective mixing angle $\sin^2\theta_{\rm eff}$) had important consequences for the indirect constraints on the Higgs-boson mass derived from the SM fit to the precision data.

Consequently, a more complete calculation of electroweak two-loop effects appears desirable, where no expansion in m_t or M_H is made. As a first step into this direction, exact results have been obtained for the Higgs-mass dependence (e.g. the quantity $M_{W,\rm subtr}(M_H) \equiv M_W(M_H) - M_W(M_H = 65\ \text{GeV})$) of the fermionic two-loop corrections to the precision observables [15]. They have been compared with the results of expanding up to $\mathcal{O}(G_\mu^2 m_t^2 M_Z^2)$ [13], analyzing specifically the effects of the m_t expansion, and good agreement has been found [16].

Beyond two-loop order, recently complete results for the pure fermion-loop corrections (i.e. contributions containing n fermion loops at n-loop order) have been obtained up to four-loop order [17]. These results contain in particular the contributions of the leading powers in $\Delta\alpha$ as well as the ones in $\Delta\rho$ and the mixed terms.

In the present paper, all fermionic two-loop corrections to Δr are calculated exactly, i.e. without an expansion in the top-quark or the Higgs-boson mass. These are all two-loop diagrams contributing to the muon decay amplitude and containing at least one closed fermion-loop (except the pure QED corrections already contained in the Fermi model result, see eq. (1)). Fig. 1 displays some typical examples. The considered class of diagrams includes the potentially large corrections both from the top/bottom doublet and from contributions proportional to $N_{\rm lf}$ and $N_{\rm lf}^2$, where $N_{\rm lf}$ is the number of light fermions (a partial result for the light-fermion contributions has been obtained in Ref. [18]). The results presented here improve on the previous results of an expansion in m_t up to next-to-leading order [13] in containing the full dependence on m_t as well as the complete light-fermion contributions at two-loop order, while in Ref. [13] higher-order corrections from light fermions have only been taken into account via a resummation of the one-loop light-fermion contribution.

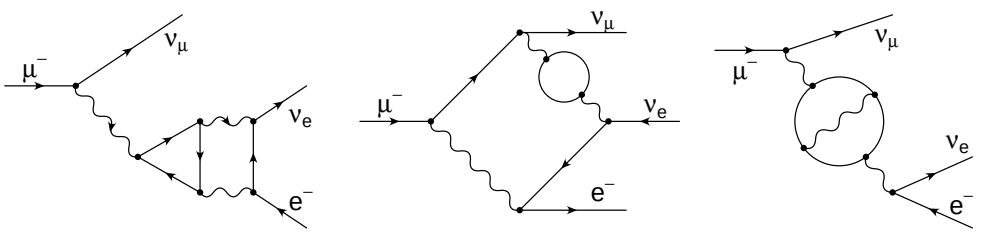

Figure 1: Examples for types of fermionic two-loop diagrams contributing to muon decay.

In the following, we briefly outline the main features of the calculation. After extracting the IR-divergent QED corrections that are already contained in the Fermi model QED factor, the generic diagrams contributing to muon decay can be reduced to vacuum-type diagrams, since the masses of the external particles and the momentum transfer are negligible. The on-shell renormalization of the gauge-boson masses, on the other hand, requires the evaluation of two-loop two-point functions with nonzero external momentum, which is more involved from a technical point of view regarding the tensor structure and the evaluation of the scalar integrals. It should be noted that this complication cannot be avoided by performing the calculation within another renormalization scheme (the $\overline{\rm MS}$ scheme, for instance), since ultimatively one is interested in the relation between the physical parameters $M_{\rm W}$, $M_{\rm Z}$, α, G_μ, rather than between their $\overline{\rm MS}$ counterparts. For this reason we have decided to use the on-shell renormalization scheme everywhere in our calculation, i.e. we use physical parameters throughout (alternatively one could of course do the calculation in a different renormalization scheme with formal parameters and perform the transition to the physical parameters in a second step). If not otherwise stated, we use the conventions of Ref. [19].

In our calculation we have made use of some computer-algebra tools. The package *FeynArts* [20] was applied for generating the Feynman amplitudes and counterterm contributions. The program *TwoCalc* [21] was applied for the algebraic evaluation of these amplitudes, which by means of two-loop tensor-integral decompositions were reduced to a set of standard scalar integrals. The calculation was carried out in a general R_ξ gauge, which allowed to test the gauge-parameter independence at the algebraic level as a highly non-trivial check. For the evaluation of the scalar one-loop integrals and the two-loop vacuum integrals we have used analytical results as given in Ref. [22], while the two-loop two-point integrals with non-vanishing external momentum have been evaluated numerically using one-dimensional integral representations with elementary functions [23]. These allow a very fast calculation of the integrals for general mass configurations.

Since we are using Dimensional Regularization [24,25] in our calculation, a careful treatment of the Dirac algebra in D dimensions involving γ_5 is necessary. While a naively anti-commuting γ_5 can safely be applied for all two-loop two-point contributions (for a discussion, see e.g. the first paper of Ref. [12]) and most of the two-loop vertex- and box-type diagrams, this is not the case for the two-loop vertex diagrams containing a triangle subgraph, shown in Fig. 2. For these graphs a naively anti-commuting γ_5, although respecting the Ward identities, would lead to an incorrect result. This is due to an inconsistent evaluation of the trace of γ_5 together with four Dirac matrices, which in four dimensions is given by $\mathrm{Tr}\,\{\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\} = 4i\epsilon^{\mu\nu\rho\sigma}$, while application of the naively anti-commuting γ_5 in D dimensions would yield zero for this trace. In order to calculate this type of diagrams, we have first evaluated the triangle subgraph with the mathematically consistent definition of γ_5 in D dimensions according to Refs. [25,26] (here we made use of the package TRACER [27] for checking). After adding appropriate counterterms, which are necessary to restore the Ward identities, the result differs from the result obtained using a naively anti-commuting γ_5 only in terms proportional to the totally antisymmetric tensor $\epsilon^{\mu\nu\rho\sigma}$. Inserting the latter contribution into the two-loop diagrams one finds that the second loop gives a finite contribution,

so that it can be evaluated in four dimensions without further complications.[1] The fermion line appearing in the second loop yields also an ϵ-tensor contribution, which after contraction with the ϵ-tensor from the triangle subgraph results in a non-vanishing contribution to the result for Δr.

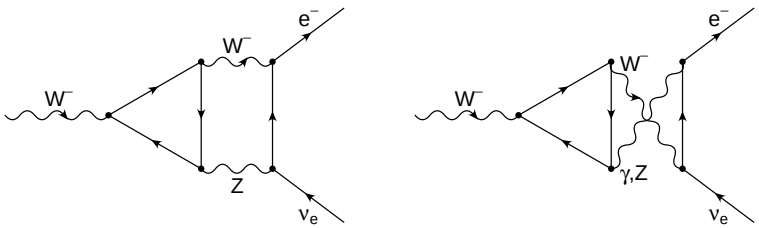

Figure 2: Two-loop vertex diagrams containing a triangle subgraph, which require a careful treatment of γ_5 in D dimensions.

As mentioned above, we perform the renormalization within the on-shell scheme. It involves a one-loop subrenormalization of the Faddeev-Popov ghost sector of the theory, which is associated with the gauge-fixing part. The gauge-fixing part is kept invariant under renormalization. For technical convenience, we manage this by a renormalization of the gauge parameters in such a way that it precisely cancels the renormalization of the parameters and fields in the gauge-fixing Lagrangian.[2] To this end, we start with the following rather general form of the bare gauge-fixing term,

$$\mathcal{L}_{\rm gf} = -\frac{1}{2}\Big((F^{\gamma})^2 + (F^{\rm Z})^2 + F^+F^- + F^-F^+\Big), \tag{4}$$

$$F^{\gamma} = (\xi_1^{\gamma})^{-\frac{1}{2}}\,\partial_\mu A^\mu + \frac{\xi^{\gamma\rm Z}}{2}\partial_\mu Z^\mu, \tag{5}$$

$$F^{\rm Z} = \left(\xi_1^{\rm Z}\right)^{-\frac{1}{2}}\partial_\mu Z^\mu + \frac{\xi^{\rm Z\gamma}}{2}\partial_\mu A^\mu - \left(\xi_2^{\rm Z}\right)^{\frac{1}{2}} M_{\rm Z}\,\chi, \tag{6}$$

$$F^{\pm} = \left(\xi_1^{\rm W}\right)^{-\frac{1}{2}}\partial_\mu W^{\pm\mu} \mp i\left(\xi_2^{\rm W}\right)^{\frac{1}{2}} M_{\rm W}\,\phi^{\pm}, \tag{7}$$

allowing two different bare gauge parameters for both W and Z, $\xi_1^{\rm W,Z}$ and $\xi_2^{\rm W,Z}$, and also mixing gauge parameters, $\xi^{\gamma\rm Z}$ and $\xi^{\rm Z\gamma}$. The renormalized parameters comply with the R_ξ gauge, with one free gauge parameter for each gauge boson.

With this prescription no counterterm contributions arise from the gauge-fixing sector. Starting at the two-loop level, counterterm contributions from the ghost sector have to be taken into account in the calculation of physical amplitudes. They follow from the variation

[1] For recent discussions of practical ways for treating γ_5 in higher-order calculations see also Refs. [28, 29].

[2] An alternative way of achieving that the gauge-fixing sector does not give rise to counterterm contributions would have been to add the gauge-fixing part to the Lagrangian only after renormalization, in which case the renormalized gauge transformations would have to be used.

of the gauge-fixing terms F^a under infinitesimal gauge transformations,

$$\mathcal{L}_{\rm FP} = \sum_{a,b=\gamma,Z,\pm} \bar{u}^a \frac{\delta F^a}{\delta \theta^b} u^b = \int d^4y \sum_{a,b} \bar{u}^a(x) \frac{\delta F^a(x)}{\delta \theta^b(y)} u^b(y). \tag{8}$$

We have derived all the counterterms arising from the ghost sector (extending the results of Ref. [30] to a general R_ξ gauge) and implemented them into the program *FeynArts*. In this way we could verify the finiteness of individual (gauge-parameter dependent) building blocks (e.g. the W- and the Z-boson self-energy) as a further check of the calculation.

Concerning the mass renormalization of unstable particles, from two-loop order on it makes a difference whether the mass is defined according to the real part of the complex pole of the S matrix,

$$\mathcal{M}^2 = \overline{M}^2 - i\overline{M}\,\overline{\Gamma}, \tag{9}$$

or according to the real pole. In eq. (9) $\mathcal{M}$ denotes the complex pole of the S matrix and $\overline{M}$, $\overline{\Gamma}$ the corresponding mass and width of the unstable particle. We use the symbol $\widetilde{M}$ for the real pole.

In the context of the present calculation, these considerations are relevant for the renormalization of the gauge-boson masses, $M_{\rm W}$ and $M_{\rm Z}$. The two-loop mass counterterms according to the definition of the mass as the real part of the complex pole are given by

$$\delta\overline{M}^2_{\rm W,(2)} = {\rm Re}\left\{\Sigma^{\rm W}_{\rm T,(2)}(M^2_{\rm W})\right\} - \delta M^2_{\rm W,(1)}\,\delta Z^{\rm W}_{(1)} + {\rm Im}\left\{\Sigma^{\rm W\prime}_{\rm T,(1)}(M^2_{\rm W})\right\} {\rm Im}\left\{\Sigma^{\rm W}_{\rm T,(1)}(M^2_{\rm W})\right\}, \tag{10}$$

$$\begin{aligned}\delta\overline{M}^2_{\rm Z,(2)} &= {\rm Re}\left\{\Sigma^{\rm ZZ}_{\rm T,(2)}(M^2_{\rm Z})\right\} - \delta M^2_{\rm Z,(1)}\,\delta Z^{\rm ZZ}_{(1)} + \frac{M^2_{\rm Z}}{4}\left(\delta Z^{\gamma\rm Z}_{(1)}\right)^2 + \frac{\left({\rm Im}\left\{\Sigma^{\gamma\rm Z}_{\rm T,(1)}(M^2_{\rm Z})\right\}\right)^2}{M^2_{\rm Z}} \\ &\quad + {\rm Im}\left\{\Sigma^{\rm ZZ\prime}_{\rm T,(1)}(M^2_{\rm Z})\right\} {\rm Im}\left\{\Sigma^{\rm ZZ}_{\rm T,(1)}(M^2_{\rm Z})\right\},\end{aligned} \tag{11}$$

where $\Sigma_{\rm T,(1)}$, $\Sigma_{\rm T,(2)}$ denote the transverse parts of the one-loop and two-loop self-energies (the terms from subloop renormalization are understood to be contained in the two-loop self-energies), and $\Sigma'_{\rm T,(1)}$ means the derivative of the one-loop self-energy with respect to the external momentum squared. Field renormalization constants are indicated as δZ^V. The relations to the mass counterterms according to the real-pole definition, $\delta\widetilde{M}^2_{\rm W,(2)}$ and $\delta\widetilde{M}^2_{\rm Z,(2)}$, are given by

$$\delta\overline{M}^2_{\rm W,(2)} = \delta\widetilde{M}^2_{\rm W,(2)} + {\rm Im}\left\{\Sigma^{\rm W\prime}_{\rm T,(1)}(M^2_{\rm W})\right\} {\rm Im}\left\{\Sigma^{\rm W}_{\rm T,(1)}(M^2_{\rm W})\right\}, \tag{12}$$

$$\delta\overline{M}^2_{\rm Z,(2)} = \delta\widetilde{M}^2_{\rm Z,(2)} + {\rm Im}\left\{\Sigma^{\rm ZZ\prime}_{\rm T,(1)}(M^2_{\rm Z})\right\} {\rm Im}\left\{\Sigma^{\rm ZZ}_{\rm T,(1)}(M^2_{\rm Z})\right\}. \tag{13}$$

It can easily be checked by direct computation that the terms in eqs. (12), (13) by which the two definitions differ are gauge-parameter dependent. Thus it is obvious that at least one of the two prescriptions leads to a gauge-dependent mass definition. While the problem of a proper definition of unstable particles in gauge theories has been addressed in the literature already many times [31], it should be noted that in the present calculation two-loop contributions of the type leading to a non-zero (and gauge-parameter dependent) difference between the two kinds of mass renormalization methods are for the first time fully included

in a computation of a physical observable in the Standard Model. Explicitly, these are contributions from light fermions and bosonic loops evaluated in a general R_ξ gauge. In the previous results for M_W, incorporating terms up to $\mathcal{O}(G_\mu^2 m_\mathrm{t}^2 M_\mathrm{Z}^2)$ [13] and M_H-dependent fermionic terms [15], the contribution $\mathrm{Im}\left\{\Sigma'_{\mathrm{T},(1)}(M^2)\right\} \mathrm{Im}\left\{\Sigma_{\mathrm{T},(1)}(M^2)\right\}$ was zero, making thus a strict distinction between the two mass definitions unnecessary at the considered order.

Since our result has been obtained within a general R_ξ gauge we can investigate the issue whether the mass renormalization is gauge-parameter independent by explicit computation. In particular, the two-loop counterterm to the weak mixing angle, $\delta s_{\mathrm{W},(2)}$, ought to be gauge-parameter independent since s_W is a physical observable (note, however, that the same argument does not hold for the mass counterterms of eq. (10) and eq. (11); see e.g. Ref. [32] for a discussion). We find that $\delta s_{\mathrm{W},(2)}$ is only gauge-parameter independent with the definition of the gauge-boson masses according to the complex pole, while the real-pole definition for the masses leads to a gauge-parameter dependent result for $\delta s_{\mathrm{W},(2)}$. This result is in accordance with what is expected from S-matrix theory, in which the complex pole is a gauge-invariant quantity [31].

We have thus adopted the complex-pole definition as given in eq. (10) and eq. (11). Using this mass definition leads to a Breit-Wigner parameterization of the resonance line shape with a constant decay width. Experimentally the gauge-boson masses are determined using a Breit-Wigner function with a running (energy-dependent) width. Connecting the latter prescription with the theoretical prediction involves the approximation $\mathrm{Im}\left\{\Sigma^{\mathrm{W,Z}}_{\mathrm{T},(1)}(s)\right\} \approx s\Gamma_{\mathrm{W,Z}}/M_{\mathrm{W,Z}}$, which is valid for the fermionic contributions to the W- and Z-boson self-energies at one-loop order. Here $\Gamma_{\mathrm{W,Z}}$ denote the W- and Z-boson widths. As a consequence of the different Breit-Wigner parameterizations there is a numerical difference between the mass parameters corresponding to the definition used in the experimental determination (denoted as M_W, M_Z henceforth) and the mass parameters in our calculation, $\overline{M}_\mathrm{W}$, $\overline{M}_\mathrm{Z}$. The shift between these parameters is given by [33] $M_{\mathrm{W,Z}} = \overline{M}_{\mathrm{W,Z}} + \Gamma^2_{\mathrm{W,Z}}/(2M_{\mathrm{W,Z}})$. Since M_W and M_Z enter on a different footing in our computation — M_Z is an experimental input parameter, while M_W is calculated — in order to evaluate the mass shifts we use the experimental value for the Z-boson width, $\Gamma_\mathrm{Z} = 2.944 \pm 0.0024$ GeV [4], and the theoretical value for the W-boson width, which is given by $\Gamma_\mathrm{W} = 3G_\mu M_\mathrm{W}^3/(2\sqrt{2}\pi)(1 + 2\alpha_\mathrm{s}/(3\pi))$ in a sufficiently good approximation. This results in $M_\mathrm{Z} \approx \overline{M}_\mathrm{Z} + 34.1$ MeV and in the mass shifts $M_\mathrm{W} \approx \overline{M}_\mathrm{W} + 27.4$ MeV and $M_\mathrm{W} \approx \overline{M}_\mathrm{W} + 27.0$ MeV for $M_\mathrm{W} = 80.4$ GeV and $M_\mathrm{W} = 80.2$ GeV, respectively.[3]

We now turn to the numerical discussion of our result for Δr. It should be noted that our definition of Δr according to eq. (2) is based on the expanded form $(1 + \Delta r)$ with

[3]The difference whether for Γ_W the tree-level result parameterized with α, the improved Born result parameterized with G_μ, or the improved Born result including QCD corrections (which we have used) is taken, is formally of higher order (i.e. beyond $\mathcal{O}(\alpha^2)$) in the calculation of M_W. Its numerical effect is nevertheless not completely negligible; it changes the shift in M_W by about -2.9 MeV if the tree-level result for Γ_W parameterized with α is used and by about -1.4 MeV if the G_μ parameterization of the Born width (without QCD corrections) is employed.

$\Delta r = \Delta r^{(\alpha)} + \Delta r^{(\alpha^2)} + \ldots$ rather than on the resummed form $1/(1-\Delta r)$ indicating a resummation of leading one-loop contributions. The terms consistently taken into account at two-loop order with such a resummation are explicitly contained in our two-loop contribution to Δr. The result for Δr contains the following contributions

$$\Delta r = \Delta r^{(\alpha)} + \Delta r^{(\alpha\alpha_s)} + \Delta r^{(\alpha\alpha_s^2)} + \Delta r^{(N_f\alpha^2)} + \Delta r^{(N_f^2\alpha^2)}, \tag{14}$$

where $\Delta r^{(\alpha)}$ is the one-loop result, eq. (3), $\Delta r^{(\alpha\alpha_s)}$ and $\Delta r^{(\alpha\alpha_s^2)}$ are the two-loop [10] and three-loop [11] QCD corrections, while $\Delta r^{(N_f\alpha^2)}$ is the new electroweak two-loop result. The notation $(N_f\alpha^2)$ symbolizes the contribution of all diagrams containing one fermion-loop, where N_f stands both for the top/bottom contribution and for all light-fermion species. The term $\Delta r^{(N_f^2\alpha^2)}$ contains the pure fermion-loop contributions in two-loop order. Since the pure fermion-loop contributions in three- and four-loop order have been found to be numerically small, as a consequence of accidental numerical cancellations, with a net effect of only about 1 MeV in M_W (using the real-pole definition of the gauge-boson masses) [17], we have not included them here.

In Fig. 3 the different contributions to Δr are shown as a function of M_H. Here M_W is kept fixed at its experimental central value, $M_W = 80.419$ GeV, and $m_t = 174.3$ GeV [34] is used. The QCD corrections and the electroweak two-loop corrections are of similar magnitude and both amount to more than 10% of the one-loop result. We have compared the Higgs-mass dependence of Δr with the result previously obtained in Ref. [15] and found perfect agreement.

The prediction for M_W is obtained from the input parameters by solving eq. (2). Since Δr itself depends on M_W this is technically done using an iterative procedure. The prediction for M_W based on the results of eq. (14) is shown in Fig. 4 as a function of M_H for $m_t = 174.3 \pm 5.1$ GeV [34] and $\Delta\alpha = 0.05954 \pm 0.00065$ [35]. The current experimental value, $M_W^{exp} = 80.419 \pm 0.038$ GeV [4], and the experimental 95% C.L. lower bound on M_H of $M_H = 107.9$ GeV [36] from the direct search are also indicated. The plot shows the well-known preference for a light Higgs boson within the SM. Confronting the theoretical prediction (allowing a variation of m_t, which at present dominates the theoretical uncertainty, and $\Delta\alpha$ within 1σ) with the 1σ region of M_W^{exp} and the 95% C.L. lower bound on M_H, only a rather small region in the plot (corresponding to 107.9 GeV $< M_H \lesssim 140$ GeV) matches all three constraints.

We have compared our results with those of an expansion for asymptotically large values of m_t up to $\mathcal{O}(G_\mu^2 m_t^2 M_Z^2)$ [13, 37]. The results are shown in Tab. 1 for different values of M_H. For the input parameters the values of Ref. [13] have been chosen, i.e. $m_t = 175$ GeV, $M_Z = 91.1863$ GeV, $\Delta\alpha = 0.0594$, $\alpha_s(M_Z) = 0.118$. Relatively good agreement is found, with maximal deviations in M_W of about 4 MeV. If we had chosen a different parameterization of Γ_W in the above calculation of the shift between the masses corresponding to the fixed and the running width definition, a somewhat larger deviation to the result of Ref. [13] would have been obtained.

The deviations in the last column of Tab. 1 can of course not exclusively be attributed to differences in the two-loop top-quark and light-fermion contributions, because the results also differ by a slightly different treatment of those higher-order terms that are not yet under

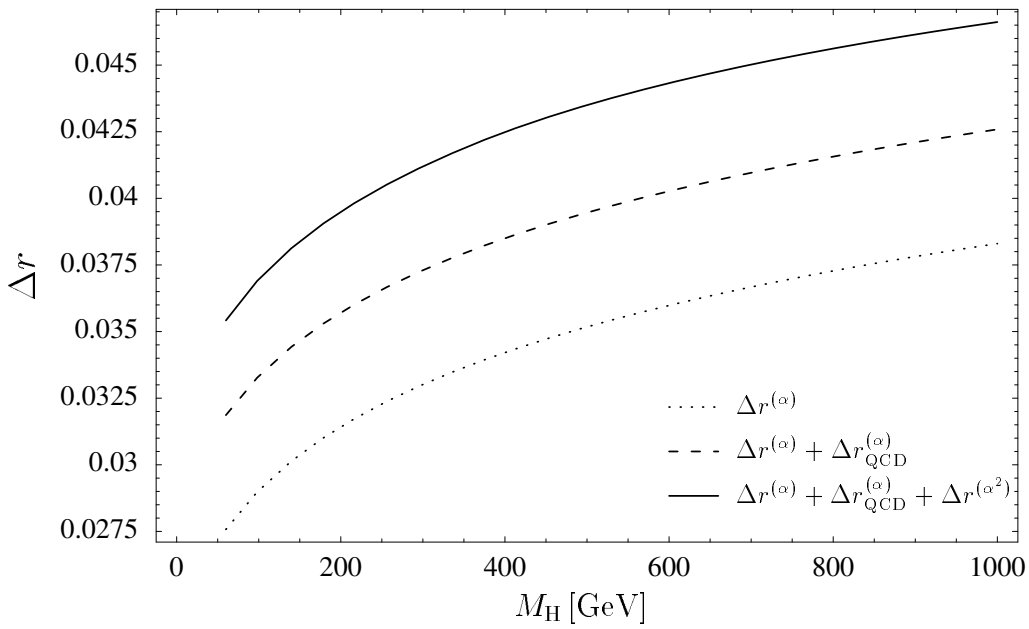

Figure 3: Different contributions to Δr as a function of $M_{\rm H}$. The one-loop contribution, $\Delta r^{(\alpha)}$, is supplemented by the two-loop and three-loop QCD corrections, $\Delta r^{(\alpha)}_{\rm QCD} \equiv \Delta r^{(\alpha\alpha_{\rm s})} + \Delta r^{(\alpha\alpha_{\rm s}^2)}$, and the fermionic electroweak two-loop contributions, $\Delta r^{(\alpha^2)} \equiv \Delta r^{(N_{\rm f}\alpha^2)} + \Delta r^{(N_{\rm f}^2\alpha^2)}$.

control, such as purely bosonic two-loop contributions and effects from scheme dependence. In a further analysis, we have aimed at reducing the latter deviations as far as possible in order to focus on the effects from the two-loop top-quark and light-fermion contributions (see also the discussion in Ref. [16]). While the result of Ref. [13] contains a term $\left(\Delta r^{(\alpha)}_{\rm bos}\right)^2$ generated from the purely bosonic one-loop contributions by writing $\Delta r^{(\alpha)}$ into the denominator, no such term is included in our result. Adding this term to our result and removing the QCD corrections from both results eliminates two possible sources of deviations. We have furthermore repeated our calculation with the definition of the gauge-boson mass according to the real pole of the S matrix, see eqs. (12), (13), using the Feynman gauge. With these modifications the deviation between the results at $M_{\rm H} = 65$ GeV is reduced from -4.2 MeV to -3.4 MeV, while the maximal difference in the Higgs-mass dependence $M_{\rm W}(M_{\rm H}) - M_{\rm W}(M_{\rm H} = 65\ {\rm GeV})$ is reduced from 4.7 MeV in Tab. 1 to 1.9 MeV. A detailed discussion of the remaining theoretical uncertainties from unknown higher-order corrections will be given in a forthcoming publication.

Following Ref. [38], we also provide a simple numerical parameterization of our result for $M_{\rm W}$. It is given by

$$M_{\rm W} = M_{\rm W}^0 - c_1\,{\rm dH} - c_5\,{\rm dH}^2 + c_6\,{\rm dH}^4 - c_2\,{\rm d}\alpha + c_3\,{\rm dt} - c_7\,{\rm dH}\,{\rm dt} - c_4\,{\rm d}\alpha_{\rm s}, \qquad (15)$$

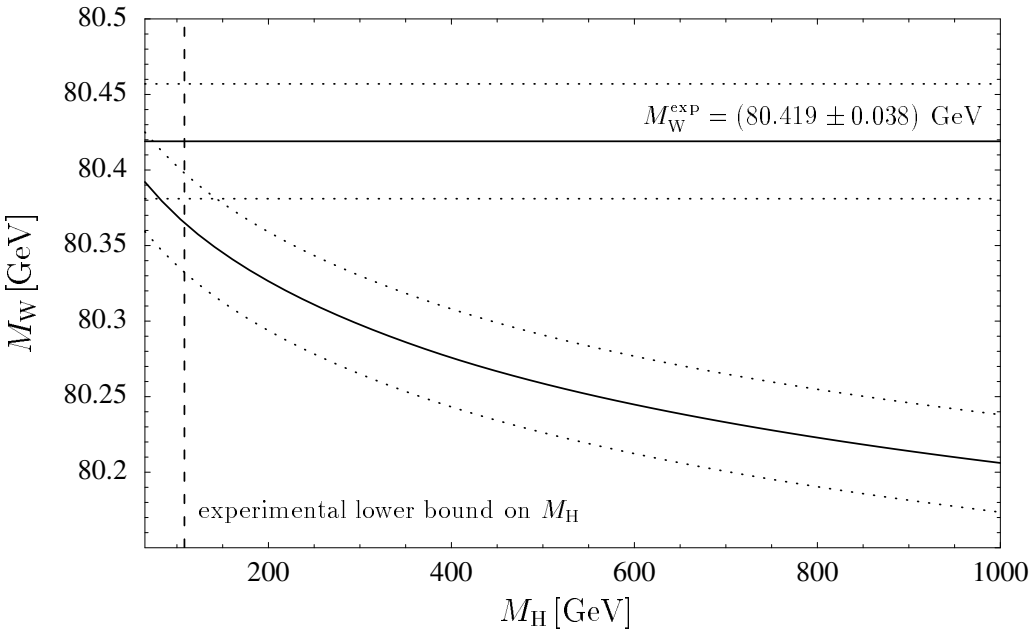

Figure 4: The SM prediction for $M_{\rm W}$ as a function of $M_{\rm H}$ for $m_{\rm t} = 174.3 \pm 5.1$ GeV is compared with the current experimental value, $M_{\rm W}^{\rm exp} = 80.419 \pm 0.038$ GeV [4], and the experimental 95% C.L. lower bound on the Higgs-boson mass, $M_{\rm H} = 107.9$ GeV [36].

where

$$\mathrm{dH} = \ln\left(\frac{M_{\rm H}}{100\ \mathrm{GeV}}\right),\ \mathrm{dt} = \left(\frac{m_{\rm t}}{174.3\ \mathrm{GeV}}\right)^2 - 1,\ \mathrm{d}\alpha = \frac{\Delta\alpha}{0.05924} - 1,\ \mathrm{d}\alpha_{\rm s} = \frac{\alpha_{\rm s}(M_{\rm Z})}{0.119} - 1, \quad (16)$$

and $M_{\rm Z} = 91.1871$ GeV [4] has been used. For the coefficients $c_1, \ldots, c_7$ we have obtained via a least square fit $M_{\rm W}^0 = 80.3762$ GeV, $c_1 = 0.0561$, $c_2 = 1.081$, $c_3 = 0.5235$, $c_4 = 0.0763$, $c_5 = 0.00936$, $c_6 = 0.000546$, $c_7 = 0.00573$. The parameterization of eq. (15) approximates our full result for $M_{\rm W}$ within 0.4 MeV for 65 GeV $\leq M_{\rm H} \leq$ 1 TeV.

In summary, we have evaluated the complete fermionic two-loop contributions to the W-boson mass within the electroweak Standard Model. Our result improves on previous results as it does not involve any approximations in the top-quark and the Higgs-boson mass and also contains the contributions of all light fermions in the Standard Model. Within our calculation we have defined the gauge-boson masses according to the complex pole of the S matrix, which ensures the gauge-parameter independence of the mass definition. We have provided a simple numerical parameterization of our result which approximates the full result with sufficient accuracy for all values of $M_{\rm H}$ up to 1 TeV. In comparison with the previous result for $M_{\rm W}$ obtained by an expansion for asymptotically large values in $m_{\rm t}$ up to next-to-leading order we find slightly lower values of $M_{\rm W}$, sharpening thus the tendency towards a light Higgs boson within the Standard Model.

$M_{\rm H}$/ GeV	$M_{\rm W}$/ GeV	$M_{\rm W}^{\rm expa}$/ GeV	$\Delta M_{\rm W}$/ MeV
65	80.3997	80.4039	−4.2
100	80.3771	80.3805	−3.4
300	80.3051	80.3061	−1.0
600	80.2521	80.2521	0.0
1000	80.2134	80.2129	0.5

Table 1: The two-loop result for $M_{\rm W}$ based on eq. (14) is compared with the results of an expansion in $m_{\rm t}$ up to $\mathcal{O}(G_\mu^2 m_{\rm t}^2 M_{\rm Z}^2)$ [13,37], $M_{\rm W}^{\rm expa}$. The last column indicates the difference between the two results.

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References

[1] R.E. Behrends, R.J. Finkelstein and A. Sirlin, *Phys. Rev.* **101** (1956) 866;
S.M. Berman, *Phys. Rev.* **112** (1958) 267;
T. Kinoshita and A. Sirlin *Phys. Rev.* **113** (1959) 1652.

[2] T. van Ritbergen and R.G. Stuart, *Phys. Rev. Lett.* **82** (1999) 488;
T. van Ritbergen and R.G. Stuart, *Nucl. Phys.* **B 564** (2000) 343;
M. Steinhauser and T. Seidensticker, *Phys. Lett.* **B 467** (1999) 271.

[3] A. Sirlin, *Phys. Rev.* **D 22** (1980) 971;
W.J. Marciano and A. Sirlin, *Phys. Rev.* **D 22** (1980) 2695.

[4] A. Straessner, talk given at the *XXXVth Rencontres de Moriond*, Les Arcs, March 2000, to appear in the proceedings.

[5] ATLAS Collaboration, "Detector and Physics Performance Technical Design Report", CERN/LHCC/99-15 (1999);
CMS Collaboration, Technical Design Reports, CMS TDR 1-5 (1997/98);
S. Haywood, P.R. Hobson, W. Hollik, Z. Kunzst et al., *Electroweak Physics*, hep-ph/0003275, in CERN-YR-2000/01, eds. G. Altarelli and M.L. Mangano.

[6] G. Wilson, to appear in the proceedings of the *International Workshop on Linear Colliders*, Sitges, April/May 1999.

[7] S. Heinemeyer, Th. Mannel and G. Weiglein, DESY 99-117, hep-ph/9909538, to appear in the proceedings of the *International Workshop on Linear Colliders*, Sitges, April/May 1999;
J. Erler, S. Heinemeyer, W. Hollik, G. Weiglein and P.M. Zerwas, CERN-TH/2000-109, hep-ph/0005024, to appear in *Phys. Lett.* **B**.

[8] M. Veltman, *Nucl. Phys.* **B 123** (1977) 89.

[9] W.J. Marciano, *Phys. Rev.* **D 20** (1979) 274;
A. Sirlin, *Phys. Rev.* **D 29** (1984) 89;
M. Consoli, W. Hollik and F. Jegerlehner, *Phys. Lett.* **B 227** (1989) 167.

[10] A. Djouadi and C. Verzegnassi, *Phys. Lett.* **B 195** (1987) 265;
A. Djouadi, *Nuovo Cimento* **A 100** (1988) 357;
B.A. Kniehl, *Nucl. Phys.* **B 347** (1990) 89;
F. Halzen and B.A. Kniehl, *Nucl. Phys.* **B 353** (1991) 567;
B.A. Kniehl and A. Sirlin, *Nucl. Phys.* **B 371** (1992) 141, *Phys. Rev.* **D 47** (1993) 883;
A. Djouadi and P. Gambino, *Phys. Rev.* **D 49** (1994) 3499.

[11] L. Avdeev, J. Fleischer, S.M. Mikhailov and O. Tarasov, *Phys. Lett.* **B 336** (1994) 560;
E: *Phys. Lett.* **B 349** (1995) 597;
K. Chetyrkin, J. Kühn and M. Steinhauser, *Phys. Lett.* **B 351** (1995) 331; *Phys. Rev. Lett.* **75** (1995) 3394.

[12] J. van der Bij and F. Hoogeveen, *Nucl. Phys.* **B 283** (1987) 477;
R. Barbieri, M. Beccaria, P. Ciafaloni, G. Curci and A. Vicere, *Phys. Lett.* **B 288** (1992) 95, *erratum:* **B 312** (1993) 511; *Nucl. Phys.* **B 409** (1993) 105;
J. Fleischer, O.V. Tarasov and F. Jegerlehner, *Phys. Lett.* **B 319** (1993) 249; *Phys. Rev.* **D 51** (1995) 3820.

[13] G. Degrassi, P. Gambino and A. Vicini, *Phys. Lett.* **B 383** (1996) 219;
G. Degrassi, P. Gambino and A. Sirlin, *Phys. Lett.* **B 394** (1997) 188.

[14] J. van der Bij and F. Hoogeveen, *Nucl. Phys.* **B 283** (1987) 477;
J. van der Bij and M. Veltman, *Nucl. Phys.* **B 231** (1984) 205.

[15] S. Bauberger and G. Weiglein, *Phys. Lett.* **B 419** (1998) 333;
G. Weiglein, hep-ph/9901317, in *Proceedings of the IVth International Symposium on Radiative Corrections*, ed. J. Solà (World Scientific, Singapore, 1999), p. 410.

[16] P. Gambino, A. Sirlin and G. Weiglein, *J. High Energy Phys.* **04** (1999) 025.

[17] G. Weiglein, *Acta Phys. Pol.* **B 29** (1998) 2735;
A. Stremplat, Diploma thesis, (Univ. of Karlsruhe, 1998).

[18] P. Malde and R.G. Stuart, *Nucl. Phys.* **B 552** (1999) 41.

[19] A. Denner, *Fortschr. Phys.* **41** (1993) 307.

[20] J. Küblbeck, M. Böhm and A. Denner, *Comp. Phys. Commun.* **60** (1990) 165;
H. Eck and J. Küblbeck, *Guide to FeynArts1.0* (Univ. of Würzburg, 1992);
T. Hahn, KA-TP-5-1999, hep-ph/9905354; *FeynArts2.2 User's Guide* (Univ. of Karlsruhe, 2000)

[21] G. Weiglein, R. Scharf and M. Böhm, *Nucl. Phys.* **B 416** (1994) 606;
G. Weiglein, R. Mertig, R. Scharf and M. Böhm, in *New Computing Techniques in Physics Research 2*, ed. D. Perret-Gallix (World Scientific, Singapore, 1992), p. 617.

[22] A.I. Davydychev and J.B. Tausk, *Nucl. Phys.* **B 397** (1993) 123.

[23] S. Bauberger, F.A. Berends, M. Böhm and M. Buza, *Nucl. Phys.* **B 434** (1995) 383;
S. Bauberger, F.A. Berends, M. Böhm, M. Buza and G. Weiglein, *Nucl. Phys.* **B** *(Proc. Suppl.)* **37B** (1994) 95, hep-ph/9406404;
S. Bauberger and M. Böhm, *Nucl. Phys.* **B 445** (1995) 25.

[24] C. Bollini and J. Giambiagi, *Nuovo Cimento* **B 12** (1972) 20;
J. Ashmore, *Nuovo Cimento Lett.* **4** (1972) 289.

[25] G. 't Hooft and M. Veltman, *Nucl. Phys.* **B 44** (1972) 189.

[26] P. Breitenlohner and D. Maison, *Commun. Math. Phys.* **52** (1977) 11.

[27] M. Jamin and M.E. Lautenbacher, *Comp. Phys. Commun.* **74** (1993) 265.

[28] P.A. Grassi, T. Hurth and M. Steinhauser, BUTP-99-13, hep-ph/9907426.

[29] F. Jegerlehner, DESY 00-075, hep-th/0005255.

[30] S. Bauberger and G. Weiglein, *Nucl. Instr. Meth.* **A 389** (1997) 318;
S. Bauberger, Doctoral thesis (Univ. of Würzburg, 1997).

[31] A. Sirlin, *Phys. Rev. Lett.* **67** (1991) 2127;
R.G. Stuart, *Phys. Lett.* **B 262** (1991) 113;
S. Willenbrock and G. Valencia, *Phys. Lett.* **B 259** (1991) 373;
H. Veltman, *Z. Phys.* **C 62** (1994) 35;
M. Passera and A. Sirlin, *Phys. Rev.* **D 58** (1998) 113010;
P. Gambino and P.A. Grassi, TUM-HEP-352/99, hep-ph/9907254;
A.R. Bohm and N.L. Harshman, hep-ph/0001206.

[32] A. Denner, S. Dittmaier and G. Weiglein, *Nucl. Phys.* **B 440** (1995) 95.

[33] D. Bardin, A. Leike, T. Riemann and M. Sachwitz, *Phys. Lett.* **B 206** (1988) 539.

[34] C. Caso et al, *Eur. Phys. J.* **C 3** (1998) 1 and 1999 off-year partial update for the 2000 edition, `http://pdg.lbl.gov/`.

[35] S. Eidelman and F. Jegerlehner, *Z. Phys.* **C 67** (1995) 585.

[36] The LEP working group for Higgs boson searches, P. Bock et al., CERN-EP-2000-055.

[37] P. Gambino, private communication.

[38] G. Degrassi, P. Gambino, M. Passera and A. Sirlin, *Phys. Lett.* **B 418** (1998) 209.