On non-L^2 solutions to the Seiberg–Witten equations

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Abstract: We show that a previous paper of Freund describing a solution to the Seiberg–Witten equations has a sign error rendering it a solution to a related but different set of equations. The non-L^2 nature of Freund's solution is discussed and clarified and we also construct a whole class of solutions to the Seiberg–Witten equations.

§ 1. Introduction

With the introduction of the Seiberg–Witten equations [**1**] there come a wealth of results on four manifold theory and a new improved point of view on Donaldson theory with an Abelian gauge theory supplanting a non-Abelian one—cf. [**2**] for a review.

An important vanishing theorem of [**1**], reminiscent of the Lichernowicz–Weitzenböck vanishing theorems, shows that there are no non-trivial solutions to the Seiberg–Witten equations on four manifolds with non-negative Riemannian scalar curvature. However one can have non-trivial solutions which are singular in some way—for example one could have a non-trivial solution which was not L^2: in [**3**] Freund describes such a non-L^2 to the Seiberg–Witten equations on $\mathbf{R}^4$. Unfortunately a sign discrepancy in [**3**] means that the expressions given there obey equations which differ from the Seiberg–Witten equations in a certain sign. These other equations also admit L^2 solutions as well as non-L^2 ones and so Freund's equations are fundamentally different from the Seiberg–Witten equations.

In § 2 we describe the Seiberg–Witten equations in a fairly explicit manner so as to expose notational conventions and matters of signs. In section § 3 we give the details concerning Freund's work and then in section § 4 we give an L^2 solution of Freund's equations and a class of solutions to the Seiberg–Witten equations.

§ 2. The Seiberg–Witten equations

If M is an oriented Riemannian four manifold with metric g_{ij} then the data we need for the Seiberg–Witten equations are a $U(1)$ connection A_i on M and a local spinor field M.

If F_{ij} is the curvature of A_i, so that its self-dual part F^+_{ij} is given by $F^+_{ij} = 1/2(F_{ij} + (\sqrt{g}/2)\epsilon_{ijkl}F^{kl})$, then the Seiberg–Witten equations are

$$\begin{aligned} F^+_{ij} &= -\frac{i}{2}\overline{M}\Gamma_{ij}M \\ \Gamma^i D_i M &= 0 \end{aligned} \tag{2.1}$$

where Γ_i are the gamma matrices* satisfying $\{\Gamma_i, \Gamma_j\} = 2g_{ij}I$, and D_i and Γ_{ij} are given by

$$D_i = \partial_i + iA_i, \qquad \Gamma_{ij} = \frac{1}{2}\left[\Gamma_i, \Gamma_j\right] \tag{2.2}$$

In [**1**] Witten also quotes the equations using the two component spinor formalism of L. Witten and Penrose, cf. [**4**], where the Gamma matrices make no explicit appearance. In this spinor form the equations are

$$\begin{aligned} \mathcal{F}_{A'B'} &= \frac{i}{2}\left(\mathcal{M}_{A'}\widetilde{\mathcal{M}}_{B'} + \mathcal{M}_{B'}\widetilde{\mathcal{M}}_{A'}\right) \\ \mathcal{D}_{AA'}\mathcal{M}^{A'} &= 0 \end{aligned} \tag{2.3}$$

We now give a short summary of the relevant properties of the spinor formalism that we need here.

With a Riemannian metric of signature $(+,+,+,+)$ the 4 components of a 4-vector $v_a \equiv (v_0, v_1, v_2, v_3)$ are represented by a 2×2 matrix which is denoted by $v_{AA'}$ and given by

$$v_{AA'} = \frac{1}{\sqrt{2}}\begin{pmatrix} v_0 + iv_3 & iv_1 + v_2 \\ iv_1 - v_2 & v_0 - iv_3 \end{pmatrix} \tag{2.4}$$

This expression for $v_{AA'}$ can be written as a linear combination of what are known as the Infeld–van der Waerden matrices $g^a_{AA'}$ defined by

$$g^0_{AA'} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad g^i_{AA'} = \frac{i}{\sqrt{2}}\sigma^i,\ i = 1,2,3 \tag{2.5}$$

where σ^i are the usual Pauli matrices so that

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.6}$$

Using the $g^a_{AA'}$'s the linear combination mentioned above is given by

$$v_{AA'} = v_a\, g^a_{AA'} \tag{2.7}$$

and more generally if we have a tensor $T_{a_1 a_2 \ldots a_n}$ it becomes $\mathcal{T}_{A_1 A'_1 A_2 A'_2 \ldots A_n A'_n}$ where

$$\mathcal{T}_{A_1 A'_1 A_2 A'_2 \ldots A_n A'_n} = T_{a_1 a_2 \ldots a_n}\, g^{a_1}_{A_1 A'_1} g^{a_2}_{A_2 A'_2} \cdots g^{a_n}_{A_n A'_n} \tag{2.8}$$

In this formalism spinor indices are raised and lowered with the matrix ϵ_{AB} defined by

$$\epsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \epsilon^{AB} \tag{2.9}$$

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* Our conventions for the Γ_i are: $\Gamma_0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, $\Gamma_i = \begin{pmatrix} 0 & -i\sigma^i \\ i\sigma^i & 0 \end{pmatrix}$, and $\Gamma_5 = -\Gamma_0\Gamma_1\Gamma_2\Gamma_3$ where σ^i are the Pauli matrices.

For example, for one spinor index, one can write

$$v^A = \epsilon^{AB} v_B, \quad v_B = v^A \epsilon_{AB} \tag{2.10}$$

An involution can also be defined on a spinor v^A taking it to a spinor $\widetilde{v}^A$ defined by

$$v^A = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad \widetilde{v}^A = \begin{pmatrix} -\bar{\beta} \\ \bar{\alpha} \end{pmatrix} \tag{2.11}$$

where bar means complex conjugate.

Simplification occurs if the tensor is antisymmetric such as the curvature tensor F_{ij}: in that case one can verify that its spinor version $\mathcal{F}_{AA'BB'}$ is a linear combination of $\epsilon_{A'B'}$ and ϵ_{AB}. More precisely one finds that

$$F_{ij}\, g^i_{AA'} g^j_{BB'} = \mathcal{F}_{AA'BB'} = \mathcal{F}_{AB}\epsilon_{A'B'} + \mathcal{F}_{A'B'}\epsilon_{AB} \tag{2.12}$$

Moreover it also turns out that $\mathcal{F}_{A'B'}$ and $\mathcal{F}_{AB}$ are the spinor projections of the self-dual and anti-self-dual parts of F_{ij} respectively; i.e. one can check that

$$F^+_{ij}\, g^i_{AA'} g^j_{BB'} = \mathcal{F}_{A'B'}\epsilon_{AB}, \quad F^-_{ij}\, g^i_{AA'} g^j_{BB'} = \mathcal{F}_{AB}\epsilon_{A'B'} \tag{2.13}$$

Now we return to the Seiberg–Witten equations and carry out the translation from the conventional to the spinor form. Starting with the Dirac equation we write

$$M = \begin{pmatrix} \alpha \\ \beta \\ 0 \\ 0 \end{pmatrix} \equiv \begin{pmatrix} \mathcal{M}^{A'} \\ 0 \\ 0 \end{pmatrix} \tag{2.14}$$

and then the Dirac equation

$$\Gamma^i D_i M = 0 \tag{2.15}$$

becomes

$$\sqrt{2} g^i_{AA'} D_i \mathcal{M}^{A'} = 0 \tag{2.16}$$

which we rewrite as

$$\mathcal{D}_{AA'}\mathcal{M}^{A'} = 0, \quad \text{with} \quad \mathcal{D}_{AA'} = \frac{1}{\sqrt{2}} \begin{pmatrix} D_0 + iD_3 & iD_1 + D_2 \\ iD_1 - D_2 & D_0 - iD_3 \end{pmatrix} \tag{2.17}$$

which is the desired form. Moving on to the other equation

$$F^+_{ij} = -\frac{i}{2}\overline{M}\Gamma_{ij}M \tag{2.18}$$

we first display this equation in full as

$$\frac{1}{2}\begin{pmatrix} 0 & F_{01}+F_{23} & F_{02}-F_{13} & F_{03}+F_{12} \\ -F_{01}-F_{23} & 0 & F_{03}+F_{12} & F_{13}-F_{02} \\ F_{13}-F_{02} & -F_{12}-F_{03} & 0 & F_{01}+F_{23} \\ -F_{12}-F_{03} & F_{02}-F_{13} & -F_{01}-F_{23} & 0 \end{pmatrix} = \tag{2.19}$$

$$\frac{1}{2}\begin{pmatrix} 0 & \bar\beta\alpha+\bar\alpha\beta & i\bar\beta\alpha-i\bar\alpha\beta & |\alpha|^2-|\beta|^2 \\ -\bar\beta\alpha-\bar\alpha\beta & 0 & |\alpha|^2-|\beta|^2 & -i\bar\beta\alpha+i\bar\alpha\beta \\ -i\bar\beta\alpha+i\bar\alpha\beta & -|\alpha|^2+|\beta|^2 & 0 & \bar\beta\alpha+\bar\alpha\beta \\ -|\alpha|^2+|\beta|^2 & i\bar\beta\alpha-i\bar\alpha\beta & -\bar\beta\alpha-\bar\alpha\beta & 0 \end{pmatrix}$$

and then find that

$$F^+_{ij}\, g^i_{AA'} g^j_{BB'} = -\frac{i}{2}\overline{M}\Gamma_{ij} M\, g^i_{AA'} g^j_{BB'} \tag{2.20}$$

becomes

$$\mathcal{F}_{A'B'}\epsilon_{AB} = \mathcal{T}_{A'B'}\epsilon_{AB} \tag{2.21}$$

i.e.

$$\mathcal{F}_{A'B'} = \mathcal{T}_{A'B'} \tag{2.22}$$

where

$$\mathcal{F}_{A'B'} = \frac{1}{2}\begin{pmatrix} iF_{01}+iF_{23}+F_{13}-F_{02} & -iF_{12}-iF_{03} \\ -iF_{12}-iF_{03} & -iF_{01}-iF_{23}+F_{13}-F_{02} \end{pmatrix} \tag{2.23}$$

and

$$\mathcal{T}_{A'B'} = \frac{1}{2}\begin{pmatrix} 2i\bar\alpha\beta & -i|\alpha|^2+i|\beta|^2 \\ -i|\alpha|^2+i|\beta|^2 & -2i\alpha\bar\beta \end{pmatrix} \tag{2.24}$$

On can now readily inspect equations 2.19 and 2.22, 2.23, 2.24 and confirm that the conventional and the spinor form of the equations agree.

Also we can compute the matrix of components $(i/2)(\mathcal{M}_{A'}\widetilde{\mathcal{M}}_{B'} + \mathcal{M}_{B'}\widetilde{\mathcal{M}}_{A'})$ and verify that it is equal to $\mathcal{T}_{A'B'}$. Doing this we find that, if we start with $\mathcal{M}^{A'} = (\alpha, \beta)$ and use 2.10 and 2.11, we obtain

$$\begin{aligned} \frac{i}{2}((\mathcal{M}_{A'}\widetilde{\mathcal{M}}_{B'} + \mathcal{M}_{B'}\widetilde{\mathcal{M}}_{A'}) &= \frac{1}{2}\begin{pmatrix} 2i\bar\alpha\beta & -i|\alpha|^2+i|\beta|^2 \\ -i|\alpha|^2+i|\beta|^2 & -2i\alpha\bar\beta \end{pmatrix} \\ &= \mathcal{T}_{A'B'}, \end{aligned} \tag{2.25}$$

as it should. We now turn to the explicit Fermion and gauge field considered by Freund.

§ 3. Freund's equations

In **[3]** Freund chooses

$$A_i = \frac{1}{2r(r-z)}\begin{pmatrix} 0 \\ -y \\ x \\ 0 \end{pmatrix}, \quad \text{and} \quad \mathcal{M}^{A'} = \frac{1}{2r\sqrt{r(r-z)}}\begin{pmatrix} x-iy \\ r-z \end{pmatrix} \tag{3.1}$$

$$\widetilde{\mathcal{M}}^{A'} = \frac{1}{2r\sqrt{r(r-z)}}\begin{pmatrix} -(r-z) \\ x+iy \end{pmatrix}$$

for which one readily verifies that

$$\mathcal{D}_{AA'}\mathcal{M}^{A'} = 0 \tag{3.2}$$

so that $\mathcal{M}^{A'}$ is indeed a zero mode.

To check the other equation we compute the curvature and find that, if $F_{ij} = \partial_i A_j - \partial_j A_i$, one has

$$\begin{aligned} F_{0i} = 0, \quad F_{12} = -\frac{z}{2r^3}, \quad F_{13} = \frac{y}{2r^3}, \quad F_{23} = -\frac{x}{2r^3} \\ \Rightarrow \mathcal{F}_{A'B'} = \frac{1}{4r^3}\begin{pmatrix} y - ix & iz \\ iz & y + ix \end{pmatrix} \end{aligned} \tag{3.3}$$

On the other hand one also finds that

$$\begin{aligned} \frac{i}{2}(\mathcal{M}_{A'}\widetilde{\mathcal{M}}_{B'} + \mathcal{M}_{B'}\widetilde{\mathcal{M}}_{A'}) &= \frac{1}{4r^3}\begin{pmatrix} -y + ix & -iz \\ -iz & -y - ix \end{pmatrix} \\ &= -\mathcal{F}_{A'B'} \end{aligned}$$

so that $\mathcal{F}_{A'B'} \neq (i/2)(\mathcal{M}_{A'}\widetilde{\mathcal{M}}_{B'} + \mathcal{M}_{B'}\widetilde{\mathcal{M}}_{A'})$ and Freund's equations are

$$\begin{aligned} \mathcal{F}_{A'B'} &= -\frac{i}{2}\left(\mathcal{M}_{A'}\widetilde{\mathcal{M}}_{B'} + \mathcal{M}_{B'}\widetilde{\mathcal{M}}_{A'}\right) \\ \mathcal{D}_{AA'}\mathcal{M}^{A'} &= 0 \end{aligned} \tag{3.4}$$

The Seiberg–Witten's equations are known to admit no non-trivial regular L^2 solutions in flat space (or spaces of positive scalar curvature) so Freund was concerned to point out that his fields provide an example of a non-trivial solution which is not L^2. Unfortunately, as we have seen, Freund's fields, though not L^2, are not solutions to the Seiberg–Witten equations.

Since Freund's fields are static and have a connection with $A_0 = 0$ it is natural to consider them in $\mathbf{R}^3$. We now do this letting $\mathbf{A} = (A_1, A_2, A_3)$ be the connection in $\mathbf{R}^3$ and denote its curvature components by $\widehat{F}_{ab}$, $a, b = 1..3$. We obtain thereby the three dimensional Freund equations

$$\begin{aligned} \widehat{F}_{ab} &= -\epsilon_{abc}\overline{M}\sigma^c M, \quad a, b = 1..3 \\ \partial\!\!\!/_A M &= 0, \quad \text{where } \partial\!\!\!/_A = i\sigma^a(\partial_a + iA_a) \end{aligned} \tag{3.5}$$

In similar fashion we could also have obtained the three dimensional Seiberg–Witten equations—cf. [**2**]—and, as in four dimensions, these differ from Freund's only in the sign of the quadratic Fermion term. They are

$$\begin{aligned} \widehat{F}_{ab} &= \epsilon_{abc}\overline{M}\sigma^c M, \quad a, b = 1..3 \\ \partial\!\!\!/_A M &= 0, \quad \text{where } \partial\!\!\!/_A = i\sigma^a(\partial_a + iA_a) \end{aligned} \tag{3.6}$$

There is also a vanishing theorem which does not allow non trivial solutions in flat space so that there are no regular L^2 solutions to the equations 3.6 in $\mathbf{R}^3$. However there is no such restriction on the Freund's equations 3.5. In the next section we show how to construct examples of *singular* non-L^2 solutions to the Seiberg–Witten equations and *regular* solutions to of Freund's equations which are L^2 in $\mathbf{R}^3$.

§ 4. The Freund and Seiberg–Witten equations in three dimensions

First of all we simply note that Freund's equations 3.5 (or indeed 3.4) admit the following regular solution which is L^2 in $\mathbf{R}^3$.

$$\begin{aligned} \mathcal{M}^{A'} &= \frac{\sqrt{12}(\mathbf{1} + i\vec{\sigma}\cdot\vec{r})}{(1+r^2)^{3/2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ A_i &= -\frac{3}{(1+r^2)^2} \begin{pmatrix} 2xz - 2y \\ 2yz + 2x \\ 1 - r^2 + 2z^2 \end{pmatrix} \end{aligned} \tag{4.1}$$

as may be checked easily.

Finally we would like to display some (necessarily singular) solutions to the three dimensional Seiberg–Witten equations 3.6; they will also of course be solutions of the full Seiberg–Witten equations 2.1 or 2.3. In fact we construct a whole class of such solutions parametrised by an arbitrary holomorphic function.

First we need some facts about the Dirac equation in 3.6. A spinor M that obeys the Dirac equation $\partial\!\!\!/_A M = 0$ of 3.6 must obey the condition

$$\partial_a \Sigma^a = 0, \quad \text{where } \Sigma^a = \overline{M}\sigma^a M \tag{4.2}$$

The connection A_i in the Dirac equation can be expressed in terms of the zero mode M by writing [**5**]

$$\begin{aligned} A_i &= -\frac{1}{\sqrt{\Sigma^a\Sigma^a}} \left(\frac{1}{2}\epsilon_{ijk}\partial_j\Sigma_k + Im\,\overline{M}\partial_i M \right) \\ &= -\frac{1}{2}\epsilon_{ijk}\left(\partial_j \ln\sqrt{\Sigma^a\Sigma^a}\right) N_k - \frac{1}{2}\epsilon_{ijk}\partial_j N_k - Im\,\overline{\widehat{M}}\partial_i\widehat{M} \\ \text{where} \quad N^a &= \frac{\Sigma^a}{\sqrt{\Sigma^b\Sigma^b}}, \quad \text{and } \widehat{M} = \frac{M}{\sqrt{\overline{M}M}} \end{aligned} \tag{4.3}$$

Now, if χ is the complex variable

$$\chi = \frac{x+iy}{r^2} \tag{4.4}$$

and $G \equiv G(\chi,\bar{\chi})$ is a function of χ and $\bar{\chi}$, we obtain a new class of zero modes M^G where

$$\begin{aligned} M^G &= e^{G/2} M^0 \\ \text{and} \quad M^0 &= \frac{1}{r^3}\begin{pmatrix} z \\ x+iy \end{pmatrix} \end{aligned} \tag{4.5}$$

The corresponding connection A_i^G is found using formula 4.3 and is given by

$$A_i^G = -\frac{1}{2}\epsilon_{ijk}\partial_j G N_k \tag{4.6}$$

We note that the spinor M^0 is singular and non-L^2. For doing calculations it is also useful to note that M^0 is a solution of the free Dirac equation

$$\not{\partial} M^0 = 0 \tag{4.7}$$

and that the spin density for M^G satisfies

$$\begin{aligned} \overline{M^G}\sigma^a M^G &= e^G\,\overline{M^0}\sigma^a M^0 \\ \text{where} \quad \overline{M^0}\sigma^a M^0 &= \frac{N^a}{r^4} = \frac{1}{2i}(\partial\bar{\chi})\times(\partial\chi) \end{aligned} \tag{4.8}$$

The corresponding curvature $\widehat{F}_{ij}^G$ is

$$\begin{aligned} \widehat{F}_{ij}^G = -\frac{\epsilon_{ijk}}{2}\,[&G_{,\chi}(\chi_{,kl}N_l + \chi_{,k}N_{l,l} - \chi_{,ll}N_k - \chi_{,l}N_{k,l}) + \\ &G_{,\bar{\chi}}(\bar{\chi}_{,kl}N_l + \bar{\chi}_{,k}N_{l,l} - \bar{\chi}_{,ll}N_k - \bar{\chi}_{,l}N_{k,l}) - \\ &(G_{,\chi\chi}\chi_{,l}\chi_{,l} + G_{,\bar{\chi}\bar{\chi}}\bar{\chi}_{,l}\bar{\chi}_{,l} + 2G_{,\chi\bar{\chi}}\chi_{,l}\bar{\chi}_{,l})N_k] \end{aligned} \tag{4.9}$$

After some tedious algebra we find that only the coefficient of $G_{,\chi\bar{\chi}}$ is non-zero and that

$$\chi_{,k}\bar{\chi}_{,k} = \frac{2}{r^4} \tag{4.10}$$

and hence

$$\widehat{F}_{ij}^G = \frac{2\epsilon_{ijk}}{r^4}G_{,\chi\bar{\chi}}N_k \tag{4.11}$$

But to have a solution of the Seiberg–Witten equations we must require that

$$\widehat{F}_{ij}^G = \epsilon_{ijk}\overline{M^G}\sigma^k M^G \tag{4.12}$$

and this means that

$$\begin{aligned} \frac{2}{r^4}G_{,\chi\bar{\chi}}N_k &= \overline{M^G}\sigma^k M^G \\ \Rightarrow G_{,\chi\bar{\chi}} &= \frac{1}{2}e^G \end{aligned} \tag{4.13}$$

as may be easily checked. But this equation for G is nothing other than the Liouville equation in the "target space coordinate" χ with the "wrong" sign; that is to say that the sign leads to the general singular solution

$$G = \frac{|f'(\chi)|^2}{(1-f\bar{f})^2} \tag{4.14}$$

where f is an arbitrary holomorphic function of χ.

Hence any pair (M^G, A_i^G) with G given by 4.14 is a solution the Seiberg–Witten equations 3.6. In fact, these solutions resemble the two-dimensional solutions of the Seiberg–Witten equations that were discussed in [**6**]. Their solutions emerged as solutions to the same Liouville equation 4.14, however the coordinate space variable $x_+ = x + iy$ appears rather than the target space variable χ used here.

Finally we want to briefly describe the geometry of the spin density term $\overline{M}\sigma^a M$. It is clearly rotationally symmetric around the z axis and the integral curves of $\overline{M^0}\sigma^a M^0$ are circles that touch the z axis at the point $z = 0$. If we restrict to the $x - z$ plane, then these integral curves are the field lines of a dipole in two dimensions, and the vector field $\overline{M^0}\sigma^a M^0$ restricted to that plane is a scalar function times the field of a dipole in two dimensions.

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