January 18, 2001

Zero modes in finite range magnetic fields

C. Adam*
Institut für theoretische Physik, Universität Karlsruhe, 76128 Karlsruhe

B. Muratori**, C. Nash***
Department of Mathematical Physics, National University of Ireland, Maynooth

Abstract

We find a class of Fermion zero modes of Abelian Dirac operators in three dimensional Euclidean space where the gauge potentials and the related magnetic fields are nonzero only in a finite space region.

*)email address: adam@particle.uni-karlsruhe.de, adam@pap.univie.ac.at
**)email address: bmurator@fermi1.thphys.may.ie
***)email address: cnash@stokes2.thphys.may.ie

1 Introduction

Fermion zero modes of the Abelian Dirac operator in three dimensional Euclidean space (i.e., of the Pauli operator) are a rather young subject of study, and still many features remain unknown. The first example of such a zero mode of the Pauli operator has been given in [1], where it was used to prove that one-electron atoms with sufficiently high nuclear charge in an external magnetic field are unstable, see [2]. Further examples of zero modes were discussed in [3], where their relevance for QED was briefly mentioned (this point is discussed in more detail in [4, 5]). In [6, 7] it was proven by explicit construction of a class of examples that the phenomenon of zero mode degeneracy (i.e., Pauli operators with more than one zero mode) occurs, and a relation between the number of zero modes of a Pauli operator and the Hopf index of the corresponding magnetic field was established. This point was further elaborated in [8].

In [9] an example of a zero mode was given where the corresponding gauge potential (and magnetic field) are non-zero only within a finite region of space (within a ball with finite radius). In fact, this example belongs to the types of zero modes that were discussed in [3]. Here we want to construct a whole class of zero modes of Pauli operators where the related gauge potentials and magnetic fields vanish outside a finite region of space. This demonstrates that the possibility of having zero modes in magnetic fields of finite range is not just a curiosity that is related to some very special examples, but a rather generical feature of the Pauli operator. We think that this observation is interesting from a physical perspective as well, because magnetic fields with finite range are precisely the types of magnetic fields that may be realised experimentally.

2 Construction of the zero modes

We want to study specific solutions of the equation

$$-i\sigma_i\partial_i\Psi(x) = A_i(x)\sigma_i\Psi(x) \tag{1}$$

(here $x = (x_1, x_2, x_3)$, $r = |x|$, and σ_i are the Pauli matrices) where Ψ is square-integrable, $\Psi(x) \in L^2(\mathbf{R}^3)$, A_i and $B_i = \epsilon_{ijk}\partial_j A_k$ are non-singular everywhere in $\mathbf{R}^3$ and are different from zero only in a finite region of space. Further Ψ, A_i and B_i have to be smooth everywhere. For this purpose, let us first observe that the spinor ($x_\pm \equiv x_1 \pm ix_2$)

$$\Psi^0(x) = \frac{i}{r^3}\begin{pmatrix} x_3 \\ x_+ \end{pmatrix} \tag{2}$$

solves the free Dirac equation

$$-i\vec{\sigma}\vec{\partial}\Psi^0 = 0 \tag{3}$$

(the i in (2) is chosen for later convenience). The spinor (2) is singular at $r = 0$ but it is well behaved for large r. So we might ask whether there exist spinors that are equal to Ψ^0 outside a ball of radius $r = R$ (where they solve the free Dirac equation, i.e., $A_i = 0$

for $r > R$), whereas they differ from Ψ^0 inside $r = R$. Inside the ball they are supposed to solve the Dirac equation for some nonzero A_i such that they are nonsingular and smooth everywhere. We shall find a whole class of such zero modes among the zero modes that were discussed in [3], therefore we want to review the results of [3] briefly. There the ansatz

$$\Psi = g(r)\exp(if(r)\frac{\vec{x}}{r}\vec{\sigma})\begin{pmatrix}1\\0\end{pmatrix} = g(r)[\cos f(r)\mathbf{1} + i\sin f(r)\frac{\vec{x}}{r}\vec{\sigma}]\begin{pmatrix}1\\0\end{pmatrix} \tag{4}$$

for the spinor lead to a zero mode for the gauge field

$$A_i = h(r)\frac{\Psi^\dagger\sigma_i\Psi}{\Psi^\dagger\Psi} \tag{5}$$

provided that $g(r)$ and $h(r)$ are given in terms of the independent function $f(r)$ as $('\equiv d/dr)$

$$g' = -\frac{2}{r}\frac{t^2}{1+t^2}g. \tag{6}$$

$$h = (1+t^2)^{-1}(t' + \frac{2}{r}t) \tag{7}$$

where

$$t(r) := \tan f(r). \tag{8}$$

A sufficient condition on $t(r)$ leading to smooth, non-singular and L^2 spinors and smooth, non-singular gauge potentials with finite energy $(\int(\vec{B})^2)$ and finite Chern–Simons action $(\int\vec{A}\vec{B})$ is

$$t(0) = 0\,, t(r) \sim c_1 r + o(r^2) \text{ for } r \to 0 \tag{9}$$

$$t(\infty) = \infty \tag{10}$$

which we shall assume in the sequel. Observe that in the limit $t \to \infty$ $\vec{A}$ vanishes whereas Ψ becomes Ψ^0. Therefore, if we find some t that become infinite at some finite $r = R$ in a smooth way and stay infinite for $r > R$, we have found precisely what we want. To get a more manageable condition, let us re-express things in terms of

$$c(r) := \cos f(r) = \frac{1}{(1+t(r)^2)^{1/2}} \tag{11}$$

which leads to

$$g' = -\frac{2}{r}(1-c^2)g \tag{12}$$

$$h = c(-\frac{c'}{(1-c^2)^{1/2}} + \frac{2}{r}(1-c^2)^{1/2}). \tag{13}$$

Further c has to behave like

$$c(r) \sim 1 - c_2 r^2 + \ldots \quad \text{for} \quad r \to 0. \tag{14}$$

Now let us assume that c approaches zero in a smooth way for $r = R$ and stays zero for $r \geq R$, and further $c'(r = R) = 0$, $|c''(r = R)| < \infty$. This implies that $\vec{A} = 0$ for $r > R$ and that

$$g(r) = kr^{-2}\,, \quad k = \exp(-2\int_0^R dr \frac{1 - c(r)^2}{r}) \quad \text{for} \quad r \geq R \tag{15}$$

which precisely leads to $\Psi = \mathrm{const} \cdot \Psi^0$ for $r > R$, see (4).

Finally, let us give some examples, where we choose $R = 1$ for convenience. A first example is

$$c(r) = (1 - r^2)^2 \quad \text{for} \quad r < 1\,, \quad c(r) = 0 \quad \text{for} \quad r \geq 1 \tag{16}$$

leading to

$$g(r) = \exp(-4r^2 + 3r^4 - \frac{4}{3}r^6 + \frac{1}{4}r^8) \quad \text{for} \quad r < 1$$
$$g(r) = \exp(-\frac{25}{12})\,r^{-2} \quad \text{for} \quad r \geq 1 \tag{17}$$

and

$$h(r) = \frac{2(1 - r^2)^2(2 - 4r^2 + 4r^4 - r^6)}{(4 - 6r^2 + 4r^4 - r^6)^{1/2}} \quad \text{for} \quad r < 1$$
$$h(r) = 0 \quad \text{for} \quad r \geq 1. \tag{18}$$

Another example is

$$c(r) = \exp(\frac{r^2}{r^2 - 1}) \quad \text{for} \quad r < 1\,, \quad c(r) = 0 \quad \text{for} \quad r \geq 1. \tag{19}$$

There is one difference between example (16) and example (19). Both lead to L^2 zero modes (i.e., bound states), and both lead to magnetic fields with are smooth, non-singular and have finite energy. However, higher derivatives of c in (16) are discontinuous, whereas all derivatives of c in (19) are smooth. In some situations (or for mathematical reasons) it may be preferable to have only such c that have only smooth higher derivatives, then functions c like in (16) may be treated as follows with the help of functions like (19). Define a function c_a

$$c_a(r) = (1 - r^2)^2 \exp(\frac{ar^2}{r^2 - 1}) \quad \text{for} \quad r < 1\,, \quad c(r) = 0 \quad \text{for} \quad r \geq 1 \tag{20}$$

where $a \geq 0$. For $a \neq 0$ c_a is a C^∞ function. In the limit $a \to 0$ c_a is equal to the c of (16). Further, for a sufficiently small, c_a approximates the c of (16) with arbitrary precision. Therefore, the function c_a with a sufficiently small a may be used as a C^∞ substitute for (16).

3 Summary

As should be clear from the above discussion, there is an infinite number of possible functions $c(r)$, therefore already for the special ansatz (4) there exists a whole class of zero modes in finite range magnetic fields. One obvious generalisation of the above result is the existence of zero modes in magnetic fields that are non-zero inside a ball $r < R_1$, zero between R_1 and $R_2 > R_1$, non-zero again between R_2 and $R_3 > R_2$, etc., forming an onion-like structure.

We started from the spherically symmetric ansatz (4), therefore the finite regions where $\vec{B} \neq 0$ are all spherically symmetric balls. It is plausible to assume that by relaxing the symmetry condition on the zero modes one could find zero modes for magnetic fields which vanish outside finite regions of different shapes.

References

[1] M. Loss and H.-Z. Yau, Comm. Math. Phys. 104 (1986) 283

[2] J. Fröhlich, E. Lieb and M. Loss, Comm. Math. Phys. 104 (1986) 251

[3] C. Adam, B. Muratori and C. Nash, Phys. Rev. D60 (1999) 125001

[4] M. Fry, Phys. Rev. D54 (1996) 6444

[5] M. Fry, Phys. Rev. D55 (1997) 968

[6] C. Adam, B. Muratori and C. Nash, hep-th/9910139

[7] C. Adam, B. Muratori and C. Nash, hep-th/0001164

[8] L. Erdos and J. Solovej, math-ph/0001036

[9] D. Elton, preprint “New examples of zero modes”