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Torsion, topology and CPT anomaly in two-dimensional chiral $U(1)$ gauge theory

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Abstract

We consider the CPT anomaly of two-dimensional chiral $U(1)$ gauge theory on a torus with topologically nontrivial zweibeins corresponding to the presence of spacetime torsion. The resulting chiral determinant can be expressed in terms of the standard chiral determinant without torsion, but with modified spinor boundary conditions. This implies that the two-dimensional CPT anomaly can be moved from one spin structure to another by choosing appropriate zweibeins. Similar results apply to higher-dimensional chiral gauge theories.

Key words: Torsion, Topology, Chiral gauge theory, CPT violation
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1 Introduction

Recently, it has been shown that there is a violation of CPT invariance in certain (non-)Abelian chiral gauge theories defined on nonsimply connected spacetime manifolds [1]. The well-known CPT theorem [2,3] is evaded by the breakdown of Lorentz invariance due to the quantum effects of the chiral fermions [1,4].

The Abelian CPT anomaly is particularly obvious on the two-dimensional torus [5] where the chiral determinant can be computed exactly (see Ref. [6] and references therein). For this reason, we will consider in this paper primarily two-dimensional chiral $U(1)$ gauge theory defined over the torus. More

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precisely, we study the CPT anomaly on the torus in the presence of spacetime torsion. That is, we consider the effects of a nontrivial configuration of zweibeins, which gives rise to a nonvanishing torsion tensor [7,8]. (Zweibeins are the two-dimensional analogs of vierbeins or tetrads in four dimensions.)

The main goal of the present paper is then to understand better the role of topology and spin structure for the two-dimensional CPT anomaly, by studying the response of the chiral determinant to the introduction of topologically nontrivial zweibeins on the torus. For a general discussion of spinors over non-simply connected spacetime manifolds, we refer the reader, in particular, to Refs. [9,10].

The paper is organized as follows. In Section 2, we discuss some aspects of the geometry of the two-dimensional torus with torsion and establish our notation. Specifically, we mention two consequences of torsion at the level of the spacetime structure. Namely, parallelograms need not close and extremal and autoparallel curves need not coincide. We also comment on some interesting properties of topologically nontrivial zwei- (or vier-)beins on the torus and their possible origin.

In Section 3, we show that one can relate a fermionic Lagrangian with nontrivial zweibeins to a Lagrangian with trivial zweibeins by a simple spinor redefinition. This field redefinition can, however, change the spinor boundary conditions.

In Section 4, we use this property of the fermionic Lagrangian to express the chiral determinant for topologically nontrivial zweibeins in terms of the chiral determinant for trivial zweibeins but modified spinor boundary conditions. We also give a heuristic argument for the result. A similar calculation is done in Appendix A for the Dirac determinant of a vector-like $U(1)$ gauge theory.

In Section 5, we discuss the role of torsion for the two-dimensional chiral CPT anomaly and find that only the topologically nontrivial part of the zweibeins affects the anomaly.

In Section 6, we summarize our results and briefly comment on the four-dimensional case.

2 Topology, geometry and torsion

2.1 Zweibeins on the torus

The Cartesian coordinates $x^\mu \in [0, L]$, $\mu = 1, 2,$ are taken to parameterize a particular two-dimensional torus $T^2[i]$, with modulus (Teichmüller parameter) $\tau = i$. This torus can be thought of as a square with flat Euclidean metric $g_{\mu\nu}(x) = \delta_{\mu\nu} \equiv \mathrm{diag}(1, 1)$ and opposite sides identified; see Fig. 1.

Zweibeins locally define an orthonormal basis of one-forms

$$e^a(x) = e^a_\mu(x)\, \mathrm{d}x^\mu \,. \tag{2.1}$$

Here, Latin indices (a, b, ...) refer to the local frame and Greek indices (μ, ν,

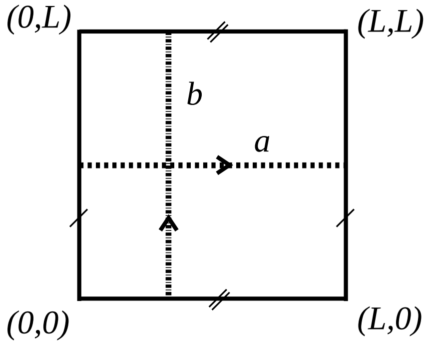

Fig. 1. The torus $T^2[i]$ is represented as a square with opposite sides identified. Two distinct noncontractible curves are labeled a and b.

...) to the base space. Throughout this paper, summation over equal upper and lower indices is understood. The inverse zweibeins $e_a^{\mu}(x)$ are defined by

$$\delta_a^{\ b} = e_a^{\mu}(x)\, e_{\mu}^{b}(x)\ . \qquad (2.2)$$

In terms of the zweibeins, the metric can be expressed as follows (see, for example, Refs. [7,8]):

$$g_{\mu\nu}(x) = e_{\mu}^{a}(x)\, e_{\nu}^{b}(x)\, \delta_{ab}\ . \qquad (2.3)$$

Inversely, this equation defines the zweibeins, but only up to a space-dependent orthogonal transformation.

In the following, we consider zweibeins $e_{\mu}^{a}(x)$ taking values in a matrix representation of the group $SO(2)$:

$$\begin{pmatrix} e_{\mu}^{1}(x) \\ e_{\mu}^{2}(x) \end{pmatrix} \equiv \delta_{\mu}^{1} \begin{pmatrix} \cos\varphi(x) \\ \sin\varphi(x) \end{pmatrix} + \delta_{\mu}^{2} \begin{pmatrix} -\sin\varphi(x) \\ \cos\varphi(x) \end{pmatrix}\ , \qquad (2.4)$$

parametrized by the real function $\varphi(x)$. The corresponding metric is flat, $g_{\mu\nu}(x) = \delta_{\mu\nu}$. Obviously, the choice $\varphi(x) = 0$ yields the trivial zweibeins $e_{\mu}^{a}(x) = \delta_{\mu}^{a}$. The extra degree of freedom $\varphi(x)$ in the zweibeins (2.4) can, however, generate torsion, as will be shown in the next subsection.

The zweibeins (2.4) need to be defined in a consistent way on $T^2[i]$, namely

$$e_{\mu}^{a}(0,x^2) = e_{\mu}^{a}(L,x^2)\ , \qquad e_{\mu}^{a}(x^1,0) = e_{\mu}^{a}(x^1,L)\ . \qquad (2.5)$$

This requirement naturally leads to a decomposition of the $SO(2)$ rotation angle $\varphi(x)$ into topologically trivial and nontrivial parts:

$$\varphi(x) \equiv \omega(x) + \chi(x)\ , \qquad (2.6)$$

where the real function $\omega(x)$ is taken to be strictly periodic in x^1 and x^2, with period L. The other real function $\chi(x)$ is associated with the two generating curves a and b of the homology group $\mathcal{H}^1(T^2, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$; cf. Ref. [8]. Specifically, the function $\chi(x)$ is given by

$$\chi(x) \equiv (2\pi/L)(m\, x^1 + n\, x^2) , \qquad m, n \in \mathbb{Z} , \tag{2.7}$$

for the two distinct noncontractible curves a and b shown in Fig. 1. [The notation $\chi(x; m, n)$ would, of course, be more accurate.]

2.2 Connection

The condition for parallel transport of an arbitrary vector field $C^\mu(x)$ along the infinitesimal path $(x, x + \delta x)$ is

$$C^\lambda(x)\, e^a_\lambda(x) = \Big(C^\lambda(x) + \delta C^\lambda(x) \Big)\, e^a_\lambda(x + \delta x) . \tag{2.8}$$

With the linear *Ansatz* [11]

$$\delta C^\lambda(x) \;\equiv\; -\hat{\Gamma}^\lambda_{\mu\nu}(x)\, C^\mu(x)\, \delta x^\nu , \tag{2.9}$$

condition (2.8) enables us to express the Riemann–Cartan connection $\hat{\Gamma}^\lambda_{\mu\nu}(x)$ in terms of the zweibeins:

$$\hat{\Gamma}^\lambda_{\mu\nu}(x) = e^\lambda_a(x)\, e^a_{\mu,\nu}(x) = (\delta^\lambda_2 \delta^1_\mu - \delta^\lambda_1 \delta^2_\mu)\, \partial_\nu \varphi(x) , \tag{2.10}$$

for the particular zweibeins (2.4) which have vanishing metric Christoffel symbol [8,9]. As usual, the notation $\phi_{,\nu}$ stands for $\partial\phi/\partial x^\nu$. The result (2.10) demonstrates that the freedom in choosing a particular connection on $T^2[i]$ is a consequence of the fact that for a fixed metric $g_{\mu\nu}(x) = \delta_{\mu\nu}$ the zweibeins are defined only up to an orthogonal transformation.

Since the zweibeins considered take values in a two-dimensional matrix representation of the group $SO(2)$, we can write the transformation of the connection under an $SO(2)$ transformation $\tilde{e}^\mu_a(x)$ as

$$\hat{\Gamma}'^{\,a}_{\;\mu b} = \tilde{e}^a_\lambda\, \hat{\Gamma}^\lambda_{\mu\nu}\, \tilde{e}^\nu_b + (\partial_\mu \tilde{e}^a_\nu)\, \tilde{e}^\nu_b . \tag{2.11}$$

While the connection itself does not transform as a tensor, its antisymmetric component in the two lower indices does. This object is called the torsion tensor [7,8],

$$T^\lambda_{\mu\nu}(x) \equiv \hat{\Gamma}^\lambda_{\mu\nu}(x) - \hat{\Gamma}^\lambda_{\nu\mu}(x) . \tag{2.12}$$

For the connection (2.10), we obtain

$$T^\lambda_{12}(x) = -T^\lambda_{21}(x) = \partial^\lambda \varphi(x) , \tag{2.13}$$

where the function $\varphi(x)$ parameterizes the zweibeins (2.4).

Parallel transport enables us to give an "operational definition" of torsion (see also Refs. [7,11]). Consider the parallelogram spanned by the line elements

$(x, E_1 \cdot x)$ and $(x, E_2 \cdot x)$, where the infinitesimal displacements of x are given by

$$
\begin{aligned}
(E_1 \cdot x)^\mu &= x^\mu + \epsilon_1^\mu(x) \,, \quad \epsilon_1^\mu(x) \;\equiv\; \epsilon_1^a\, e_a^\mu(x) \,, \\
(E_2 \cdot x)^\mu &= x^\mu + \epsilon_2^\mu(x) \,, \quad \epsilon_2^\mu(x) \;\equiv\; \epsilon_2^a\, e_a^\mu(x) \,,
\end{aligned}
\tag{2.14}
$$

with real infinitesimal coefficients ϵ_1^a and ϵ_2^a. There is then torsion if the parallelogram does not close, i.e. $[E_1 E_2 - E_2 E_1] \cdot x \neq 0$. In fact, a short calculation yields

$$
\begin{aligned}
(\,[E_2 E_1 - E_1 E_2] \cdot x)^\mu &= [\,\hat\Gamma^\mu_{\rho\sigma}(x) - \hat\Gamma^\mu_{\sigma\rho}(x)\,]\, \epsilon_2^\rho(x)\, \epsilon_1^\sigma(x) \\
&= T^\mu_{\rho\sigma}(x)\; \epsilon_2^\rho(x)\, \epsilon_1^\sigma(x) \,,
\end{aligned}
\tag{2.15}
$$

where Eq. (2.9) has been used for the parallel transport of $\epsilon_1^\mu(x)$ and $\epsilon_2^\mu(x)$.

2.3 Extremal and autoparallel curves

Apart from the fact that parallelograms do not close in the presence of torsion, there is a further consequence of torsion at the level of the spacetime structure: extremal and autoparallel curves do not necessarily coincide (see also Ref. [7]).

The equations for an extremal curve (shortest or longest line) on the two-dimensional flat spacetime manifold $T^2[i]$ are given by

$$
\ddot x^1(\tau) \;=\; 0 \,, \qquad \ddot x^2(\tau) \;=\; 0 \,,
\tag{2.16}
$$

where the dot denotes differentiation with respect to the affine parameter τ. Extremal curves on $T^2[i]$ may or may not close. An example of an extremal curve that does *not* close is given by $x^1 = \tau$ and $x^2 = \sqrt{2}\,\tau$, since $\sqrt{2}$ is an irrational number. Recall that the ratio of the periodicities in x^1 and x^2 is exactly 1 for the particular torus $T^2[i]$ considered; see Fig. 1.

The equations for an autoparallel curve (straightest line) can be deduced by requiring that autoparallel curves are always tangent to the zweibeins. The substitution $C^\mu = \mathrm{d}x^\mu(\tau)/\mathrm{d}\tau \equiv \dot x^\mu(\tau)$ in Eq. (2.9) gives the following result:

$$
\ddot x^\lambda + \hat\Gamma^\lambda_{\mu\nu}(x)\, \dot x^\mu \dot x^\nu \;=\; 0 \,.
\tag{2.17}
$$

Note that the Riemann–Cartan connection $\hat\Gamma^\lambda_{\mu\nu}$ in Eq. (2.10) also has a symmetric part in $\mu\nu$. For the special case of $\omega(x) = 0$ in Eq. (2.6), the equations become

$$
\begin{aligned}
\ddot x^1 - (2\pi/L)\,(m\, \dot x^1 + n\, \dot x^2)\, \dot x^2 &\;=\; 0 \,, \\
\ddot x^2 + (2\pi/L)\,(m\, \dot x^1 + n\, \dot x^2)\, \dot x^1 &\;=\; 0 \,.
\end{aligned}
\tag{2.18}
$$

Since the general solution of these coupled differential equations can be quite involved, we only discuss the class of solutions satisfying

$$m\,\dot{x}^1 + n\,\dot{x}^2 \;=\; 0\;. \tag{2.19}$$

In this case, the Eqs. (2.18) reduce to Eqs. (2.16), but with the additional constraint (2.19) on the constant velocities $\dot{x}^\mu$. One easily recognizes that Eq. (2.19) describes autoparallel curves by noting that solutions to this equation are curves of constant χ; see Eq. (2.7).

In contrast to extremal curves, which may or may not close on the torus $T^2[i]$, the autoparallel curves given by Eqs. (2.18)-(2.19) always close, since their slopes are rational ($\dot{x}^2/\dot{x}^1 \;=\; -m/n$). More specifically, the extremal curve $(x^1, x^2) \;=\; (\tau, \sqrt{2}\,\tau)$ mentioned a few lines below Eq. (2.16) does *not* solve the Eqs. (2.18) and is, therefore, not autoparallel, as long as the torsion parameter function is topologically nontrivial, $\varphi(x) = \chi(x) \neq 0$. This clearly shows the difference of the two types of curves in the presence of torsion.

2.4 Topologically nontrivial vielbeins

In this last subsection on geometry, we elaborate on the special nature of topologically nontrivial zweibeins (2.4)–(2.7) with $(m, n) \neq (0, 0)$ and $\omega(x) = 0$. For these zweibeins, namely, the torsion tensor (2.13) would be constant (and nonzero) over the whole spacetime manifold $T^2[i]$. This would then correspond to a new local property of spacetime. One manifestation would be the nonclosure of parallelograms as discussed in Section 2.2.

For topologically nontrivial zweibeins (2.4)–(2.7) with $(m, n) = (1, 1)$ and $\omega(x) = 0$, a typical parallelogram obtained by parallel transport, with lengths $|\epsilon_1^\mu| = |\epsilon_2^\mu| = \ell$, would fail to close by a distance (2.15) of order

$$(2\pi/L)\,\ell^{\,2} \sim 10^{-25}\,\mathrm{m}\;(10^{10}\,\mathrm{lyr}/L)\,(\ell/\mathrm{m})^2\;, \tag{2.20}$$

which would still be 10 orders of magnitude above the Planck length $l_\mathrm{P} \equiv \sqrt{\hbar\, G/c^3}$. As will be discussed in Section 6, a similar torsion effect may occur in four (or more) dimensions, for appropriate vier- (or viel-)beins. The level of accuracy indicated by Eq. (2.20) might be, in principle, within reach of experiment. [We have in mind a rapidly rotating ($\sim$ 100 Hz) experimental setup in a free-fall environment (e.g. in a drag-free satellite). See, for example, Eq. (3.35) in Ref. [12] for the optimal sensitivity of a resonant-bar detector for periodic gravitational waves.]

Throughout this paper, we consider the zwei- or vierbeins as fixed classical background fields. Let us, however, briefly remark on the possible origin of translation-invariant torsion resulting from topologically nontrivial vierbeins (see also Section 6). The crucial point is that this type of torsion would not have to be generated dynamically by a local spin density, but could perhaps arise as a kind of boundary condition (most likely, set at the beginning of our universe).[1] Moreover, the spin density can only be expected to give a negligible contribution to the torsion tensor for the present cosmological number densities n of protons or electrons. In fact, the order of magnitude to be compared with Eq. (2.20) is

$$(G\,c^{-3})\,(\hbar\,n)\,\ell^2 \sim n\,l_{\rm P}^2\,\ell^2 \sim 10^{-70}\,{\rm m}\;(n/{\rm m}^{-3})\,(\ell/{\rm m})^2\;. \tag{2.21}$$

(See also Section V A 3 of Ref. [7].) Hence, the translation-invariant torsion from topologically nontrivial vierbeins may at present be an extremely weak effect, but the effect is still many orders of magnitude above that expected from the ordinary matter of the universe.

3 Fermionic Lagrangian

We use a "chiral" basis for the two-dimensional Dirac matrices

$$\gamma^1 \;\equiv\; \begin{pmatrix} 0 & +1 \\ +1 & 0 \end{pmatrix}\,, \quad \gamma^2 \;\equiv\; \begin{pmatrix} 0 & +i \\ -i & 0 \end{pmatrix}\,, \quad \gamma_{\rm S} \;\equiv\; i\gamma^1\gamma^2 = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}\,, \tag{3.1}$$

where $\gamma_{\rm S}$ anticommutes with γ^1 and γ^2. (The suffix S stands for "strong reflection," originally introduced by Pauli for the proof of the CPT theorem [2,3].)

For trivial zweibeins $e^a_\mu(x) = \delta^a_\mu \equiv 1\!\!1$, the manifestly Hermitian Lagrangian is given by

$$\mathcal{L}[\bar\Psi, \Psi, A, 1\!\!1] = (i/2)\,\bar\Psi\,\gamma^\mu(\overrightarrow{\partial}_\mu + i A_\mu)\,\Psi - (i/2)\,\bar\Psi\,(\overleftarrow{\partial}_\mu - i A_\mu)\gamma^\mu\,\Psi\;, \tag{3.2}$$

with constant gamma matrices

$$\gamma^\mu \;\equiv\; \delta^\mu_a\,\gamma^a. \tag{3.3}$$

This Lagrangian is invariant under global $SO(2)$ transformations (with $\gamma_{\rm S}$ as defined above),

$$\Psi(x) \;\longrightarrow\; e^{-i\kappa\gamma_{\rm S}/2}\,\Psi(x')\;, \quad \bar\Psi(x) \;\longrightarrow\; \bar\Psi(x')\,e^{i\kappa\gamma_{\rm S}/2}, \tag{3.4}$$

and local $U(1)$ gauge transformations,

$$\Psi(x) \;\longrightarrow\; {\rm e}^{i\xi(x)}\;\Psi(x)\;, \quad \bar\Psi(x) \;\longrightarrow\; \bar\Psi(x)\,{\rm e}^{-i\xi(x)}, \tag{3.5}$$

[1] Note that the two-dimensional gravitational Einstein–Cartan action [7], based on the Ricci scalar defined in terms of the connection (2.10), gives the field equation $\partial^2\varphi = 0$, which is trivially solved by the configuration (2.7). The situation in four dimensions is less satisfactory, as will be discussed in Section 6.

supplemented by the usual transformations of the gauge field $A_\mu(x)$. For the basis of gamma matrices (3.1), the two (independent) Dirac spinors can be decomposed into four one-component Weyl spinors:

$$\Psi(x) \equiv \begin{pmatrix} \psi_R(x) \\ \psi_L(x) \end{pmatrix} \;, \quad \bar\Psi(x) \equiv \begin{pmatrix} \bar\psi_R(x) \; \bar\psi_L(x) \end{pmatrix} \;, \tag{3.6}$$

where $(1 \mp \gamma_{\mathrm{S}})/2$ projects on the left- and right-moving subspaces of solution space.

Throughout this paper, we consider only topologically trivial gauge potentials $A_\mu(x)$. We therefore take the $U(1)$ gauge potential $A_\mu(x)$ to be periodic in x^1 and x^2, with period L. The spinors are allowed to have either periodic or antiperiodic boundary conditions:

$$\begin{aligned} \Psi(x^1+L,x^2) &= -\,\mathrm{e}^{2\pi i\theta_1}\;\; \Psi(x^1,x^2) \;, \\ \Psi(x^1,x^2+L) &= -\,\mathrm{e}^{2\pi i\theta_2}\;\; \Psi(x^1,x^2) \;. \end{aligned} \tag{3.7}$$

(The adjoint spinors $\bar\Psi(x)$ obey the same boundary conditions.) The variables $\theta_1, \theta_2 \in \{0, 1/2\}$ then fix the spinor boundary conditions, with $(\theta_1,\theta_2) = (1/2, 1/2)$ corresponding to doubly-periodic boundary conditions and $(\theta_1,\theta_2) = (0,0)$ to doubly-antiperiodic boundary conditions. Mixed spinor boundary conditions correspond to $(\theta_1,\theta_2) = (1/2,0)$ or $(0,1/2)$. The four possible combinations of (θ_1,θ_2) are said to define the four spin structures over the torus.

For the general zweibeins (2.4), the Lagrangian (3.2) becomes

$$\mathcal{L}[\bar\Psi,\Psi,A,e] = (i/2)\,\bar\Psi\,\hat\gamma^\mu D_\mu \Psi + \mathrm{H.c.}\;, \tag{3.8}$$

with space-dependent gamma matrices

$$\hat\gamma^\mu(x) \;\equiv\; e^\mu_a(x)\,\gamma^a \tag{3.9}$$

and covariant derivatives

$$D_\mu \Psi \;\equiv\; (\partial_\mu + iA_\mu + i\Omega_\mu)\Psi \;. \tag{3.10}$$

The Lagrangian (3.8) is invariant under gauged $SO(2)$ transformations (3.4) due to the presence of the spin connection [8,9]

$$\begin{aligned} \Omega_\mu(x) \equiv \hat\Gamma^{ab}_\mu(x)\,\sigma_{ab}/2 \;&=\; e^a_{\nu,\mu} e^{b\nu}\,\sigma_{ab}/2 \;=\; -\gamma_{\mathrm{S}}\,\partial_\mu\varphi(x)/2\;, \\ \sigma_{ab} \equiv i\,[\gamma_a,\gamma_b]/4\;, \end{aligned} \tag{3.11}$$

provided Eq. (2.11) is used for the transformations. Here, the generator of $SO(2)$ transformations is given by $\sigma_{12} = -\sigma_{21} = \gamma_{\mathrm{S}}/2$, for the representation of gamma matrices chosen.

It is now possible to rewrite the Lagrangian (3.8) as follows:

$$\mathcal{L}[\bar{\Psi}, \Psi, A, e] = \mathcal{L}\left[\bar{\Psi}\, \mathrm{e}^{i\varphi\gamma_5/2}, \mathrm{e}^{-i\varphi\gamma_5/2}\, \Psi, A, \mathbb{1}\right] , \tag{3.12}$$

for zweibeins $e^a_\mu(x)$ and parameter function $\varphi(x)$ given by Eqs. (2.4) and (2.6), respectively. The effects of the nontrivial zweibeins (2.4) can therefore be absorbed by the simple spinor redefinition

$$\Psi'(x) \;\equiv\; \mathrm{e}^{-i\varphi(x)\gamma_5/2}\; \Psi(x)\,, \quad \bar{\Psi}'(x) \;\equiv\; \bar{\Psi}(x)\; \mathrm{e}^{i\varphi(x)\gamma_5/2}\;. \tag{3.13}$$

Obviously, this field redefinition changes the spinor boundary conditions according to the values of m and n in the function $\varphi(x)$; see Eqs. (2.6) and (2.7). Note, however, that this field redefinition is not a proper $SO(2)$ spinor redefinition on $T^2[i]$, in the sense that contractible loops of spinor rotations need not correspond to contractible loops of coordinate rotations (see Ref. [9] for further details). Physical consequences of changed spinor boundary conditions are, for example, the difference of the vacuum energy density [10] and the occurrence of the CPT anomaly (see Section 5).

4 Chiral determinant with torsion

In this section, we express the two-dimensional chiral determinant with torsion in terms of the standard chiral determinant without torsion. This can be done by use of the identity (3.12) and field redefinition (3.13).

The chiral determinant with torsion is then given by the following path integral:

$$D^{\{\theta_1,\theta_2\}}[A, e] = \int \left[\mathcal{D}\bar{\psi}_R\, \mathcal{D}\psi_L\right]_{(\theta_1,\theta_2)} \exp\left\{-S\left[\bar{\psi}_R\; \mathrm{e}^{i\varphi/2}, \mathrm{e}^{i\varphi/2}\;\psi_L, A\right]\right\} , \tag{4.1}$$

in terms of the standard action for the one-component Weyl spinor $\psi_L(x)$ and its conjugate $\bar{\psi}_R(x)$,

$$S[\bar{\psi}_R, \psi_L, A] = \int_{T^2[i]} \mathrm{d}^2x\; \bar{\psi}_R(x)\, \sigma^\mu \Big(\partial_\mu + i A_\mu(x)\Big)\, \psi_L(x)\,, \tag{4.2}$$

with $\sigma^\mu \equiv (1, i)$. The parameters θ_1 and θ_2 in Eq. (4.1) denote the spinor boundary conditions for the compact dimensions; see Eq. (3.7). Recall also that the zweibeins $e^a_\mu(x)$ are given by Eqs. (2.4)–(2.7) and the corresponding torsion tensor by Eq. (2.13).

The easiest way to calculate the chiral determinant with torsion is to perform the field redefinition (3.13):

$$D^{\{\theta_1,\theta_2\}}[A, e] = \int \left[\mathcal{D}(\bar{\psi}'_R\; \mathrm{e}^{-i\varphi/2})\, \mathcal{D}(\mathrm{e}^{-i\varphi/2}\;\psi'_L)\right]_{(\theta'_1,\theta'_2)} \exp\left\{-S[\bar{\psi}'_R, \psi'_L, A]\right\} , \tag{4.3}$$

where θ_1' and θ_2' indicate the boundary conditions of the transformed spinor fields (see below). Now, we only need to compute the relevant Jacobians and the next subsection reviews a convenient method.

4.1 Jacobians for infinitesimal phase transformations

We propose to use Fujikawa's method [13] to compute the Jacobians of the spinor redefinition (3.13) for the case of an infinitesimal phase $\varphi(x) = \alpha(x)$. Note, however, that our spinor redefinition is *not* a chiral transformation, as was the case in Fujikawa's original calculation.

The relevant Hermitian Dirac operator $(i\not{D}) = (i\not{D})^\dagger$ is given by

$$i\not{D} \;\equiv\; i\hat{\gamma}^\mu(x)D_\mu \;=\; i\,\mathrm{e}^{i\varphi\gamma_5/2}\;\gamma^a\delta_a^\mu(\partial_\mu + iA_\mu)\;\mathrm{e}^{-i\varphi\gamma_5/2} \;\equiv\; \begin{pmatrix} 0 & i\not{d}_R \\ i\not{d}_L & 0 \end{pmatrix}\,, \tag{4.4}$$

with

$$i\not{d}_R \;\equiv\; (i\not{d}_L)^\dagger \;\equiv\; \mathrm{e}^{i\varphi/2}\;i\sigma^\mu(\partial_\mu + iA_\mu)\;\mathrm{e}^{i\varphi/2}\;. \tag{4.5}$$

Since we intend to compute the Jacobians for the left- and right-moving chiral fermions separately, we can work with the following Hamiltonian:

$$H \equiv (i\not{D})(i\not{D})^\dagger = \begin{pmatrix} -\not{d}_R\not{d}_L & 0 \\ 0 & -\not{d}_L\not{d}_R \end{pmatrix} \equiv \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}\,, \tag{4.6}$$

which has the advantage of being Hermitian for each chirality separately, $H_\pm = (H_\pm)^\dagger$. Explicitly, its components are given by

$$H_\pm = -D_\pm^\mu D_{\pm\,\mu} \mp F\,, \tag{4.7}$$

with the further definitions

$$D_{\pm\mu} \;\equiv\; \partial_\mu + iA_\mu \mp i\partial_\mu\varphi/2\,, \quad F \;\equiv\; \partial_1 A_2 - \partial_2 A_1\,. \tag{4.8}$$

Following Ref. [13], we introduce normalized eigenfunctions of $H_\pm$,

$$H_\pm\,\phi_{\pm,k}(x) \;=\; \lambda_k^2\,\phi_{\pm,k}(x)\,, \quad \int_{T^2[i]} \mathrm{d}^2x\;\phi_{\pm,k}^\dagger(x)\,\phi_{\pm,l}(x) \;=\; \delta_{kl}\,, \tag{4.9}$$

for $k, l \in \mathbb{Z}$. The four independent Weyl spinors are then expanded as follows:

$$\begin{aligned} \psi_R(x) &= \sum_k a_k\,\phi_{+,k}(x)\,, \qquad & \bar{\psi}_R(x) &= \sum_k \overline{a}_k\,\phi_{+,k}^\dagger\,, \\ \psi_L(x) &= \sum_k b_k\,\phi_{-,k}(x)\,, \qquad & \bar{\psi}_L(x) &= \sum_k \overline{b}_k\,\phi_{-,k}^\dagger\,, \end{aligned} \tag{4.10}$$

with Grassmann numbers $a_k, \bar{a}_k, b_k, \bar{b}_k$. Note that the eigenfunctions $\phi_{\pm,k}(x)$ have been assigned to the Weyl spinors in order to diagonalize H:

$$< \bar{\Psi}|H|\Psi > = \sum_k \lambda_k^2 \, (\bar{a}_k \, a_k + \bar{b}_k \, b_k) \; . \tag{4.11}$$

For the field redefinition used in Eq. (4.3) and with the definition

$$\psi'_R(x) \equiv \sum_k a'_k \phi_{+,k}(x) \; , \tag{4.12}$$

the Grassmann variable a_k changes to

$$a'_k = \sum_l \left(\int_{T^2[i]} \mathrm{d}^2 x \; \phi^\dagger_{+,k}(x) \; \mathrm{e}^{-i\varphi/2} \; \phi_{+,l}(x) \right) a_l \equiv \sum_l V_{+,kl}[-\varphi/2] \; a_l \; . \tag{4.13}$$

The changes for the Grassmann variables $\bar{a}_k, b_k$ and $\bar{b}_k$ are analogous, but with matrices $V_+^{\mathrm{T}}[\varphi/2]$, $V_-[\varphi/2]$ and $V_-^{\mathrm{T}}[-\varphi/2]$ replacing $V_+[-\varphi/2]$ in Eq. (4.13). The superscript T indicates the transpose of the matrix.

Formally, the functional measure can be written as a product over the differentials $\mathrm{d}\bar{a}_k$ and $\mathrm{d}b_l$:

$$\mathcal{D}\bar{\psi}_R \, \mathcal{D}\psi_L \equiv \prod_{k\in\mathbb{Z}} \mathrm{d}\bar{a}_k \prod_{l\in\mathbb{Z}} \mathrm{d}b_l \; . \tag{4.14}$$

Under a spinor redefinition, the change of the functional measure is then given by the corresponding Jacobians,

$$\mathcal{D}\bar{\psi}'_R \, \mathcal{D}\psi'_L \;=\; \bar{J}_R \, J_L \, \mathcal{D}\bar{\psi}_R \, \mathcal{D}\psi_L \; , \tag{4.15}$$

with

$$\bar{J}_R \;\equiv\; \left(\det V_+^{\mathrm{T}}[\varphi/2]\right)^{-1} , \qquad J_L \;\equiv\; \left(\det V_-[\varphi/2]\right)^{-1} . \tag{4.16}$$

For the moment, the Jacobians in Eq. (4.15) are only considered as formal expressions.

The regularized determinant arising from a phase transformation (3.13) with an infinitesimal parameter $\varphi(x) = \alpha(x)$ can be calculated using the plane-wave method of Ref. [13]. The result is

$$\det V_\pm[\alpha] \;=\; \exp\left\{ i \, \mathcal{A}_\pm[\alpha, M] \right\} \; , \tag{4.17}$$

with

$$\mathcal{A}_\pm[\alpha, M] \;\equiv\; \frac{M^2}{4\pi} \int_{T^2[i]} \mathrm{d}^2 x \; \alpha(x) \; \mathrm{e}^{\pm F(x)/M^2} \; , \tag{4.18}$$

where the regulator mass M is to be taken to infinity at the end of the calculation. Equation (4.17) will be adopted as the proper definition of the determinant of the matrices $V_\pm[\alpha]$ for infinitesimal $\alpha(x)$. For later convenience, we establish two further identities:

$$\det V_{\pm}[\alpha]\,\det V_{\pm}[\beta] = \det V_{\pm}[\alpha+\beta]\;, \tag{4.19}$$

and

$$\det V_{\pm}^{\mathrm{T}}[\alpha] = \det V_{\pm}^{*}[-\alpha] = \Big(\det V_{\pm}[-\alpha]\Big)^{*} = \det V_{\pm}[\alpha]\;, \tag{4.20}$$

for infinitesimal $\alpha(x)$ and $\beta(x)$.

4.2 Chiral determinant

The method used in the previous subsection holds for infinitesimal phase transformations. Here, we simply *define* the determinant of a finite phase transformation (3.13) to be

$$\det V_{\pm}[\varphi/2] \equiv \lim_{N\to\infty}\Big(\det V_{\pm}[\varphi/(2N)]\Big)^{N}\;. \tag{4.21}$$

For topologically trivial functions $\varphi(x) = \omega(x)$ as given in Eq. (2.6), this definition is unproblematic. For topologically nontrivial functions $\varphi(x) = \chi(x)$ as given in Eq. (2.7), on the other hand, the result turns out to break translation invariance; cf. Eq. (2.5). The corresponding phase factor is, nevertheless, well-behaved for the appropriate limit $M \to \infty$, as will become clear shortly.

From Eqs. (4.17) and (4.21), the combined regularized Jacobian (4.15) for a left-moving fermion and its conjugate is found to be given by

$$\bar{J}_R\, J_L = \Big(\det V_{+}^{\mathrm{T}}[\varphi/2]\Big)^{-1}\Big(\det V_{-}[\varphi/2]\Big)^{-1} = \exp\left\{-iW[\varphi,F,M]\right\}\;, \tag{4.22}$$

with

$$W[\varphi,F,M] \;\equiv\; \frac{M^2}{4\pi}\int_{T^2[i]} \mathrm{d}^2x\;\varphi(x)\,\cosh\Big(F(x)/M^2\Big)\;, \tag{4.23}$$

and F as defined in Eq. (4.8). For the topologically nontrivial part $\chi(x) = (2\pi/L)\,(m\,x^1 + n\,x^2)$ of $\varphi(x)$, the corresponding phase factor (4.22) approaches 1 for $M^2 = 8\pi N/L^2$ with integer $N \to \infty$. The remaining (translation-invariant) phase factor depends only on the topologically trivial part $\omega(x)$ of $\varphi(x)$.

We can now express the chiral determinant (4.1) for nontrivial zweibeins ($e^a_\mu \neq 1\!\!1$) in terms of the chiral determinant for trivial zweibeins ($e^a_\mu = 1\!\!1$):

$$D^{\{\theta_1,\theta_2\}}[A,e] = \exp\left\{iW[\varphi,F,M]\right\}\;D^{\{\theta_1',\theta_2'\}}[A,1\!\!1]\;, \tag{4.24}$$

with the definitions

$$2\,\theta_1' \;\equiv\; (2\,\theta_1 + m)\ \mathrm{mod}\ 2\;, \qquad 2\,\theta_2' \;\equiv\; (2\,\theta_2 + n)\ \mathrm{mod}\ 2 \tag{4.25}$$

and the understanding that M has to be taken to infinity in the way discussed in the previous paragraph. In fact, the regulator dependence drops out in the limit $M \to \infty$ for the physically relevant ratio of chiral determinants:

$$\frac{D^{\{\theta_1,\theta_2\}}[A,e]}{D^{\{\theta_1,\theta_2\}}[B,e]} = \frac{D^{\{\theta_1',\theta_2'\}}[A,1\!\!1]}{D^{\{\theta_1',\theta_2'\}}[B,1\!\!1]}\;, \tag{4.26}$$

with modified spinor boundary conditions given by the parameters θ_1' and θ_2' of Eq. (4.25). Here, B is considered to be a fixed reference field, for example $B_1(x) = B_2(x) = (2\pi/L)/\sqrt{2}$ [this particular choice is motivated by Eq. (4.32) below]. Equation (4.26) is the main result of this section.

Two remarks on the result (4.26) are in order. The first remark is that, in the end, only the topologically nontrivial part $\chi(x)$ of $\varphi(x)$ contributes to this ratio of chiral determinants, i.e. the dependence on the function $\omega(x)$ from Eq. (2.6) drops out. The second remark is that the torsion does not affect the translation invariance of the normalized Euclidean effective action (defined as minus the logarithm of the chiral determinant), because the right-hand side of Eq. (4.26) is translation-invariant by construction [6].

4.3 Heuristic argument

We now present a heuristic argument [5] for the change of the chiral determinant due to the presence of torsion, restricting ourselves to the case of a constant gauge potential

$$A_\mu = (2\pi/L)\, h_\mu \tag{4.27}$$

and constant torsion tensor (2.13) determined by

$$\varphi(x) = \chi(x) = (2\pi/L)\,(m\, x^1 + n\, x^2)\;. \tag{4.28}$$

The path integral (4.1) to be calculated is then

$$D^{(\theta_1,\theta_2)}[h_\mu, m, n] \;\equiv\; \int \Big[\mathcal{D}\bar\psi_R\, \mathcal{D}\psi_L\Big]_{(\theta_1,\theta_2)} \exp\Big\{-\mathcal{S}[\bar\psi_R, \psi_L, h_\mu, m, n]\Big\}\;, \tag{4.29}$$

with the simplified action

$$\mathcal{S}[\bar\psi_R, \psi_L, h_\mu, m, n] = \int_{T^2[i]} \mathrm{d}^2x\; \bar\psi_R(x)\; \mathrm{e}^{i\chi(x)/2}\, \sigma^\mu(\partial_\mu + i\, 2\pi h_\mu/L)\, \mathrm{e}^{i\chi(x)/2}\, \psi_L(x)\;. \tag{4.30}$$

The one-component spinors ψ_L and $\bar\psi_R$ with boundary conditions determined by θ_1 and θ_2 can be expressed in a Fourier basis as follows:

$$\begin{aligned}
\psi_L &\equiv \sum_{p_1,p_2\in\mathbb{Z}} b_{p_1p_2}\,\exp\left\{+\frac{2\pi i}{L}\left(\left(p_1+\tfrac{1}{2}+\theta_1\right)x^1+\left(p_2+\tfrac{1}{2}+\theta_2\right)x^2\right)\right\}\,, \\
\bar{\psi}_R &\equiv \sum_{q_1,q_2\in\mathbb{Z}} \bar{a}_{q_1q_2}\,\exp\left\{-\frac{2\pi i}{L}\left(\left(q_1+\tfrac{1}{2}+\theta_1\right)x^1+\left(q_2+\tfrac{1}{2}+\theta_2\right)x^2\right)\right\}\,,
\end{aligned} \tag{4.31}$$

with Grassmann variables $b_{p_1p_2}$ and $\bar{a}_{q_1q_2}$. The measure of the path integral (4.29) can be written as in Eq. (4.14), but with k and l replaced by pairs of integers (q_1, q_2) and (p_1, p_2).

Using this plane wave decomposition, the path integral of Eq. (4.29) is formally given by the following infinite product:

$$\prod_{p_1,p_2\in\mathbb{Z}}\Big(p_1+1/2+\theta_1+m/2+h_1+i\big(p_2+1/2+\theta_2+n/2+h_2\big)\Big)\,, \tag{4.32}$$

up to a constant overall factor. Although this expression needs regularization, we can already infer that the introduction of torsion only changes the spinor boundary conditions. Namely, $m/2$ appears together with θ_1 and $n/2$ with θ_2. Hence, the chiral determinant of a $U(1)$ gauge theory on $T^2[i]$, with torsion determined by $\varphi(x) = \chi(x)$ and with constant gauge potentials, is proportional to the chiral determinant of the theory without torsion and new spinor boundary conditions θ_1' and θ_2' given by Eq. (4.25). This explains the result found in the previous subsection, at least for the particular gauge potentials (4.27) and torsion parameter function (4.28).

In Appendix A, we discuss the zeta-function regularization of a product similar to the one of Eq. (4.32), which occurs for the vector-like $U(1)$ gauge theory. Again, the spinor boundary conditions are found to be changed according to Eq. (4.25).

5 CPT anomaly with torsion

The calculation of Section 4.2 has shown how to relate the chiral determinant with torsion to the standard chiral determinant without torsion. The result (4.26) demonstrates that the introduction of the topologically nontrivial zweibeins (2.4)–(2.7) can effectively change the spinor boundary conditions according to the constants m and n appearing in the torsion tensor (2.13).

It has been shown in Ref. [5] that the CPT anomaly of chiral $U(1)$ gauge theory on the torus without torsion appears only for *doubly-periodic* spinor boundary conditions, at least for a particular class of regularizations that respect modular invariance. Under a CPT transformation of the gauge potential,

$$A_\mu(x) \quad\longrightarrow\quad A_\mu^{\mathrm{CPT}}(x) \,\equiv\, -A_\mu(-x)\,, \tag{5.1}$$

the CPT anomaly on $T^2[i]$ manifests itself as a sign change of the chiral determinant,

$$D^{\{1/2,1/2\}}[A^{\mathrm{CPT}}, \mathbb{1}] = -D^{\{1/2,1/2\}}[A, \mathbb{1}] \;, \tag{5.2}$$

for the case of trivial zweibeins ($e_\mu^a = \delta_\mu^a \equiv \mathbb{1}$).

In this paper, we only consider a *single* charged chiral fermion. The chiral $U(1)$ gauge anomaly needs, however, to be cancelled between different species of chiral fermions. There is then the CPT anomaly (5.2), as long as the *total* number N_F of charged chiral fermions is *odd*. Note that even if there is no net CPT anomaly (that is, for N_F even), there may still be Lorentz noninvariance; see Ref. [5] for further details.

A consequence of our result (4.24) is that the CPT anomaly can be moved to different spinor boundary conditions by choosing appropriate zweibeins. (The additional phase factor (4.22) is CPT-even.) For example, we can now have the CPT anomaly for *doubly-antiperiodic* spinor boundary conditions,

$$D^{\{0,0\}}[A^{\mathrm{CPT}}, \bar{e}] = -D^{\{0,0\}}[A, \bar{e}] \;, \tag{5.3}$$

provided the zweibeins $\bar{e}_\mu^a(x)$ have topologically nontrivial torsion determined by *odd* constants m and n in Eq. (2.7).

According to the heuristic argument of Section 4.3, the chiral determinant is formally proportional to the infinite product (4.32). It is then easy to understand that there is a CPT anomaly if both $2\,\theta_1 + m$ and $2\,\theta_2 + n$ are odd. Start, for example, with purely antiperiodic spinor boundary conditions. Now, the introduction of torsion with odd m and odd n formally leads to the infinite product

$$\prod_{p_1', p_2' \in \mathbb{Z}} (p_1' + h_1 + i p_2' + i h_2) \;, \tag{5.4}$$

which equals the chiral determinant of a torsionless theory with doubly-periodic spinor boundary conditions. Under a CPT transformation, $h_\mu \to -h_\mu$, the single factor with $p_1' = p_2' = 0$ is CPT-odd, whereas the other factors combine into a CPT-even product (which still needs to be regularized). Hence, for torsion determined by odd m and odd n, the CPT anomaly has been moved to the doubly-antiperiodic spin structure. Analogous arguments apply to the other cases.

To summarize, the CPT anomaly for chiral $U(1)$ gauge theory with an odd number of charged chiral fermions on the torus $T^2[i]$ occurs only if the following conditions hold:

$$(2\,\theta_1 + m) = 1 \bmod 2 \;, \qquad (2\,\theta_2 + n) = 1 \bmod 2 \;, \tag{5.5}$$

at least for the regularizations used in Refs. [5,6]. Here, θ_1 and θ_2 determine the fermion boundary conditions (3.7) and m and n are constants appearing in the topologically nontrivial zweibeins (2.4)–(2.7).

6 Discussion

For two-dimensional chiral $U(1)$ gauge theory, we have presented in this paper a calculation of the chiral determinant on the torus $T^2[i]$ with nontrivial zweibeins corresponding to the presence of torsion on the spacetime manifold.

In Section 4.2 we have shown how to relate the chiral determinant with torsion to the chiral determinant without torsion by the spinor redefinition (3.13). The Jacobian of this redefinition turns out to be a gauge-invariant and CPT-even phase factor (4.22), which cancels in the ratio of the chiral determinants (4.26). The chiral determinant with torsion is then proportional to the chiral determinant without torsion, but with spinor boundary conditions changed according to Eq. (4.25). This result was confirmed in Section 4.3 by a heuristic argument for a particular choice of gauge potentials and torsion. Hence, the CPT anomaly can effectively be moved from one spin structure to another by choosing topologically nontrivial zweibeins.

The calculations of the present paper demonstrate that the two-dimensional CPT anomaly is a genuine effect for chiral $U(1)$ gauge theory on the torus. The CPT anomaly can be moved around between the different spin structures by taking appropriate zweibeins; see Eq. (5.5). But the anomaly cannot be removed completely from the general theory, which is a sum over all spin structures [9].

Alternatively, we can fix the spinor boundary conditions (for example, antiperiodic boundary conditions (3.7) with $\theta_1 = \theta_2 = 0$) and consider different classes ($m, n \in \mathbb{Z}$) of zweibeins (2.4)–(2.7), with the corresponding torsion tensor (2.13). It is quite remarkable that topologically nontrivial spacetime torsion, which is not visible in the metric and the curvature, can affect the local physics of chiral $U(1)$ gauge theory in the same way as different spinor boundary conditions would do for the case of trivial zweibeins (i.e. vanishing torsion).

But, as we have shown in Section 2, there are more consequences of torsion than just modified spinor boundary conditions. There is, for example, the fact that parallelograms do not close and that extremal and autoparallel curves need not coincide if torsion is present. Moreover, these local manifestations of torsion can already occur for topologically trivial zweibeins with $m = n = 0$ in Eq. (2.7), whereas the boundary-like effects require topologically nontrivial zweibeins ($m \neq 0$ or $n \neq 0$). Still, topologically nontrivial zweibeins may have a special status, as discussed in Section 2.4.

In this paper, we have focused on two-dimensional chiral $U(1)$ gauge theory, because the chiral determinant is known exactly [6]. But our discussion of the effects of torsion can be readily extended to higher-dimensional orientable manifolds. Consider, for example, the flat spacetime manifold $\mathbb{R}^2 \times T^2$, with noncompact coordinates $x^0, x^3 \in \mathbb{R}$ and periodic coordinates $x^1, x^2 \in [0, L]$. The zweibeins (2.4)–(2.7) can then be embedded in the vierbeins $e^A_M(x)$ as follows:

$$
e^A_M(x) = \begin{cases} e^a_\mu(x)\,, & \text{for } A = a \in \{1,2\}\,,\ M = \mu \in \{1,2\}\,, \\ \delta^A_M\,, & \text{otherwise}\,, \end{cases} \tag{6.1}
$$

with indices A and M running over 0, 1, 2, 3. The $SO(2)$ angle $\varphi(x)$ which enters the nontrivial zweibein part of Eq. (6.1) is taken to be purely topological, namely $\varphi(x) = \chi(x^1, x^2)$ with χ as given by Eq. (2.7).

The metric resulting from the vierbeins (6.1) is flat, $g_{MN}(x) = \delta_{MN}$. Note, however, that the vierbeins (6.1) with $\chi \neq 0$ do not solve the vacuum field equations of the Einstein–Cartan theory [7], in contrast with the situation in two dimensions as mentioned in Footnote 1. These vierbeins could play the role of prior-geometric fields (that is, non-dynamical fields); see, for example, the discussion in Ref. [14]. For the moment, let us just continue with the particular vierbeins (6.1), regardless of their origin.

In order to be specific, we also take the particular chiral gauge theory corresponding to the well-known $SO(10)$ grand-unified theory with three families of quarks and leptons. The CPT anomaly now gives two Chern–Simons-like terms [1] for the hypercharge $U(1)$ gauge field in the effective action, again provided condition (5.5) holds. [2] These Chern–Simons-like terms affect the local physics, making the propagation of photons birefringent for example [19,20]. This last phenomenon is all the more remarkable, since at tree level torsion does not couple to the photons because of gauge invariance [7].

To summarize, a topological component of a (prior-geometric) torsion field could modify the propagation of photons via the CPT anomaly. Inversely, the propagation of photons could perhaps inform us about the structure of spacetime.

A Dirac determinant for two-dimensional $U(1)$ gauge theory with torsion

In this appendix, we evaluate the regularized fermionic determinant of a two-dimensional $U(1)$ gauge theory with a single Dirac fermion, i.e. the vector-like $U(1)$ gauge theory. The spacetime manifold considered is the torus $T^2[i]$ shown in Fig. 1. In order to simplify the calculation, we take, as in Section 4.3, constant gauge potentials $A_\mu(x) = (2\pi/L)\,h_\mu$, with $h_\mu \in \mathbb{R}$, and constant torsion

[2] It has been claimed in Ref. [15] that a cosmic torsion field $S_\mu(x) \equiv \epsilon_{\mu\nu\rho\sigma}\, T^{\nu\rho\sigma}(x)$ could also generate a Chern–Simons-like term for the photon field via the quantum effects of Dirac fermions coupled to both photon and torsion fields. This radiatively induced Chern–Simons-like term must, however, vanish according to an argument based on gauge invariance and analyticity [16,17] or, for constant S_μ in particular, causality [18]. Note that the CPT anomaly necessarily involves chiral (Weyl) fermions, not Dirac fermions [1,5].

tensor components (2.13) determined by $\varphi(x) = \chi(x) = (2\pi/L)\,(m\,x^1 + n\,x^2)$, with $m, n \in \mathbb{Z}$.

For a single Dirac fermion, the fermionic determinant (exponent of minus the Euclidean effective action) is the infinite product of the following eigenvalues:

$$\lambda_{p_1 p_2} \equiv (2\pi/L)^2 \left((p_1 + a_1)^2 + (p_2 + a_2)^2 \right) , \tag{A.1}$$

with quantum numbers $p_1, p_2 \in \mathbb{Z}$ and (noninteger) parameters

$$a_1 \;\equiv\; 1/2 + \theta_1 + m/2 + h_1 \;, \qquad a_2 \;\equiv\; 1/2 + \theta_2 + n/2 + h_2 \;. \tag{A.2}$$

Compare with the product (4.32) for a single chiral fermion.

This product of eigenvalues can be regularized using zeta-function techniques (see Refs. [21,22] and references therein). For $\vec{a} \equiv (a_1, a_2)$, we define the regularized Dirac determinant as follows:

$$D^{\{\theta_1,\theta_2\}}_{\text{Dirac}}[h_\mu, m, n] \;\equiv\; \exp\left\{ -\zeta_E'(s, \vec{a})\Big|_{s=0} \right\} \;, \tag{A.3}$$

with the generalized Epstein zeta function

$$\zeta_E(s, \vec{a}) \;\equiv\; \sum_{p_1, p_2 \in \mathbb{Z}} \left((p_i + a_i) g^{ij} (p_j + a_j) \right)^{-s} \;, \tag{A.4}$$

for $g^{ij} \equiv (2\pi/L)^2\,\delta^{ij}$ and $\mathrm{Re}\,(s) > 1$. The prime in Eq. (A.3) denotes differentiation with respect to the variable s (which is set to 0 afterwards).

Our evaluation of the sum (A.4) essentially repeats the calculation of Ref. [22], to which the reader is referred for further details. In the rest of this appendix, g_{ij} will stand for the inverse of the matrix g^{ij} and we will set $p^i \equiv p_i$.

By writing the generalized Epstein zeta function (A.4) as a Mellin transform,

$$\zeta_E(s, \vec{a}) = \frac{1}{\Gamma(s)} \sum_{p_1, p_2 \in \mathbb{Z}} \int_0^\infty \mathrm{d}t\; t^{s-1} \exp\left\{ -t\,\lambda_{p_1 p_2} \right\} \;, \tag{A.5}$$

we can apply the generalized Poisson resummation formula,

$$\begin{aligned} &\sum_{p_1, p_2 \in \mathbb{Z}} \exp\left\{ -\pi (p_i + a_i) g^{ij} (p_j + a_j) \right\} = \\ &\sqrt{\det(g_{ij})} \sum_{p_1, p_2 \in \mathbb{Z}} \exp\left\{ -\pi\, p^i g_{ij}\, p^j + 2\pi i\, p^j a_j \right\} \;, \end{aligned} \tag{A.6}$$

to the integrand of Eq. (A.5). The result is given by

$$\zeta_E(s, \vec{a}) = \frac{\Gamma(1-s)}{\Gamma(s)}\, \pi^{s-1} \sqrt{\det(g_{ij})} \;{\sum_{p_1, p_2 \in \mathbb{Z}}}' (p^i g_{ij}\, p^j)^{s-1} \exp\left\{ 2\pi i\, p^j a_j \right\} \;, \tag{A.7}$$

with the prime on the sum indicating that the modes $p_i = 0$ are excluded, since they do not contribute for the region $\mathrm{Re}\,(s) > 1$ where the original sum

(A.4) is convergent. Analytic continuation to $s = 0$ then yields

$$\begin{aligned} \zeta_E(0,\vec{a}) &= 0 \,, \\ \zeta_E^{'}(0,\vec{a}) &= \pi^{-1}\,\sqrt{\det(g_{ij})}\,\sum_{p_1,p_2\in\mathbb{Z}}{}^{'}(p^i g_{ij}\,p^j)^{-1}\exp\left\{2\pi i\,p^j a_j\right\}\,. \end{aligned} \tag{A.8}$$

It is a remarkable fact [22] that one can express this $\zeta_E^{'}(0,\vec{a})$ in terms of the Riemann theta function and Dedekind eta function

$$\zeta_E^{'}(0,\vec{a}) = -\log\left|\frac{1}{\eta(\tau)}\,\vartheta\begin{bmatrix}1/2-a_1\\ 1/2+a_2\end{bmatrix}(0,\tau)\right|^2\,, \tag{A.9}$$

with modulus $\tau = i$ for the particular matrix $g^{ij} \propto \delta^{ij}$ of Eq. (A.4). Here, the Riemann theta function with characteristics a and b is defined as in Ref. [23],

$$\vartheta\begin{bmatrix}a\\ b\end{bmatrix}(z,\tau) \;\equiv\; \sum_{n\in\mathbb{Z}}\exp\left\{i\pi\tau(n+a)^2 + 2\pi i(n+a)(z+b)\right\}\,, \tag{A.10}$$

and the Dedekind eta function is given by

$$\eta(\tau) \;\equiv\; \mathrm{e}^{i\pi\tau/12}\,\prod_{m=1}^{\infty}(1-\mathrm{e}^{2\pi i\tau m})\,. \tag{A.11}$$

Note that the regularization method used has eliminated the L-dependence present in Eq. (A.1) and produced the result (A.9) which does not depend on L; cf. Refs. [21,22].

The theta functions (A.10) obey the following identity:

$$\vartheta\begin{bmatrix}a+N\\ b+M\end{bmatrix}(z,\tau) = \mathrm{e}^{2\pi i a M}\;\vartheta\begin{bmatrix}a\\ b\end{bmatrix}(z,\tau)\,, \tag{A.12}$$

for arbitrary integers N and M. In addition, there are some further properties for the special case of $z = 0$ and $\tau = i$, which allow us to write Eqs. (A.3) and (A.9) as

$$\log D^{\{\theta_1,\theta_2\}}_{\text{Dirac}}[h_\mu,m,n] = -\zeta_E^{'}(0,\vec{a}) = \log\left|\frac{1}{\eta(i)}\,\vartheta\begin{bmatrix}\theta_1'+h_1\\ \theta_2'+h_2\end{bmatrix}(0,i)\right|^2\,, \tag{A.13}$$

with

$$2\,\theta_1' \;\equiv\; (2\,\theta_1+m)\bmod 2\,,\qquad 2\,\theta_2' \;\equiv\; (2\,\theta_2+n)\bmod 2\,. \tag{A.14}$$

This shows that the effect of torsion (parameters m and n) for the regularized Dirac determinant can be entirely absorbed by a change of spinor boundary conditions, as given by Eq. (A.14). Note also that the identity (A.12) implies the gauge invariance of (A.13) under $h_\mu \to h_\mu + n_\mu$, for $n_\mu \in \mathbb{Z}$.

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