CPT violation: mechanism and phenomenology*

Frans R. Klinkhamer†
Institut für Theoretische Physik, Universität Karlsruhe, D–76128 Karlsruhe, Germany

Abstract

A new mechanism for T and CPT violation is reviewed, which relies on chiral fermions, gauge interactions and nontrivial spacetime topology. Also discussed are the possible effects on the propagation of electromagnetic waves *in vacuo*, in particular for the cosmic microwave background radiation.

I. INTRODUCTION

The CPT theorem [1–5] states that any local relativistic quantum field theory is invariant under the combined operation of charge conjugation (C), parity reflection (P) and time reversal (T). Very briefly, the main inputs are (cf. Ref. [4]):

- the Minkowski spacetime;
- the invariance under transformations of the proper orthochronous Lorentz group $\mathcal{L}_+^\uparrow$ and spacetime translations;
- the normal spin-statistics connection;
- the locality and hermiticity of the Hamiltonian.

An extensive discussion of this theorem and its consequences can be found in Refs. [6,7].

Here, we go further and ask the following question: *can* CPT invariance be violated at all in a physical theory and, if so, *is* it in the real world? It is clear that something "unusual" is required for this to be the case. Two possibilities, in particular, have been discussed in the literature.

First, there is *quantum gravity*, which may or may not lead to CPT violation; cf. Ref. [8]. The point is, of course, that Poincaré invariance does not hold in general. Still, a CPT theorem can be "proven" in the Euclidean formulation for asymptotically-flat spacetimes [9]. In the canonical formulation, on the other hand, there is an indication that certain

---

*to appear in: *Proceedings 7-th International Wigner Symposium*, edited by Y.S. Kim and M.E. Noz.

†Email address: frans.klinkhamer@physik.uni-karlsruhe.de KA–TP–28–2001, hep-th/0110135

semiclassical (weave) states could affect the Lorentz invariance of Maxwell theory at the Planck scale and break CPT invariance [10].

Second, there is *superstring theory*, which may or may not lead to CPT violation; cf. Ref. [11]. The point, now, is the (mild) nonlocality of the theory. There exists, however, no convincing calculation showing the necessary violation of CPT.

In this talk, we discuss a third possibility: *for certain spacetime topologies and classes of chiral gauge theories, Lorentz and CPT invariance are broken by quantum effects.* The main paper on this "CPT anomaly" is Ref. [12]. (The relation with earlier work on sphalerons, spectral flow, and anomalies, is explained in Ref. [13].) Further aspects of the CPT anomaly have been considered in Refs. [14–17] and a brief review has already appeared in Ref. [18].

The outline of the present write-up is as follows. In Sec. II, a realistic (?) example of the CPT anomaly is given and some general properties are emphasized. In Sec. III, the CPT anomaly is established for a class of exactly solvable two-dimensional models (specifics are relegated to Appendix A). In Sec. IV, the CPT anomaly is obtained perturbatively for a class of four-dimensional chiral gauge theories, which includes the example of Sec. II. The resulting modification of Maxwell theory is briefly discussed (the important issue of microcausality is dealt with in Appendix B). In Sec. V, certain phenomenological aspects of the modified Maxwell theory are mentioned, in particular as regards the propagation of light over large distances (e.g., the cosmic microwave background radiation). In Sec. VI, finally, some conclusions are drawn.

II. EXAMPLE AND GENERAL REMARKS

The CPT anomaly is best illustrated by a concrete example. Consider the spacetime manifold M with metric g given by

$$(\mathrm{M};\mathrm{g}) = (\mathbb{R}^3 \times S^1; \eta_{\mu\nu}^{\text{Minkowski}}) \,, \tag{1}$$

and coordinates

$$x^0 \equiv c\,t,\; x^1,\; x^2 \in \mathbb{R} \quad \text{and} \quad x^3 \in [0, L] \,. \tag{2}$$

Take also the chiral gauge field theory with group G and left-handed fermion representation R_L given by

$$(G\,; R_L) = (SO(10)\,; \mathbf{16} + \mathbf{16} + \mathbf{16}) \,, \tag{3}$$

which incorporates the Standard Model (SM) with three families of quarks and leptons. Then, quantum effects give *necessarily* CPT violation [12], with a typical mass scale

$$m_G \equiv \frac{\alpha_G\, \hbar}{L\, c} \sim 10^{-35}\, \mathrm{eV}/c^2 \, \left(\frac{\alpha_G}{0.01}\right) \left(\frac{2\; 10^{10}\, \mathrm{lyr}}{L}\right), \tag{4}$$

where $\alpha_G \equiv g^2/(4\pi)$ is defined in terms of the dimensionless $SO(10)$ gauge coupling constant g and L is the size of the compact dimension. As mentioned in the Introduction, this phenomenon has been called the CPT anomaly. Further discussion of this particular case will be postponed till Sec. IV. Here, we continue with some general remarks.

The CPT anomaly occurs for the case of

$$SO(10) + (\mathbf{16}_L)^3 \quad \text{over} \quad \mathbb{R}^3 \times S^1 \quad \text{or} \quad \mathbb{R} \times S^2 \times S^1 \,, \tag{5}$$

but not for

$$SO(10) + (\mathbf{16}_L)^3 \quad \text{over} \quad \mathbb{R} \times S^3 \,, \tag{6}$$

where the space manifold S^3 is simply connected. The CPT anomaly also does not occur for the case of

$$\text{QED} \quad \text{over} \quad \mathbb{R}^3 \times S^1 \,, \tag{7}$$

where QED stands for the vector-like gauge theory of photons and electrons (Quantum Electrodynamics), with $G = U(1)$ and $R_L = 1 + (-1)$. Hence, *both* topology and parity violation are crucial ingredients of the CPT anomaly. (The precise conditions for the occurrence of the CPT anomaly have been given in Ref. [12]; see also Sec. IV below.)

As regards the role of topology, the CPT anomaly resembles the Casimir effect, with the local properties of the vacuum depending on the boundary conditions; cf. Ref. [19]. Note that the actual topology of our universe is unknown [20], but theoretically there may be some constraints [21]. Interestingly, the modification of the local physics due to the CPT anomaly (see Sec. V) would allow for an indirect observation of the global spacetime structure.

Clearly, it is important to be sure of this surprising effect and to understand the mechanism. In the next section, we, therefore, turn to a relatively simple theory, the Abelian chiral gauge theory in two spacetime dimensions (2D). From now on, we put $\hbar = c = 1$, except when stated otherwise.

III. EXACT RESULT IN 2D

Consider chiral $U(1)$ gauge theory over the torus $T^2 \equiv S^1 \times S^1$, with a Euclidean metric $g_{\mu\nu}(x) = \delta_{\mu\nu} \equiv \text{diag}(1,1)$. In order to be specific, take the gauge-invariant model with five left-handed fermions of charges $(1,1,1,1,-2)$. Furthermore, impose doubly-periodic boundary conditions on the fermions. The corresponding spin structure is denoted by PP.

The effective action $\Gamma_{\text{PP}}^{1111\overline{2}}[a]$ for the $U(1)$ gauge field $a_\mu(x)$, which is defined by the functional integral

$$\exp\left(-\Gamma_{\text{PP}}^{1111\overline{2}}[a]\right) = \int \prod_{f=1}^{5} \left(\mathcal{D}\bar{\psi}_{Rf}\mathcal{D}\psi_{Lf}\right)_{\text{PP}} \, \exp\left(-S_{\text{Weyl}}^{T^2}\left[\bar{\psi}_{Rf}, \psi_{Lf}, a\right]\right) \,, \tag{8}$$

is known *exactly* (see Ref. [22] and references therein). Specifically, the effective action is given in terms of Riemann theta functions; see Appendix A for details.

It can now be checked explicitly that the CPT transformation

$$a_\mu(x) \;\rightarrow\; -a_\mu(-x) \tag{9}$$

does *not* leave the effective action invariant:

$$\Gamma_{\mathrm{PP}}^{1111\overline{2}}[a] \;\to\; \Gamma_{\mathrm{PP}}^{1111\overline{2}}[a] + \pi i \quad (\mathrm{mod}\; 2\pi i)\,. \tag{10}$$

This result [14], which can also be understood heuristically (see Appendix A), shows unambiguously the existence of a CPT anomaly in this particular two-dimensional chiral $U(1)$ gauge theory, the crucial ingredients being the doubly-periodic (PP) boundary conditions and the odd number (here, 5) of Weyl fermions.

IV. PERTURBATIVE RESULT IN 4D

Return to 3+1 dimensions. Take, again, the $SO(10)$ chiral gauge theory (3) with $N_{\mathrm{fam}} = 3$ and the cylindrical spacetime manifold $\mathrm{M} = \mathbb{R}^3 \times S^1$ with metric $g_{\mu\nu}(x) = \eta_{\mu\nu} \equiv \mathrm{diag}(-1,1,1,1)$. For other possibilities, see Sec. 5 of Ref. [12].

The effective action $\Gamma[A]$, for $A \in \mathrm{so}(10)$, is, of course, not known exactly. But the crucial term has been identified perturbatively:

$$\Gamma_{\mathrm{CS-like}}^{\mathbb{R}^3\times S^1}[A] = \int_{\mathbb{R}^3} \mathrm{d}x^0\mathrm{d}x^1\mathrm{d}x^2 \int_0^L \mathrm{d}x^3\; \frac{n\,\pi}{L}\,\omega_{\mathrm{CS}}[A_0(x), A_1(x), A_2(x)]\,, \tag{11}$$

with the Chern–Simons density

$$\omega_{\mathrm{CS}}[A_0, A_1, A_2] \equiv \frac{1}{16\,\pi^2}\,\epsilon^{3klm}\,\mathrm{tr}\Big(F_{kl}\,A_m - \tfrac{2}{3}\,A_k A_l A_m\Big), \tag{12}$$

in terms of the Yang–Mills field strength

$$F_{kl} \equiv \partial_k A_l - \partial_l A_k + A_k A_l - A_l A_k\,. \tag{13}$$

Here, the gauge field takes values in the Lie algebra, $A_m(x) \equiv A_m^a(x)\,T^a$, with normalization $\mathrm{tr}\,(T^aT^b) = (-1/2)\,\delta^{ab}$, and $\epsilon^{\kappa\lambda\mu\nu}$ is the completely antisymmetric Levi-Civita symbol, normalized by $\epsilon^{0123} = -1$. The Latin indices in Eq. (12) run over $0,1,2$, but the fields depend on *all* coordinates x^μ, for $\mu = 0,1,2,3$. Note that the effective action term (11) is called Chern–Simons-like, because a genuine topological Chern–Simons term exists only in an odd number of dimensions. At this point we can make three basic observations.

First, the local Chern–Simons-like term (11) is manifestly Lorentz noninvariant and CPT-odd, in contrast to the Yang–Mills action,

$$S_{\mathrm{YM}}^{\mathbb{R}^3\times S^1}[A] = \int_{\mathbb{R}^3} \mathrm{d}x^0\mathrm{d}x^1\mathrm{d}x^2 \int_0^L \mathrm{d}x^3\; \tfrac{1}{2}\,g^{-2}\,\mathrm{tr}\,\Big(\eta^{\kappa\mu}\,\eta^{\lambda\nu}\,F_{\kappa\lambda}(x)\,F_{\mu\nu}(x)\Big)\,. \tag{14}$$

More precisely, the Lorentz and CPT transformations considered are active transformations on fields of local support, as discussed in Sec. IV of Ref. [14]. In physical terms, the wave propagation from the action (14) is isotropic, whereas the term (11) makes the propagation anisotropic; see also Sec. V below. Moreover, both the quadratic and cubic terms in the Chern–Simons-like term (11) can be seen to be T-odd and C- and P-even. (That both terms transform in the same way under the discrete symmetries is consistent with the observation that both terms are needed to make the action (11) invariant under infinitesimal non-Abelian gauge transformations; cf. Sec. 3, p. 239, of Ref. [13].)

Second, the integer n in the effective action term (11) is a remnant of the ultraviolet regularization:

$$n \equiv \sum_{f=1}^{N_{\rm fam}} (2\,k_{0f}+1)\,, \quad k_{0f} \in \mathbb{Z}\,, \tag{15}$$

with $N_{\rm fam}=3$ for the case considered. Since the sum of an odd number of odd numbers is odd, one has

$$n \neq 0\,, \quad \text{for} \quad N_{\rm fam}=3\,, \tag{16}$$

and the anomalous term (11) is necessarily present in the effective action. For $N_{\rm fam}=3$, the regularization of Ref. [12] gives simply

$$n = (1-1+1)\,\Lambda_0\,/\,|\Lambda_0| = \pm 1\,, \tag{17}$$

with Λ_0 an ultraviolet Pauli–Villars cutoff for the x^3-independent modes of the fermionic fields contributing to the effective action. The effective action term (11) has, therefore, a rather weak dependence on the small-scale structure of the theory, as shown by the factor $\Lambda_0/|\Lambda_0|$ in Eq. (17). (This weak dependence on the ultraviolet cutoff was first seen in the so-called "parity" anomaly of three-dimensional gauge theories [23], which underlies the four-dimensional CPT anomaly discussed here [13].)

Third, for the $SO(10)$ theory with *three* identical irreducible representations (irreps), the CPT anomaly *must* occur [the integer n is odd and therefore nonzero; cf. Eqs. (15) and (16)]. For the Standard Model, the CPT anomaly *may or may not* occur, depending on the ultraviolet regularization. The reason is that the SM irreps come in *even* number (for example, four left-handed isodoublets per family), so that the integer n is not guaranteed to be nonzero [n is even]. For further details on this subtle point, see again Sec. 5 of Ref. [12].

Next, consider the electromagnetic $U(1)$ gauge field $a_\mu(x)$ embedded in the $SO(10)$ gauge field $A_\mu(x)$ and take $n=-1$. After the appropriate rescaling of $a_\mu(x)$, the effective action at low energies then contains the following local terms:

$$S_{\rm MCS}^{\mathbb{R}^3\times S^1}[a] = \int_{\mathbb{R}^3} {\rm d}x^0{\rm d}x^1{\rm d}x^2 \int_0^L {\rm d}x^3 \,\Big(\mathcal{L}_{\rm Maxwell}[a] + \mathcal{L}_{\rm CS-like}[a]\Big)\,, \tag{18}$$

$$\mathcal{L}_{\rm Maxwell}[a] = -\tfrac{1}{4}\,\eta^{\kappa\mu}\,\eta^{\lambda\nu}\,f_{\kappa\lambda}\,f_{\mu\nu}\,, \tag{19}$$

$$\mathcal{L}_{\rm CS-like}[a] = +\tfrac{1}{4}\,m\,\epsilon^{3\kappa\lambda\mu}\,f_{\kappa\lambda}\,a_\mu\,, \tag{20}$$

with the definitions

$$f_{\mu\nu} \equiv \partial_\mu a_\nu - \partial_\nu a_\mu\,, \quad m \sim \alpha/L\,, \quad \alpha \equiv e^2/(4\pi)\,. \tag{21}$$

The precise numerical factor in the definition of m depends on the details of the unification and the running of the coupling constant.

Now focus on the Maxwell–Chern–Simons (MCS) theory (18) *per se*. The action $S_{\rm MCS}^{\mathbb{R}^3\times S^1}$ is gauge invariant [24], provided the electric and magnetic fields in f_{kl} vanish fast enough as $(x^0)^2+(x^1)^2+(x^2)^2 \to \infty$. (This observation makes clear that the parameter m is not simply the mass of the photon [25], it affects the propagation in a different way; cf. Sec. V.)

On the other hand, there is known to be a close relation [5,6] between CPT invariance and microcausality, i.e., the commutativity of local observables with spacelike separations. How about causality in the CPT-violating MCS theory? Remarkably, microcausality (locality) can be established also in the MCS theory [16]. Details are given in Appendix B.

V. PHENOMENOLOGY: PROPAGATION OF LIGHT

The propagation of light in the Maxwell–Chern–Simons (MCS) theory (18) makes clear that C and P are conserved, but T *not*. An example is provided by the behavior of pulses of circularly polarized light, as will now be discussed briefly.

The dispersion relation for plane electromagnetic waves in the MCS theory is given by [16,24,26]

$$\omega_\pm^2 \equiv k_1^2+k_2^2+\left(q_3 \pm m/2\right)^2 \,, \quad q_3 \equiv \sqrt{k_3^2+m^2/4}\,, \tag{22}$$

where the suffix $\pm$ labels the two different modes. The phase and group velocities are readily calculated from this dispersion relation,

$$\vec{v}_{\rm ph}^{\,\pm} \equiv (k_1,k_2,k_3)\,\frac{\omega_\pm}{|\vec{k}\,|^2}\,, \quad \vec{v}_{\rm g}^{\,\pm} \equiv \left(\frac{\partial}{\partial k_1},\frac{\partial}{\partial k_2},\frac{\partial}{\partial k_3}\right)\omega_\pm\,. \tag{23}$$

The magnitudes of the group velocities turn out to be given by

$$|\vec{v}_{\rm g}^{\,\pm}(k_1,k_2,k_3)|^2 = \frac{k_1^2+k_2^2+\left(q_3\pm m/2\right)^2 k_3^2/q_3^2}{k_1^2+k_2^2+\left(q_3\pm m/2\right)^2} \leq 1\,, \tag{24}$$

with equality for $m=0$ (recall $c\equiv 1$). Strictly speaking, the wave vector component k_3 is discrete ($k_3 = 2\pi n_3/L$, with $n_3 \in \mathbb{Z}$), but here k_3 is considered to be essentially continuous.

For our purpose, it is necessary to give the electric and magnetic fields of the two modes in detail. As long as the propagation of the plane wave is not exactly along the x^3 axis, the radiative electric field can be expanded as follows ($\Re$ denotes taking the real part):

$$\vec{E}_\pm(\vec{x},t) = \Re\left(c_1^\pm\left(\hat{e}_3-(\hat{e}_3\cdot\hat{k})\,\hat{k}\right)+c_2^\pm\left(\hat{e}_3\times\hat{k}\right)+c_3^\pm\,\hat{k}\,\sin\theta\right)\,\exp\left[\,i(\vec{k}\cdot\vec{x}-\omega_\pm\,t)\,\right]\,, \tag{25}$$

with unit vector $\hat{e}_3$ in the compact x^3 direction, unit vector $\hat{k}$ corresponding to the wave vector $\vec{k}$, polar angle θ of the wave vector (so that $k_3 \equiv \vec{k}\cdot\hat{e}_3 = |\vec{k}|\,\cos\theta$), and complex coefficients $c_1^\pm$, $c_2^\pm$, and $c_3^\pm$ (at this point, the overall normalization is arbitrary). The vacuum MCS field equations [24] then give the following polarization coefficients for the two modes:

$$\begin{pmatrix} c_1^\pm \\ c_2^\pm \\ c_3^\pm \end{pmatrix} = \begin{pmatrix} \cos\theta\left(\sqrt{\cos^2\theta+\mu_\pm^2\sin^4\theta}\;\pm\;\mu_\pm\,\sin^2\theta\right)^{-1} \\ \pm\, i \\ \mp\, 2\,\mu_\pm\,\sin\theta \end{pmatrix}\,, \quad \mu_\pm \equiv \frac{m}{2\,\omega_\pm}\,, \tag{26}$$

with the positive frequencies $\omega_\pm$ from Eq. (22). The corresponding magnetic field is

$$\vec{B}_\pm = \Re\left(\vec{k} \times \vec{E}_\pm\right)/\,\omega_\pm\ . \tag{27}$$

As long as the $\mu_\pm \sin^2\theta$ terms in Eq. (26) are negligible compared to $|\cos\theta|$, the transverse electric field consists of the usual circular polarization modes (see below). For the opposite case, $|\cos\theta|$ negligible compared to $\mu_\pm \sin^2\theta$, the transverse polarization $(c_1^\pm, c_2^\pm)$ becomes effectively linear, which agrees with the general remarks in Sec. IV B of Ref. [26].

Now consider the propagation of light pulses close to the x^2 axis. For $k_1 = 0$ and $0 < m \ll 2\pi/L \ll |k_3| \ll |k_2|$ in particular, we can identify the $\pm$ modes of the dispersion relation (22) with left- and right-handed circularly polarized modes (L and R; see Ref. [27]), depending on the sign of $k_3 \equiv |\vec{k}|\cos\theta$. From Eqs. (25) and (26), one obtains that $+/-$ corresponds to R/L for $k_3 > 0$ and to L/R for $k_3 < 0$.

Equation (24) then gives the following relations for the group velocities of pulses of circularly polarized light:

$$|\,\vec{v}_{\rm g}^{\,L}(0, k_2, k_3)| = |\,\vec{v}_{\rm g}^{\,R}(0, -k_2, -k_3)|\ , \tag{28}$$

and

$$|\,\vec{v}_{\rm g}^{\,L}(0, k_2, k_3)| \neq |\,\vec{v}_{\rm g}^{\,L}(0, -k_2, -k_3)|\ , \tag{29}$$

as long as $m \neq 0$. Recall at this point that the time-reversal operator T reverses the direction of the wave vector and leaves the helicity unchanged, whereas the parity-reflection operator P flips both the wave vector and the helicity. The equality (28) therefore implies parity invariance and the inequality (29) time-reversal noninvariance for this concrete physical situation (see Fig. 1).

Furthermore, the vacuum has become *optically active*, with left- and right-handed monochromatic plane waves traveling at different speeds [24], as follows from Eq. (22) above (see also Fig. 1). As mentioned in Ref. [12], this may lead to observable effects of the CPT anomaly in the cosmic microwave background: the polarization pattern around hot- and cold-spots is modified, due to the action of the Chern–Simons-like term (20) on the electromagnetic waves traveling between the last-scattering surface (redshift $z \sim 10^3$) and the detector ($z = 0$). Figure 2 gives a sketch of this cosmic birefringence, which may be looked for by NASA's Microwave Anisotropy Probe and ESA's Planck Surveyor. See Ref. [28] for a pedagogical review of the expected cosmic microwave background polarization and Ref. [29] for further details on the possible signatures of cosmic birefringence.

It is important to realize that the optical activity from the CPT anomaly, as illustrated by Fig. 2, is essentially frequency independent, in contrast to the quantum gravity effects suggested by the authors of Ref. [10], for example. In general, quantum gravity effects on the photon propagation can be expected to become more and more important as the photon frequency increases towards $M_{\rm Planck} \equiv (\hbar\, c/G)^{1/2} \sim 10^{19}\,\text{GeV}$. The potential CPT anomaly effect at the relatively low frequencies ($\sim 10^{-4}\,\text{eV}$) of the cosmic microwave background is, therefore, quite remarkable; see also the comments below Eq. (17).

VI. CONCLUSIONS

The possible influence of the spacetime topology on the local properties of quantum field theory has long been recognized (e.g., the Casimir effect). As discussed in the present talk, it now appears that nontrivial topology may also lead to CPT noninvariance for chiral gauge field theories such as the Standard Model with an odd number of families. This holds even for flat spacetime manifolds, that is, without gravity.

As to the physical origin of the CPT anomaly, many questions remain (the same can be said about chiral anomalies in general). It is, however, clear that the gauge-invariant second-quantized vacuum state plays a crucial role in connecting the global spacetime structure to the local physics [12,13]. In a way, this is also the case for the Casimir effect [19]. New here is the interplay of parity violation (chiral fermions) and gauge invariance. Work on this issue is in progress, but progress is slow.

As to the potential applications of the CPT anomaly, we can mention:

- the *optical activity of the vacuum* (leading to a polarization effect for the cosmic microwave background; cf. Fig. 2);
- the *fundamental arrow-of-time* (possibly playing a role at the beginning of the universe; cf. Refs. [12,30]).

An important property of the four-dimensional CPT anomaly is the ultraviolet/infrared connection, exemplified by the factor n/L in the effective action term (11). Perhaps this allows us to get a handle on the small-scale structure of spacetime (wormholes, spacetime foam, spin network, ...) by studying the long-range propagation of photons.

ACKNOWLEDGMENTS

It is a pleasure to thank the organizers of the conference for their hospitality, J. Pullin for correspondence, C. Adam for help with the signs, and C. Mayer for help with the figures.

APPENDIX A: EFFECTIVE ACTION FOR 2D CHIRAL $U(1)$ GAUGE THEORY

The two-dimensional Euclidean action for a single one-component Weyl field $\psi(x)$ of unit electric charge over the particular torus T^2 with modulus $\tau = i$ is given by

$$S^{T^2}_{\text{Weyl}}\left[\bar{\psi}, \psi, a\right] = -\int_0^L \mathrm{d}x^1 \int_0^L \mathrm{d}x^2 \; e\; \bar{\psi}\; e^{\mu}_{a}\, \sigma^a (\partial_\mu + i a_\mu)\, \psi \,, \tag{A1}$$

with

$$(\sigma^1, \sigma^2) = (1, i)\,, \quad e^{\mu}_{a} = \delta^{\mu}_{a}\,, \quad e \equiv \det\left(e^{a}_{\mu}\right) = 1\,. \tag{A2}$$

The $U(1)$ gauge potential can be decomposed as follows:

$$a_\mu(x) = \epsilon_{\mu\nu}\, \delta^{\nu\rho}\, \partial_\rho \phi(x) + 2\pi h_\mu / L + \partial_\mu \chi(x)\,, \tag{A3}$$

with $\phi(x)$ and $\chi(x)$ real periodic functions and h_1 and h_2 real constants. Here, $\chi(x)$ corresponds to the gauge degree of freedom. The related gauge transformations on the fermion fields are

$$\psi(x) \to \exp[-i\chi(x)]\, \psi(x)\,, \quad \bar{\psi}(x) \to \exp[+i\chi(x)]\, \bar{\psi}(x)\,. \tag{A4}$$

Next, impose doubly-periodic boundary conditions on the fermions,

$$\psi(x^1+L, x^2) = +\,\psi(x^1, x^2)\,, \quad \psi(x^1, x^2+L) = +\,\psi(x^1, x^2)\,. \tag{A5}$$

This spin structure will be denoted by PP, where P stands for periodic boundary conditions. (The other spin structures are AA, AP, and PA, where A stands for antiperiodic boundary conditions.)

The effective action $\Gamma_{\rm PP}\,[a]$ of this $(1111\overline{2})$-model, defined by the functional integral (8), is found to be given by [22]

$$\exp\left(-\Gamma^{1111\overline{2}}_{\rm PP}\,[a]\right) \equiv D^{1111\overline{2}}_{\rm PP}\,[a] = \left(D_{\rm PP}\,[a]\right)^4\, \overline{\left(D_{\rm PP}[2a]\right)}\,, \tag{A6}$$

with the single chiral determinant

$$D_{\rm PP}\,[a] = \hat{\vartheta}(h_1+\tfrac{1}{2}, h_2+\tfrac{1}{2})\, \exp\left(i\,\frac{\pi}{2}\,(h_1-h_2)\right)\, \exp\left(\frac{1}{4\pi}\int_{T^2} \mathrm{d}^2x\, \left(\phi\,\partial^2\phi + i\phi\,\partial^2\chi\right)\right). \tag{A7}$$

Here, the complex function

$$\hat{\vartheta}(x,y) \equiv \exp\left(-\pi y^2 + i\pi xy\right)\, \vartheta(x+iy; i)/\eta(i)\,, \quad \text{for} \quad x, y \in \mathbb{R}\,, \tag{A8}$$

is defined in terms of the Riemann theta function $\vartheta(z;\tau)$ and Dedekind eta function $\eta(\tau)$, for modulus $\tau = i$. The bar on the right-hand side of Eq. (A6) denotes complex conjugation.

The gauge invariance of the effective action (A6) can be readily verified. In fact, the gauge degree of freedom $\chi(x)$ appears only in the exponential of Eq. (A7), namely in the term proportional to $i\phi\,\partial^2\chi$, and cancels out for the full expression (A6), since $4\times 1^2 - 1\times 2^2 = 0$. The invariance under large gauge transformations $h_\mu \to h_\mu + n_\mu$, for $n_\mu \in \mathbb{Z}$, requires a little bit more work.

The CPT anomaly (10) follows directly from the ϑ function properties [14]. The relevant properties of $\vartheta(z;\tau)$ are its periodicity under $z \to z+1$ and quasi-periodicity under $z \to z+\tau$, together with the symmetry $\vartheta(-z;\tau) = \vartheta(z;\tau)$. But the anomaly can also be understood heuristically from the product of eigenvalues. For gauge fields (A3) with $\phi(x) = \chi(x) = 0$ and infinitesimal harmonic pieces h_μ, one has, in fact,

$$D^{1111\overline{2}}_{\rm PP}\,[h_1, h_2] = c\,(h_1 + ih_2)^3\,(h_1^2 + h_2^2) + \mathrm{O}(h^7)\,, \tag{A9}$$

with a nonvanishing complex constant c. Clearly, this expression changes sign under the transformation $h_\mu \to -h_\mu$, which corresponds to the CPT transformation (9). See Ref. [14] for further details.

By choosing topologically nontrivial zweibeins $e^a_\mu(x)$ [still with a flat metric $g_{\mu\nu}(x) \equiv e^a_\mu(x)\; e^b_\nu(x)\; \delta_{ab} = \delta_{\mu\nu}$] and including the spin connection term in the covariant derivative of the fermionic action (A1), the CPT anomaly can be moved to the spin structures AA, AP, and PA. These topologically nontrivial zweibeins correspond, however, to the presence of spacetime torsion, which may be of interest in itself. See Ref. [15] for further details on role of topologically nontrivial torsion.

APPENDIX B: MICROCAUSALITY IN 4D MCS THEORY

For the Maxwell–Chern–Simons (MCS) theory (18) in the Coulomb gauge $\nabla\cdot\vec{a}=0$ (with vector indices running over 1, 2, 3, and $\hbar=c=1$), the following commutators have been found for the electric field $\vec{e}\equiv\partial_0\,\vec{a}-\nabla\,a_0$ and magnetic field $\vec{b}\equiv\nabla\times\vec{a}$:

$$[e_i(x),e_j(0)]=\Big((\delta_{ij}\,\partial_0^2-\partial_i\partial_j)\,(\partial_0^2-\nabla^2)+m^2\,\delta_i^3\delta_j^3\,\partial_0^2 \\ -m\,\epsilon_{ij3}\,\partial_0^3+m\,(\epsilon_{ia3}\,\partial_j-\epsilon_{ja3}\,\partial_i)\,\partial_a\partial_0\Big)\,iD_{\rm MCS}(x)\,, \tag{B1}$$

$$[e_i(x),b_j(0)]=\Big(\epsilon_{ijl}\,\partial_l\partial_0\,(\partial_0^2-\nabla^2)-m^2\,\delta_i^3\,\epsilon_{j3a}\,\partial_a\partial_0 \\ +m\,\left(\delta_{ij}\,\partial_0^2-\partial_i\partial_j\right)\partial_3-m\,\delta_j^3\,\partial_i(\partial_0^2-\nabla^2)\Big)\,iD_{\rm MCS}(x)\,, \tag{B2}$$

$$[b_i(x),b_j(0)]=\Big((\delta_{ij}\nabla^2-\partial_i\partial_j)\,(\partial_0^2-\nabla^2)-m\,\epsilon_{ijl}\,\partial_l\partial_0\partial_3 \\ +m^2\left(\delta_{ij}\,(\nabla^2-\partial_3^2)-\partial_i\partial_j-\delta_i^3\delta_j^3\,\nabla^2+(\delta_i^3\,\partial_j+\delta_j^3\,\partial_i)\,\partial_3\right)\Big)\,iD_{\rm MCS}(x)\,, \tag{B3}$$

with the commutator function

$$D_{\rm MCS}(x)\equiv(2\pi)^{-4}\oint_C {\rm d}p_0\int {\rm d}^3p\;\frac{\exp\left[\,ip_0\,x^0+i\vec{p}\cdot\vec{x}\,\right]}{\left(|\vec{p}\,|^2-p_0^2\right)^2+m^2\,(p_1^2+p_2^2-p_0^2)}\quad, \tag{B4}$$

for a contour C that encircles all four poles of the integrand in the counterclockwise direction. Note that the derivatives on the right-hand side of Eqs. (B1)–(B3) effectively bring down powers of the momenta in the integrand of Eq. (B4). [The calculation of the commutators (B1)–(B3) is rather subtle: a_0, for example, does not vanish in the Coulomb gauge but is determined by a nontrivial equation of motion, $a_0=i\,m\,|\vec{p}\,|^{-2}\,\epsilon_{3kl}\,a_k\,p_l$ in momentum space.]

The Lorentz noninvariance of the MCS theory is illustrated by the denominator of the integrand in Eq. (B4) and the fact that commutators (B1) and (B3) differ at order m^2. Two further observations can be made:

1. The commutator function vanishes for spacelike separations,

   $$D_{\rm MCS}(x^0,\vec{x}\,)=0\,,\quad \text{for }\;|x^0|<|\vec{x}\,|\,, \tag{B5}$$

   as follows by direct calculation.

2. Even though the commutators of the vector potentials $\vec{a}\,(x)$ have poles of the type $|\vec{p}\,|^{-2}$, these poles, which spoil causality, are absent for the commutators (B1)–(B3) of the physical (gauge-invariant) electric and magnetic fields.

Hence, the locality results of QED [31] carry over to the MCS theory, at least for the "spacelike" Chern–Simons term (20) considered [16]. This MCS theory has, however, new uncertainty relations (e.g., for the b_1 and b_2 fields averaged over the same spacetime region).

The "timelike" MCS theory, with $\epsilon^{3\kappa\lambda\mu}$ in (20) replaced by $\epsilon^{0\kappa\lambda\mu}$, does violate microcausality, as long as unitarity is enforced [16]. This particular result may have other implications. It rules out, for example, the possibility that a Chern–Simons-like term can be radiatively induced from a CPT-violating axial-vector term in the Dirac sector [17].

REFERENCES

[1] G. Lüders, Kgl. Danske Videnskab. Selskab. Mat.-Fys. Medd. **28**, No. 5 (1954).

[2] W. Pauli, in *Niels Bohr and the Development of Physics*, edited by W. Pauli, L. Rosenfeld, and V. Weisskopf (Pergamon, London, 1955), p. 30.

[3] J.S. Bell, Proc. R. Soc. London **A 231**, 479 (1955).

[4] G. Lüders, Ann. Phys. (N. Y.) **2**, 1 (1957).

[5] R. Jost, Helv. Phys. Acta **30**, 409 (1957).

[6] N.N. Bogolubov, A.A. Logunov, A.I. Oksak, and I.T. Todorov, *General Principles of Quantum Field Theory* (Kluwer Academic, Dordrecht, 1990).

[7] J.J. Sakurai, *Invariance Principles and Elementary Particles* (Princeton Univ. Press, Princeton, 1964); N.W. Tanner and R.H. Dalitz, Ann. Phys. (N. Y.) **171**, 463 (1986).

[8] R. Wald, Phys. Rev. **D 21**, 2742 (1980); S.W. Hawking, Phys. Rev. **D 32**, 2489 (1985).

[9] L. Alvarez-Gaumé and C. Gomez, Commun. Math. Phys. **89**, 235 (1983).

[10] R. Gambini and J. Pullin, Phys. Rev. **D 59**, 124021 (1999); J. Alfaro, H.A. Morales-Técotl, and L.F. Urrutia, Phys. Rev. **D 65**, 103509 (2002).

[11] H. Sonoda, Nucl. Phys. **B 326**, 135 (1989); V.A. Kostelecký and R. Potting, Nucl. Phys. **B 359**, 545 (1991); Phys. Lett. **B 381**, 89 (1996).

[12] F.R. Klinkhamer, Nucl. Phys. **B 578**, 277 (2000) [hep-th/9912169].

[13] F.R. Klinkhamer, Nucl. Phys. **B 535**, 233 (1998) [hep-th/9805095].

[14] F.R. Klinkhamer and J. Nishimura, Phys. Rev **D 63**, 097701 (2001) [hep-th/0006154].

[15] F.R. Klinkhamer and C. Mayer, Nucl. Phys. **B 616**, 215 (2001) [hep-th/0105310].

[16] C. Adam and F.R. Klinkhamer, Nucl. Phys. **B 607**, 247 (2001) [hep-ph/0101087].

[17] C. Adam and F.R. Klinkhamer, Phys. Lett. **B 513**, 245 (2001) [hep-th/0105037].

[18] F.R. Klinkhamer, in *Proceedings International Europhysics Conference on High Energy Physics*, edited by D. Horváth et al., J. High Energy Phys. Proceedings hep2001/231.

[19] B.S. DeWitt, Phys. Rep. **19**, 295 (1975); T.H. Boyer, Ann. Phys. (N. Y.) **56**, 474 (1970).

[20] J.-P. Luminet and B.F. Roukema, in *Theoretical and Observational Cosmology*, edited by M. Lachièze-Rey (Kluwer Academic, Dordrecht, 1999), p. 117 [astro-ph/9901364].

[21] I. Ciufolini and J.A. Wheeler, *Gravitation and Inertia* (Princeton Univ. Press, Princeton, 1995), Sec. 4.3.

[22] T. Izubuchi and J. Nishimura, J. High Energy Phys. **10**, 002 (1999).

[23] A.N. Redlich, Phys. Rev. **D 29**, 2366 (1984); L. Alvarez-Gaumé and E. Witten, Nucl. Phys. **B 234**, 269 (1984); A. Coste and M. Lüscher, Nucl. Phys. **B 323**, 631 (1989).

[24] S.M. Carroll, G.B. Field, and R. Jackiw, Phys. Rev. **D 41**, 1231 (1990).

[25] A.S. Goldhaber and M.M. Nieto, Rev. Mod. Phys. **43**, 277 (1971).

[26] D. Colladay and V.A. Kostelecký, Phys. Rev. **D 58**, 116002 (1998).

[27] J.D. Jackson, *Classical Electrodynamics*, 2nd ed. (Wiley, New York, 1975), Sec. 7.2.

[28] W. Hu and M. White, New Astron. **2**, 323 (1997) [astro-ph/9706147].

[29] N.F. Lepora, [gr-qc/9812077]; see also A. Lue, L. Wang, and M. Kamionkowski, Phys. Rev. Lett. **83**, 1506 (1999).

[30] R. Penrose, in *General Relativity: An Einstein Centenary Survey*, edited by S.W. Hawking and W. Israel (Cambridge Univ. Press, Cambridge, 1979), Sec. 12.4.

[31] P. Jordan and W. Pauli, Z. Phys. **47**, 151 (1928); W. Heitler, *The Quantum Theory of Radiation*, 3rd ed. (Oxford Univ. Press, London, 1954), Sec. 9 and App. 2.

FIGURES

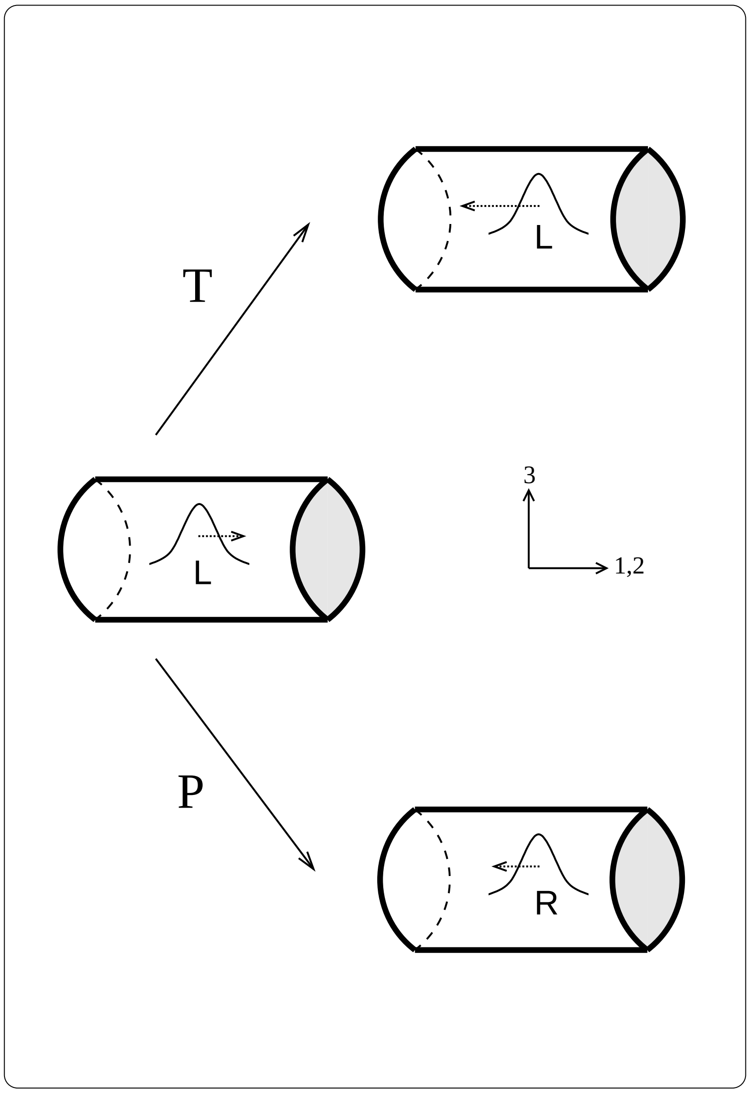

FIG. 1. Sketch of the behavior of a left-handed wave packet in the Maxwell–Chern–Simons theory (18) under the time reversal (T) and parity (P) transformations. [The charge conjugation (C) transformation acts trivially.] The dotted arrows indicate the group velocity approximately in the x^1 or x^2 direction, for the case of a compact x^3 coordinate. The magnitude of the group velocity changes under T, but not under C or P (hence, the physics is CPT-noninvariant). In addition, the vacuum is seen to be optically active, with left- and right-handed light pulses traveling to the left at different speeds (because of parity invariance, the same holds for pulses of circularly polarized light traveling to the right).

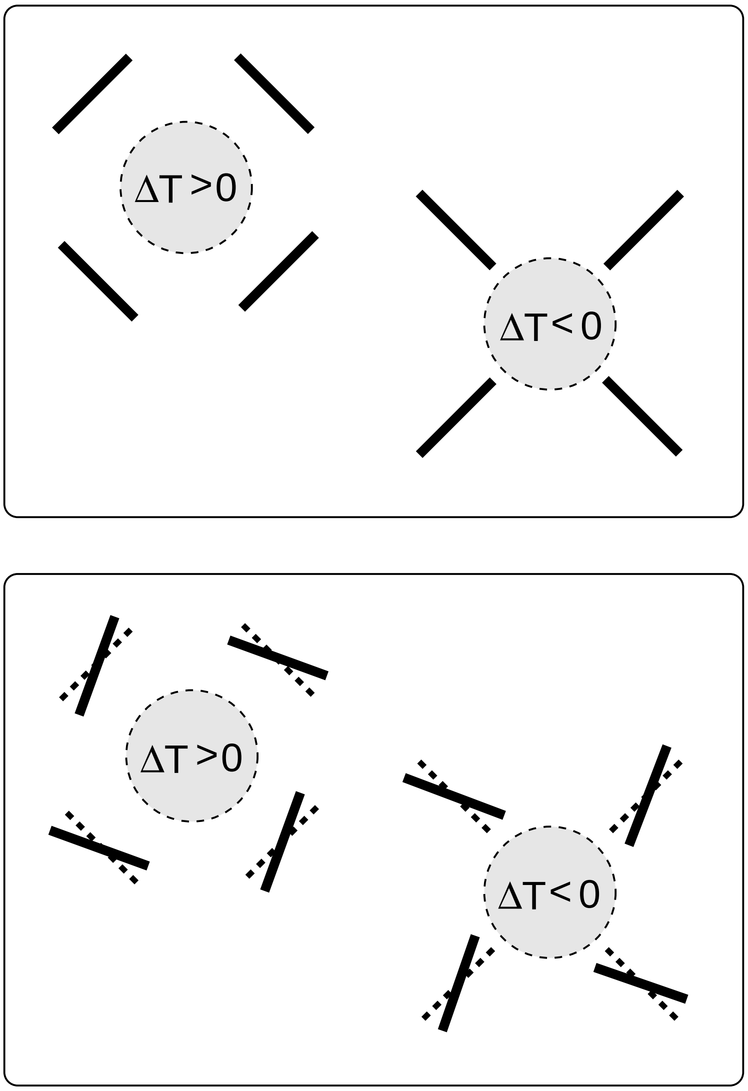

FIG. 2. Sketch of the linear polarization pattern (indicated by heavy bars) around cosmic microwave background hot- and cold spots, generated by scalar perturbations of the metric. The top panel is in a noncompact direction (corresponding to the x^1 coordinate, say). The bottom panel is in the compact direction (corresponding to the x^3 coordinate) and displays the optical activity of the Maxwell–Chern–Simons theory (18) considered. In fact, for a patch of the sky in a particular direction along the x^3 axis (bottom panel), the linear polarization pattern is rotated by a very small amount in counterclockwise direction. For a patch of the sky in the opposite direction (not shown), the rotation of the linear polarization is in the clockwise direction. Precisely which particular direction along the x^3 axis corresponds to the counterclockwise rotation and which to the clockwise rotation depends on the small-scale structure of the theory, that is, the sign of the parameter n in the Chern–Simons-like term (11) of the effective action.