# Space-time defect and elementary flatness 

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I declare that I have developed and written the enclosed thesis completely by myself, and have not used sources or means without declaration in the text.

Karlsruhe, April 22, 2015
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## Abstract

It is an unresolved question whether the space we live in is a smooth continuum down to smallest length scales or if it possesses structures, which have been elusive to our experience so far due to their smallness. Fundamental theoretical considerations lead to this presumption, though [84]. Such a quantum space-time foam could affect the propagation of particles in vacuo and be observable in this way [6]. However, in spite of great efforts, it has not been possible to date to describe gravity within the framework of a quantum theory. Thus, we can only speculate about the nature of such a space-time foam so far.

In the present thesis a space-time is investigated, which may serve as a classical model for a single topological defect of space [47]. The space-time defect can be considered as a solution of the vacuum Einstein field equations; however, it possesses a peculiar geometry, which does not conform to the common notions of local flatness of space in general relativity. The possible implications of the violation of elementary flatness by the defect are discussed. We try and answer the questions, which arose after the discovery of the space-time defect [46]. In particular, we address the question of whether the violation of elementary flatness rules out the space-time defect as a solution of general relativity.

Other interpretations of the presented space-time defect are possible, though. When assigning mass and possibly charge to the defect, it may serve as a classical model for an elementary particle itself. With regard to the unification of the fundamental forces of nature such a model is of particular physical interest.

After an introduction into the topic of Lorentzian and non-Lorentzian geometry the space-time defect and its elementary properties are presented. Subsequently the motion of test particles through the defect is investigated and classical scattering experiments in the defect space-time are described. Here it will be shown how parameters like mass and charge affect the scattering. Further, the behaviour of quantum fields near the defect core is investigated in detail. As it turns out, the behaviour even for a massless defect is fundamentally different from that in ordinary Minkowski space-time and the geometry of the defect itself is not stable against the influence of a quantum field. Subsequently, a spin- $1 / 2$ field and possible problems with respect to its interaction with other fields in the defect space-time are described. Finally, the implications of the topology of the space-time defect for physical processes are investigated.
$\qquad$

## Zusammenfassung

Es ist eine ungeklärte Frage, ob der Raum, in dem wir leben, bis hin zu kleinsten Längenskalen ein glattes Kontinuum bildet oder ob er selbst Strukturen besitzt, die unserer Erfahrung bisher aufgrund ihrer Kleinheit entgangen sind. Grundlegende theoretische Überlegungen führen jedoch zu dieser Annahme [84]. Ein solcher Quantenschaum könnte die Ausbreitung von Teilchen im Vakuum beeinflussen und auf diese Weise beobachtbar sein [6]. Jedoch ist es trotz großer Bemühungen bis heute nicht gelungen, die Gravitation im Rahmen einer Quantentheorie zu beschreiben. Über die Natur eines solchen Quantenschaums kann daher bislang nur spekuliert werden.

In der vorliegenden Arbeit wird eine Raumzeit untersucht, die als klassisches Modell für einen einzelnen topologischen Defekt des Raumes dienen kann [47]. Der Raumzeit-Defekt kann als Lösung der einsteinschen Vakuumfeldgleichungen angesehen werden, besitzt jedoch eine besondere Geometrie, die nicht den gewöhnlichen Vorstellungen von lokaler Flachheit des Raumes in der allgemeinen Relativitätstheorie entspricht. Die möglichen Auswirkungen der Verletzung von elementarer Flachheit werden beschrieben. Wir versuchen die Fragen zu beantworten, die nach der Entdeckung des Raumzeit-Defektes aufgekommen sind [46]. Insbesondere soll der Frage nachgegangen werden, ob die Verletzung elementarer Flachheit den Defekt als Lösung der allgemeinen Relativitätstheorie ausschließt.

Jedoch sind auch andere Interpretationen des vorgestellten Defekts möglich. Wird dem Defekt Masse und eventuell Ladung verliehen, kann er selbst als klassisches Modell für ein Elementarteilchen dienen. Im Hinblick auf die Vereinheitlichung der Grundkräfte ist ein solches Modell von besonderem physikalischen Interesse.

Nach einer Einführung in das Themengebiet von lorentzscher und nicht-lorentzscher Geometrie wird der Raumzeit-Defekt und seine grundlegenden Eigenschaften vorgestellt. Anschließend werden die Bewegung von Testteilchen durch den Defekt untersucht und klassische Streuversuche am Defekt beschrieben. Hierbei wird gezeigt werden, wie Parameter wie Ladung und Masse von Defekt und Testteilchen die Streuung beeinflussen. Des Weiteren wird das Verhalten von Quantenfeldern nahe des Defekts eingehend untersucht. Es wird sich zeigen, dass sich das Verhalten selbst für einen masselosen Defekt grundlegend von jenem im gewöhnlichen Minkowski-Raum unterscheidet und die Geometrie des Defektes selbst nicht stabil unter Einwirkung eines Quantenfeldes ist. Anschließend wird ein Spinorfeld und mögliche Probleme hinsichtlich seiner Wechselwirkung mit anderen Feldern in der Defekt-Raumzeit beschrieben. Zuletzt werden die Auswirkungen der Topologie des Raumzeit-Defekts auf physikalische Vorgänge untersucht.

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## Chapter 1

## Elementary flatness

In this chapter we give an introduction into some topics related to the geometry of Lorentzian manifolds and into the elementary flatness condition, which seems to have been introduced into general relativity by Einstein and Rosen in 1936, while analyzing the Silberstein solution of the Einstein field equations [25,53, 85].

### 1.1 Introduction

### 1.1.1 Silberstein solution

Previously, Ludwik Silberstein had suggested a solution, which in his view represented a situation where two mass points are placed side by side in everywhere else empty space, and yet do not attract, since the solution is static. Since such a situation does not occur in nature, he concluded that general relativity cannot be maintained when describing particles as mass points [71]:
"Here, then, is an example of a perfectly rigorous solution of Einstein's field equations which does not at all correspond to reality. If, therefore, these equations are to be retained, one cannot consider 'material particles' (mass points) as singularities of the field [...]."

Silberstein used as an Ansatz the general axially symmetric line-element for the vacuum Einstein field equations [75] and obtained the solution [we use natural units, $\hbar=c=G=1$ ]

$$
\begin{align*}
& \mathrm{d} s^{2}=-\mathrm{e}^{2 \lambda(r, z)} \mathrm{d} t^{2}+\mathrm{e}^{2(v(r, z)-\lambda(r, z))}\left(\mathrm{d} r^{2}+\mathrm{d} z^{2}\right)+r^{2} \mathrm{e}^{-2 \lambda(r, z)} \mathrm{d} \phi^{2},  \tag{1.1}\\
& \lambda(r, z)=-\frac{m_{1}}{\rho_{1}(r, z)}-\frac{m_{2}}{\rho_{2}(r, z)},  \tag{1.2a}\\
& v(r, z)=-\frac{r^{2}}{2}\left(\frac{m_{1}^{2}}{\rho_{1}^{4}(r, z)}+\frac{m_{1}^{2}}{\rho_{2}^{4}(r, z)}\right)+\frac{2 m_{1} m_{2}}{\left(z_{2}-z_{1}\right)^{2}}\left(\frac{r^{2}+\left(z-z_{1}\right)\left(z-z_{2}\right)}{\rho_{1}(r, z) \rho_{2}(r, z)}-1\right),  \tag{1.2b}\\
& \rho_{i}(r, z)=\sqrt{r^{2}+\left(z-z_{i}\right)^{2}}, \quad i \in\{1,2\} . \tag{1.2c}
\end{align*}
$$

The mass points with masses $m_{1}$ and $m_{2}$ are situated at the two points ( $r=0, z=z_{1}$ ) and ( $r=0, z=z_{2}$ ), and the corresponding Ricci tensor of the metric (1.1) appears to vanish everywhere. Also, the Riemann curvature tensor $R_{\mu \nu \rho \sigma}$ and the Kretschmann scalar $K=R_{\mu \nu \rho \sigma} R^{\mu \nu \rho \sigma}$ remain finite throughout the whole space-time.

However, Einstein and Rosen pointed out that Silberstein's solution cannot be thought of as representing a regular gravitational field of physical significance, since the circumference of a circle around the $z$ axis between $z_{1}$ and $z_{2}$ is not equal to $2 \pi$ times its radius $r$, even if $r$ tends to zero [25]. Hence, there cannot be a coordinate transformation which would make the space-time locally isometric to Minkowski space-time. It is a fundamental assumption of general relativity, however, that the laws of nature be locally reducible to those of special relativity. Requiring local isometry to Minkowski space-time or, equivalently, Lorentzian geometry, has become known as the elementary flatness condition. Regardless of this, Einstein and Rosen also put Silberstein's conclusion into question, since the solution (1.1) still contains two singularities, which, in their view, could not account for the physical properties of material particles completely.

### 1.1.2 Lorentzian manifolds as space-time model

Space-time is usually modelled by a Lorentzian manifold or, strictly speaking, by an equivalence class of Lorentzian manifolds. ${ }^{1}$ A Lorentzian manifold is a smooth, connected, four-dimensional manifold $\mathcal{M}$, satisfying the Hausdorff property and possessing a globally defined metric tensor $g_{\mu \nu}$ with three positive eigenvalues and one negative eigenvalue and non-vanishing determinant [38,63].

From linear algebra it is clear that a non-degenerate metric of the above form can always be transformed to the Minkowski metric locally: It is always possible to find a nonsingular, orthogonal matrix, which diagonalizes a real, quadratic and symmetric matrix, and according to Sylvester's law of inertia, the signs of the eigenvalues are invariants under regular coordinate transformations.

### 1.1.3 Elementary flatness on a symmetry axis

However, the standard requirement on a space-time, that the metric tensor be nowhere degenerate, i. e. have no zero eigenvalues, appears to be violated in many common cases, for example when Minkowski space-time is written in cylindrical coordinates ( $t, r, \phi, z$ ):

$$
\begin{align*}
& t \in(-\infty, \infty) ; \quad r \in[0, \infty) ; \quad \phi \in[0,2 \pi] ; \quad z \in(-\infty, \infty) ; \quad(t, r, 0, z) \equiv(t, r, 2 \pi, z) \\
& \mathrm{d} s^{2}=-\mathrm{d} t^{2}+\mathrm{d} r^{2}+r^{2} \mathrm{~d} \phi^{2}+\mathrm{d} z^{2} . \tag{1.3}
\end{align*}
$$

This by itself does not mean that Minkowski space-time is not a Lorentzian manifold, since it is, when written in standard Cartesian coordinates. Rather, in Lorentzian geometry, points where the metric tensor is degenerate are a priori not to be considered as part of the

[^0]coordinate chart, i. e. the coordinate chart associated to the coordinates of equation (1.3) does actually not cover all of Minkowski space-time.

It is necessary to include the points at $r=0$ into the underlying manifold in the above example, however, since otherwise the space-time would be geodesically incomplete. The requirement of geodesic completeness ensures that all geodesics can be extended to infinite values of the affine parameter. This in turn appears to be physically necessary, since it should be possible to compute the future development of, say, a massive test particle, for any value of the proper time. Hence, a geodesically incomplete space-time has to be considered as singular [30,38].

Since, according to the above definition, a Lorentzian space-time is a smooth manifold, we require that there exist a proper coordinate chart in the Lorentzian sense, which covers the symmetry axis $r=0$ and which is related to the coordinates (1.3) by a diffeomorphism in the region of overlap. This is the case for standard Cartesian coordinates, as one can easily check. As it will turn out, this is not everywhere possible in the Silberstein solution, which shows that there are space-times that are exact solutions of the Einstein field equations, while not being Lorentzian manifolds.

### 1.1.4 Normal coordinates

Any Riemannian and also any pseudo-Riemannian or Lorentzian manifold allows for so called normal coordinates [38,52]. We restrict ourselves to the Lorentzian case. Normal coordinates are obtained by choosing a point $p$ of the manifold and four linearly independent directions or four linearly independent basis vectors $\boldsymbol{e}_{A}$ in the tangent space at $p$, respectively, where the metric is equal to the Minkowski metric. Geodesics through space-time may then be send out, whose tangent vector at the starting point equals one of the four basis vectors chosen initially. Each geodesic may then be interpreted as a coordinate axis where the affine parameter plays the role of a coordinate. Hence, the inverse map of the so obtained geodesics is a coordinate chart, where the metric tensor is, at the point $p$, equal to the Minkowski metric. In fact, it is possible to show rigorously that by this method a coordinate chart of an open neighbourhood of $p$ can be constructed [52]. In these coordinates, the metric tensor takes the form [61]

$$
\begin{equation*}
g_{\mu v}(x)=\eta_{\mu \nu}-\left.\frac{1}{3} R_{\mu \alpha v \beta}\right|_{x=0} x^{\alpha} x^{\beta}+O\left(x^{3}\right), \tag{1.4}
\end{equation*}
$$

where $R_{\mu \alpha \nu \beta}$ denotes the Riemann curvature tensor, which is evaluated at the origin. Using the metric (1.4) in geodesic normal coordinates, it is possible to compute the volume, area and length of small geometric figures. For example, one obtains for the volume $V$ of a small three-ball $B_{r}$ around the origin of the normal coordinate system with geodesic radius $r$

$$
\begin{align*}
& V(r)=\int_{B_{r}} \sqrt{\mid(3)} g \mid  \tag{1.5}\\
& \mathrm{d} x^{1} \mathrm{~d} x^{2} \mathrm{~d} x^{3} \approx \int_{B_{r}}\left(1+\frac{1}{6} R_{k m k n} x^{m} x^{n}\right) \mathrm{d} x^{1} \mathrm{~d} x^{2} \mathrm{~d} x^{3} \\
&=\frac{4 \pi r^{3}}{3}\left(1+\frac{1}{30} R_{k n k n} r^{2}\right),
\end{align*}
$$

where a sum over the indices $k$ and $n$ from 1 to 3 is understood. On similar grounds, it can be shown for the circumference $C$ of a geodesic circle that [73]

$$
\begin{equation*}
C(r)=2 \pi r\left(1-\frac{\left.K\right|_{r=0} r^{2}}{6}+O\left(r^{3}\right)\right) \tag{1.6}
\end{equation*}
$$

where $K$ is the Gaussian curvature of the plane formed by the circle, evaluated at the origin of the geodesic normal coordinate system. This formula is known as the Bertrand-PuiseuxDiquet theorem and states, roughly speaking, that the ratio of circumference and radius of a geodesic circle approaches $2 \pi$ for small radii. Hence, if regions of a space-time exist, where a geodesic circle does not have the characteristic ratio of circumference and radius, it cannot be a Lorentzian manifold.

### 1.1.5 Regularity condition

We would like to have an efficient method for checking a space-time for the existence of such hidden singularities as in the Silberstein solution. For axisymmetric spaces, such a method is available and is given by calculating the ratio of circumference and radius of an infinitesimally small circle around the axis of symmetry, lying in a coordinate slice of the space-time manifold. This can be made mathematically precise by defining the circle as the orbit of a spacelike Killing vector field $\eta^{\mu}$, which serves as the generator of a rotation. This Killing vector then satisfies the condition [10, 54, 74]

$$
\begin{equation*}
\lim _{r \rightarrow 0} \frac{\partial_{\alpha}\left(\eta^{\mu} \eta_{\mu}\right) \partial^{\alpha}\left(\eta^{v} \eta_{\nu}\right)}{4 \eta^{\lambda} \eta_{\lambda}}=1 \tag{1.7}
\end{equation*}
$$

Assuming the possibility to expand the Killing vector $\eta^{\mu}$ in terms of local coordinates, we can easily prove the validity of equation (1.7). We choose coordinates $x^{\nu}$ in an infinitesimal neighborhood around the rotation axis. Then we can write

$$
\begin{equation*}
\eta^{\mu}(x)=\left.\eta^{\mu}\right|_{x=0}+\left.\nabla_{v} \eta^{\mu}\right|_{x=0} x^{\nu}+O\left(x^{2}\right) \tag{1.8}
\end{equation*}
$$

However, we must assume that the Killing vector field vanishes on the axis of rotation, $\left.\eta^{\mu}\right|_{x=0}=0$. This gives us the following expression for the norm of the Killing vector field:

$$
\begin{equation*}
\|\boldsymbol{\eta}\|^{2}=\eta^{\mu} \eta_{\mu}=\left.\left.x^{v} x^{\rho}\left(\nabla_{v} \eta^{\mu}\right)\right|_{x=0}\left(\nabla_{\rho} \eta_{\mu}\right)\right|_{x=0}+O\left(x^{3}\right) \equiv x^{v} x^{\rho} H_{v \rho}+O\left(x^{3}\right) \tag{1.9}
\end{equation*}
$$

We see that the norm of the Killing vector field is proportional to the distance from the rotation axis. The last equality gives rise to the definition of a projection tensor $\boldsymbol{H}$, which projects any space-time vector into the part of the tangent space orthogonal to the symmetry axis. This tensor is symmetric and naturally satisfies $H_{\alpha \rho} H_{\sigma}^{\alpha}=H_{\rho \sigma}$ if the Killing vector field is normalized properly [74]. Thus, we have

$$
\begin{equation*}
\lim _{x \rightarrow 0} \frac{\partial_{\alpha}\left(\eta^{\mu} \eta_{\mu}\right) \partial^{\alpha}\left(\eta^{v} \eta_{v}\right)}{4 \eta^{\lambda} \eta_{\lambda}}=\frac{\left.\left.2 x^{\mu}\left(H_{\alpha \mu}\right)\right|_{x=0} \cdot 2 x^{v}\left(H_{v}^{\alpha}\right)\right|_{x=0}}{\left.4 x^{\rho} x^{\sigma}\left(H_{\rho \sigma}\right)\right|_{x=0}}=1 \tag{1.10}
\end{equation*}
$$

Hence, equation (1.7) basically says that the absolute norm of the Killing vector field is proportional to the distance from the symmetry axis, and the limit (1.7) determines the ratio of circumference and radius of a circle divided by $2 \pi-$ provided that also the range of the angular coordinate is normalized properly [74].

### 1.2 Applications

### 1.2.1 Cone in Minkowski space-time

Let us investigate the validity of the regularity condition introduced above at an easy example [26]. First, we consider a cone as submanifold in Minkowski space-time as defined in (1.3) and choose, in cylindrical coordinates, the parametrization $z=r$. We then get

$$
\begin{equation*}
\mathrm{d} s^{2}=-\mathrm{d} t^{2}+\mathrm{d} r^{2}+r^{2} \mathrm{~d} \phi^{2}+\mathrm{d} z^{2}=-\mathrm{d} t^{2}+2 \mathrm{~d} r^{2}+r^{2} \mathrm{~d} \phi^{2} \tag{1.11}
\end{equation*}
$$

with the standard coordinate ranges $t \in(-\infty ; \infty), r \in[0 ; \infty)$ and $\phi \in[0 ; 2 \pi]$. In this $(2+1)$-dimensional example, the metric (1.11) allows for the Killing vector field $\eta_{\mu}=\partial_{\mu} \phi$ with $\eta^{\mu} \eta_{\mu}=r^{2}$, and the time axis can be thought of as the associated axis of rotation. We find that the regularity condition (1.7) is not satisfied,

$$
\begin{equation*}
\lim _{r \rightarrow 0} \frac{\partial_{\alpha}\left(\eta^{\mu} \eta_{\mu}\right) \partial^{\alpha}\left(\eta^{\nu} \eta_{v}\right)}{4 \eta^{\lambda} \eta_{\lambda}}=\frac{1}{2} \neq 1 \tag{1.12}
\end{equation*}
$$

Thus, at the tip of the cone, there cannot be a smooth coordinate transformation, which makes the cone locally isometric to (2+1)-dimensional Minkowski space-time.

### 1.2.2 Silberstein solution

Next, we consider the Silberstein solution from the beginning. The metric (1.1) also allows for the Killing vector field $\eta_{\mu}=\partial_{\mu} \phi$ and the "axis of rotation" is a two-dimensional timelike surface. When applying the regularity condition (1.7) for $z$ between $z_{1}$ and $z_{2}$, we get

$$
\begin{equation*}
\lim _{r \rightarrow 0} \frac{\partial_{\alpha}\left(\eta^{\mu} \eta_{\mu}\right) \partial^{\alpha}\left(\eta^{v} \eta_{v}\right)}{4 \eta^{\lambda} \eta_{\lambda}}=\exp \left(\frac{8 m_{1} m_{2}}{\left(z_{1}-z_{2}\right)^{2}}\right) \neq 1, \tag{1.13}
\end{equation*}
$$

while for $z$ outside the interval $\left[z_{1} ; z_{2}\right]$ we obtain

$$
\begin{equation*}
\lim _{r \rightarrow 0} \frac{\partial_{\alpha}\left(\eta^{\mu} \eta_{\mu}\right) \partial^{\alpha}\left(\eta^{v} \eta_{v}\right)}{4 \eta^{\lambda} \eta_{\lambda}}=1 \tag{1.14}
\end{equation*}
$$

In some sense, the space between the two point singularities is hence not empty, but rather appears to contain an infinitesimally thin material rod [68], which curves the space around it. This invalidates Silberstein's argument, that there are two singularities in empty space which do not attract, and it seems natural to require Lorentzian geometry also for other solutions of Einstein's field equations.

### 1.3 Einstein's and Rosen's modification of the field equations

Seemingly in response to Silberstein's argument cited above, Einstein and Rosen tried to remove the gravitational singularities arising through the introduction of point masses by modifying the vaccum field equations [24]. It can be shown that, in an asymptotically flat space-time and presuming Lorentzian geometry, such singularities always arise for particles of finite mass [21,23]. The deficit angle on the symmetry axis, as obtained in equation (1.13), was only later found in [25].

Einstein and Rosen suggested to describe particles by regions of the space-time manifold, where the metric tensor is degenerate and the geometry is not Lorentzian and thereby introduced the Einstein-Rosen bridge. The vanishing of the metric determinant implies that no inverse metric tensor exists at this point. However, it has been pointed out by Einstein [20,77] that the gravitational field is completely described by the covariant metric alone, since the contravariant components do not have a direct geometric meaning. The vacuum Einstein field equations,

$$
\begin{equation*}
R_{\mu \nu}=0, \tag{1.15}
\end{equation*}
$$

written in standard form, depend on the inverse metric and have a denominator proportional to $g^{2} \equiv\left(\operatorname{det}\left(g_{\mu \nu}\right)\right)^{2}$. To cancel this denominator, Einstein and Rosen introduced a modified version of the vacuum field equations,

$$
\begin{equation*}
g^{2} R_{\mu \nu}=0 . \tag{1.16}
\end{equation*}
$$

These modified field equations trivially reduce to the standard ones for metric tensors with non-vanishing determinant. ${ }^{2}$

### 1.3.1 Needlessness of modification

However, it turned out [77,78] that solutions $g_{\mu \nu}$ of equation (1.16), which are degenerate only on a submanifold of lower dimension and which are sufficiently smooth, are in some sense always also solutions of equation (1.15)-to be specific in the sense of a limiting process to the points of degeneracy. Einstein's and Rosen's modifcation is in this sense not necessary, as we will show briefly.

Let $\Sigma$ be a three-dimensional hypersurface of a space-time $\mathcal{M}$, on which the determinant of the metric tensor vanishes,

$$
\begin{equation*}
\left.g\right|_{\Sigma}=0 . \tag{1.17}
\end{equation*}
$$

The hypersurface $\Sigma$ may also be two- or one-dimensional, which does not affect the argumentation. Suppose now, we are given a solution of the modified vacuum field equations (1.16),

[^1]which is valid in a four-dimensional strip of finite width surrounding the hypersurface $\Sigma$. We can then choose coordinates $x^{\mu}$, such that the hypersurface $\Sigma$ is characterized by the equation $x^{1}=0$. Since the expression $g^{2} R_{\mu \nu}$ vanishes on an interval of finite width, clearly also all its derivatives vanish on the hypersurface $x^{1}=0$,
\[

$$
\begin{equation*}
\left.g^{2} R_{\mu v}\right|_{x_{1}=0}=\left.\frac{\partial}{\partial x^{1}} g^{2} R_{\mu v}\right|_{x_{1}=0}=\left.\partial_{x^{1}}^{2} g^{2} R_{\mu v}\right|_{x_{1}=0}=\cdots=0 \tag{1.18}
\end{equation*}
$$

\]

Applying L'Hôpital's rule repeatedly, this enables us to calculate the limit of the unmodified field equation (1.15) for zeros of $g^{2}$ of some finite order $n$ :

$$
\begin{equation*}
\lim _{x^{1} \rightarrow 0}\left(R_{\mu \nu}\right)=\lim _{x^{1} \rightarrow 0}\left(\frac{g^{2} R_{\mu v}}{g^{2}}\right)=\frac{\left.\partial_{x^{1}}^{n} g^{2} R_{\mu \nu}\right|_{x_{1}=0}}{\left.\partial_{x^{1}} 9^{2}\right|_{x_{1}=0}}=0 \tag{1.19}
\end{equation*}
$$

An analogous procedure is possible in the case where matter fields are present [50]. In this case, the modified Einstein field equations take the form

$$
\begin{equation*}
g^{p}\left(R_{\mu \nu}-\frac{1}{2} g_{\mu \nu} R-8 \pi T_{\mu \nu}\right)=0, \quad \quad p \geq 3 \tag{1.20}
\end{equation*}
$$

The standard equations are multiplied by at least $g^{3}$ in order to compensate for the denominator of the Ricci scalar $R$. In the case of an energy-momentum tensor $T_{\mu \nu}$ with vanishing denominator, it might be necessary to use a higher power $p$. This does not affect the argumentation, as long as $p$ is a finite integer. As before,

$$
\begin{align*}
\lim _{x^{1} \rightarrow 0}\left(R_{\mu \nu}-\frac{1}{2} g_{\mu \nu} R-8 \pi T_{\mu \nu}\right) & =\lim _{x^{1} \rightarrow 0}\left(\frac{g^{3}\left(R_{\mu \nu}-\frac{1}{2} g_{\mu \nu} R-8 \pi T_{\mu v}\right)}{g^{3}}\right) \\
& =\frac{\left.\partial_{x^{1}}^{n^{\prime}} g^{3}\left(R_{\mu \nu}-\frac{1}{2} g_{\mu \nu} R-8 \pi T_{\mu \nu}\right)\right|_{x_{1}=0}}{\left.\partial_{x^{1}}^{n^{\prime}} g^{3}\right|_{x_{1}=0}}=0 . \tag{1.21}
\end{align*}
$$

Hence, we shall treat any solution of the modified Einstein field equations as a proper solution of the unmodified ones in the following.

### 1.3.2 Metric signature

For a space-time to be physically acceptable, we demand that the signature, i. e. the number of positive and negative eigenvalues of the metric tensor, be fixed [50]. Otherwise it would be possible that in a region of space-time one would have, for example, two timelike coordinate axes or no timelike coordinate axis at all. Our experience does not tell us how to handle such situations, and so it would seem unscientific to speculate about them.

Hence, when allowing for vanishing metric determinants, we have to make sure, that the sign of the metric determinant does not change, when travelling along any curve through space-time. Thus, we find that the roots of the determinant $g$ can only be of even order:

$$
\begin{equation*}
g\left(x^{1}=0\right)=\left.0 \quad \Rightarrow \frac{\partial^{n} g}{\partial x^{1 n}}\right|_{x^{1}=0}=0, \quad \text { for } n<k, \text { eeven. } \tag{1.22a,b}
\end{equation*}
$$

### 1.4 Local conservation of energy and momentum

As shown above, it is in general not possible to transform a metric tensor with at a certain space-time point vanishing determinant to the standard metric of Minkowski space by a regular coordinate transformation, i. e. by a diffeomorphism. However, we should require that it be possible to transform a metric tensor to the metric of Minkowski space-time in a small neighbourhood of any point in order to ensure energy-momentum conservation to hold at least locally [38].

On a Lorentzian manifold, we may at any point introduce normal coordinates and expand the metric tensor in a neighbourhood of that point as in equation (1.4). This metric allows at any point approximately for the ten Killing vector fields of flat Minkowski space-time, which, in Cartesian coordinates ( $T, X, Y, Z$ ), are given by:

$$
\begin{array}{lcc}
\partial_{\mu} T, \quad \partial_{\mu} X, & \partial_{\mu} Y, \quad \partial_{\mu} Z, & \\
X \partial_{\mu} Y-Y \partial_{\mu} X, & Y \partial_{\mu} Z-Z \partial_{\mu} Y, & Z \partial_{\mu} X-X \partial_{\mu} Z,  \tag{1.23}\\
X \partial_{\mu} T+T \partial_{\mu} X, & Y \partial_{\mu} T+T \partial_{\mu} Y, & Z \partial_{\mu} T+T \partial_{\mu} Z .
\end{array}
$$

In accordance with Noether's theorem, we expect these Killing vector fields to be connected with local conservation laws. In order to investigate energy-momentum conservation, we now contract the components of the energy-momentum tensor with the first four of the Killig vector fields $\eta_{\mu}^{(i)}$ of equation (1.23) and integrate over the boundary $\partial V$ of a small sperical volume $V$ of radius $R$, which gives us the net flux of those components of energy and momentum,

$$
\begin{equation*}
\int_{\partial V} T^{\mu v} \eta_{v}^{(i)} \mathrm{d} \sigma_{\mu}=\int_{V}\left(T_{; \mu}^{\mu v} \eta_{v}^{(i)}+T^{\mu v} \eta_{v ; \mu}^{(i)}\right) \mathrm{d} V=\int_{V} T^{\mu v} \eta_{v ; \mu}^{(i)} \mathrm{d} V=\frac{1}{2} \int_{V} T^{\mu v}\left(\eta_{v ; \mu}^{(i)}+\eta_{\mu ; v}^{(i)}\right) \mathrm{d} V . \tag{1.24}
\end{equation*}
$$

In the first step, we used Gauß's divergence theorem, whose validity is itself only ensured in the case of Lorentzian geometry. The divergence of the energy-momentum tensor vanishes by the Bianchi identities, and it is symmetric by definition, explaining steps two and three. In a curved space-time, the integrand of the last integral will not vanish identically. However, since the volume $V$ is small and we used geodesic normal coordinates, the deviation of the Christoffel symbols from zero will approximately be linear. Thus,

$$
\begin{equation*}
=\frac{1}{2} \int_{V} T^{\mu \nu}\left(\eta_{\mu, \nu}^{(i)}+\eta_{v, \mu}^{(i)}+\Gamma_{\mu \nu}^{\kappa} \eta_{\kappa}^{(i)}+\Gamma_{\nu \mu}^{\lambda} \eta_{\lambda}^{(i)}\right) \mathrm{d} V=\int_{V} T^{\mu \nu} \Gamma_{\mu \nu}^{\kappa} \eta_{\kappa}^{(i)} \mathrm{d} V \xrightarrow{R \rightarrow 0} 0 . \tag{1.25}
\end{equation*}
$$

We obtain locally an arbitrarily precise conservation of energy and momentum, as long as the energy-momentum tensor remains finite. The Lorentzian geometry is seen to ensure local conservation of energy and momentum. A similar analysis can be carried out for angular momentum.

### 1.5 Initial value problem

When dealing with space-times with non-Lorentzian geometry, the questions of determinism, solvability of the initial value problem and the fulfillment of Mach's principle arise naturally. One aspect is the question of geodesic completeness and determinism with respect to the motion of test particles, which we will study at a concrete example in detail in chapter 3.

In the description of general relativity as geometrodynamics, giving the geometry of space in the form of a three-metric $g_{i j}$ and the matter distribution on a spacelike hypersurface and its change $\dot{g}_{i j}$ along a nowhere vanishing timelike vector field on it determines the future of the system in a small time step. This can be interpreted as a fulfillment of Mach's principle in general relativity: Remote matter governs the time development of the metric and hence the locally inertial frames at a specific space-time point [58].

There have been attempts to incorporate degeneracies of the metric tensor systematically into the general theory of relativty by rewriting the Einstein-Hilbert action

$$
\begin{equation*}
S=\frac{1}{16 \pi} \int \mathrm{~d}^{4} x R \sqrt{-g} \tag{1.26}
\end{equation*}
$$

in polynomial form, i. e. to formulate it without denominators containing the determinant of the metric tensor. The time development could then-in principle-be determined by means of a variational procedure of the action. However, it turned out that there are many different possibilities of modifications which all yield the standard Einstein field equations for non-vanishing determinants of the metric. Depending on the kind of modified action, different classes of degenerate metrics are then to be considered as regular or singular [4]. It seems that additional input is needed in order to formulate the theory.

We do not consider this any further and restrict ourselves to stating that the common solution of the initial value problem cannot be applied to a wide range of degenerate metrics.

## Chapter 2

## Nonsingular space-time defect

We now consider the nonsingular exact solution of Einstein's field equations given in [46,47], describing a space-time with the nontrivial topology $\mathcal{M}_{b}^{4}=\mathbb{R} \times \mathcal{M}_{b}^{3}=\mathbb{R} \times\left(\mathbb{R} P^{3}-\{\mathrm{pt}\}\right)$. The space-time $\mathcal{M}_{b}^{4}$ can be interpreted as a topological defect with length-scale $b$, i. e. as a small-scale structure in vacuum.

The outline for this chapter is as follows: First, we construct the underlying topological manifold of the space-time defect. Second, we will give a system of coordinate charts, which are suitable for this topology. Then we will solve the Einstein field equations and investigate some properties of this solution.

### 2.1 Construction by surgery

We start with Euclidean three-space, $E_{3}=\left(\mathbb{R}^{3}, \delta_{m n}\right)$, and construct the spatial hypersurface $\mathcal{M}_{b}^{3}$ by performing a surgery. On $E_{3}$ we introduce the standard Cartesian coordinates, in the following called $(X, Y, Z)$, and the standard spherical coordinates $(r, \theta, \phi)$,

$$
\begin{equation*}
\vec{X}=(X, Y, Z)=(r \sin (\theta) \cos (\phi), r \sin (\theta) \sin (\phi), r \cos (\theta)) . \tag{2.1}
\end{equation*}
$$

In the Euclidean space we cut out an open ball of radius $b$ and identify antipodal points on the resulting boundary of the complement space,

$$
\begin{equation*}
\mathcal{M}_{b}^{3}=\left\{\vec{X} \in \mathbb{R}^{3}:|\vec{X}| \geq b\right\} / \sim, \quad \vec{X} \sim-\vec{X} \Leftrightarrow|\vec{X}|=b \tag{2.2a,b}
\end{equation*}
$$

where $b$ is taken to be a positive real number. In figure 2.1 a two-dimensional slice containing the origin of $E_{3}$ is depicted for easier perception of the surgery.

Open neighbourhoods of points away from the defect and not reaching the defect core are simply defined by the induced topology of $\mathbb{R}^{3}$. Open neighbourhoods of points on the defect core can be defined by a construction involving the union of two neighbourhoods in $\mathbb{R}^{3}$. Equipped with this topology, it is possible to show that the manifold $\mathcal{M}_{b}^{3}$ satisfies the Hausdorff property [70]. The case $b=0$ gives the Euclidean three-space itself and shall not be included when we refer to $\mathcal{M}_{b}^{3}$ or $\mathcal{M}_{b}^{4}$, respectively.

The topology of the spatial hypersurface $\mathcal{M}_{b}^{3}$ can be determined by inverting the radial coordinate and mapping it into the unit ball in $E_{3}$, whose antipodal points on the boundary are identified [70]:

$$
\begin{equation*}
\rho: r \mapsto \rho(r)=\frac{b}{r} \tag{2.3}
\end{equation*}
$$

No point will be mapped to the origin of the unit ball, and thus no open cover of the manifold will contain a finite subcover, making the manifold noncompact. In addition, the spatial hypersurface $\mathcal{M}_{b}^{3}$ is orientable, nonsimply-connected and without boundary. It is homeomorphic to the punctured three-dimensional real-projective space,

$$
\begin{equation*}
\mathcal{M}_{b}^{3} \simeq \mathbb{R} P^{3}-\{\mathrm{pt}\} \tag{2.4}
\end{equation*}
$$

Adding the missing point gives a one-point compactification to $\mathbb{R} P^{3}$, since the closed unit ball with antipodal points on the boundary identified is one possible representation of the real projective space [59]. Moreover, the manifold is orientable, since $\mathbb{R} P^{n}$ is orientable for odd $n$ and removing one point from an orientable manifold does not change its orientability [37]. The full space-time manifold is now obtained by forming the product space of the spatial hypersurface with the real line.

### 2.2 Coordinate charts

In order to construct a space-time, it is necessary first to give a set of coordinate charts which together cover all points of the manifold. The time axis is topologically trivial, and hence we concentrate on the spatial hypersurface. We introduce three different coordinate charts $\left(y_{i}, z_{i}, x_{i}\right)$ which are related to the standard spherical coordinates (2.1) in Euclidean three-space by the following coordinate maps.

$$
\begin{align*}
& x_{1}(r, \phi)=\left\{\begin{array}{ll}
r-b, & \cos (\phi) \geq 0 \\
b-r, & \cos (\phi)<0
\end{array} ;\right.  \tag{2.5a-c}\\
& z_{1}(\theta, \phi)= \begin{cases}\theta, & \cos (\phi) \geq 0 \\
\pi-\theta, & \cos (\phi)<0\end{cases}
\end{align*}
$$



Figure 2.1. Construction of defect space-time by surgery. Points inside the white area are removed. Antipodal points on the resulting boundary (black, "defect core") are identified as indicated by the red circles. The black dot in the middle indicates the origin of the Cartesian coordinate system.


Figure 2.2. Two-dimensional slices of the two coordinate charts 1 and 2 and Cartesian coordinate axes $X$ and $Y$. The whole domain of coordinate chart 1 is obtained by rotating the shaded region around the Cartesian $X$ axis. Analogously for charts 2 and 3.

$$
\begin{align*}
& x_{2}(\phi)=\left\{\begin{array}{ll}
\phi, & 0<\phi<\pi \\
\phi-\pi, & 2 \pi>\phi>\pi
\end{array} ; \quad y_{2}(\phi)= \begin{cases}r-b, & \phi<\pi \\
b-r, & \phi>\pi\end{cases} \right. \\
& z_{2}(\theta, \phi)
\end{align*}=\left\{\begin{array}{ll}
\theta, & 0<\phi<\pi  \tag{2.6a-c}\\
\pi-\theta, & \phi>\pi
\end{array}\right] \begin{array}{ll}
x_{3}(\phi) & = \begin{cases}\phi-\pi / 2, & \cos (\phi)<0 \\
\phi-3 \pi / 2, & 3 \pi / 2<\phi<2 \pi ; \\
\phi+\pi / 2, & 0<\phi<\pi / 2\end{cases}  \tag{2.7a-c}\\
y_{3}(\phi)= \begin{cases}\theta, & \cos (\phi) \geq 0 \\
\pi-\theta, & \cos (\phi)<0\end{cases} \\
z_{3}(r, \phi) & = \begin{cases}r-b, & \cos (\phi) \geq 0 \\
b-r, & \cos (\phi)<0\end{cases}
\end{array}
$$

See also figure 2.2. The three coordinate charts $\left(y_{i}, z_{i}, x_{i}\right)$ together form a differentiable atlas of the spatial hypersurface since all transition maps between different coordinate charts are smooth functions, and hence the manifold $\mathcal{M}_{b}^{3}$ is a smooth manifold. Adding an additional time coordinate does not affect the argumentation. This proves that the space-time manifold is locally homeomorphic to subsets of $\mathbb{R}^{4}$.

The above coordinate transformations are also valid for $b=0$, in which case the coordinate charts with added time coordinate simply represent Minkowski space-time $\mathcal{M}_{0}^{4}$ in a form adapted to the coordinate charts of the defect space-time $\mathcal{M}_{b}^{4}$. Throughout this thesis, we will work with the chart- 2 coordinates defined by ( $2.6 \mathrm{a}-\mathrm{c}$ ) and drop the index ' 2 ' for brevity. In chart 2, the coordinate $y$ can be interpreted as a radial coordinate, whereas $z$ and $x$ are a longitudinal and azimuthal coordinate, respectively. The coordinate ranges are

$$
\begin{equation*}
y \in(-\infty ; \infty), \quad z \in(0 ; \pi), \quad x \in(0 ; \pi) \tag{2.8a-c}
\end{equation*}
$$

### 2.3 Solution of Einstein's field equation

In Lorentzian geometry, the most general spherically symmetric solution of Einstein's field equations can be written in the form [18]

$$
\begin{equation*}
\mathrm{d} s^{2}=-\mathrm{e}^{2 \nu(r)} \mathrm{d} t^{2}+\mathrm{e}^{2 \lambda(r)} \mathrm{d} r^{2}+r^{2}\left(\mathrm{~d} \theta^{2}+\sin ^{2}(\theta) \mathrm{d} \phi^{2}\right) \tag{2.9}
\end{equation*}
$$

In order to obtain a smooth solution, we define a second radial coordinate $\zeta \equiv \sqrt{y^{2}+b^{2}}$ and use the following modified spherically symmetric Ansatz, adopted to our special coordinates,

$$
\begin{equation*}
\mathrm{d} s^{2}=-\mathrm{e}^{2 \tilde{\nu}(\zeta)} \mathrm{d} t^{2}+\mathrm{e}^{2 \tilde{\lambda}(\zeta)} \mathrm{d} y^{2}+\zeta^{2}\left(\mathrm{~d} z^{2}+\sin ^{2}(z) \mathrm{d} x^{2}\right) \tag{2.10}
\end{equation*}
$$

Inserting this Ansatz into the vacuum Einstein field equation, $R_{\mu \nu}=0$, and using the boundary condition of an asymptotically flat space-time, the following solution can be obtained [47],

$$
\begin{align*}
& \mathrm{e}^{2 \tilde{v}(\zeta)}=1-\frac{2 M}{\zeta}=1-\frac{2 M}{\sqrt{y^{2}+b^{2}}} \equiv W(\zeta)  \tag{2.11a}\\
& \mathrm{e}^{2 \tilde{\lambda}(\zeta)}=\frac{1-b^{2} / \zeta^{2}}{1-2 M / \zeta}=\left(1-\frac{2 M}{\sqrt{y^{2}+b^{2}}}\right)^{-1} \frac{y^{2}}{y^{2}+b^{2}} \tag{2.11b}
\end{align*}
$$

The parameter $M$ is a constant of integration, and its value has important consequences for the physical properties of the space-time described by the metric (2.10). The line element now takes the form

$$
\begin{equation*}
\mathrm{d} s^{2}=-\left(1-\frac{2 M}{\sqrt{y^{2}+b^{2}}}\right) \mathrm{d} t^{2}+\left(1-\frac{2 M}{\sqrt{y^{2}+b^{2}}}\right)^{-1} \frac{y^{2}}{y^{2}+b^{2}} \mathrm{~d} y^{2}+\left(y^{2}+b^{2}\right)\left(\mathrm{d} z^{2}+\sin ^{2}(z) \mathrm{d} x^{2}\right) \tag{2.12}
\end{equation*}
$$

Notice the vanishing of the metric determinant at the points $y=0$. These points together form a three-dimensional hypersurface and, according to the explanations of section 1.3, we can call (2.12) a regular solution of the standard vacuum field equations.

For $b>2 M$, the coefficient $W(\zeta)$ in equation (2.11a) will never vanish, and the spacetime does not possess an event horizon. This case will in the following be called the defect solution. The case $M=0$ corresponds to a flat space-time and is an important special case due to its simplicity. For $b<2 M$, the coefficient $W(\zeta)$ does vanish at $y= \pm \sqrt{4 M^{2}-b^{2}}$ and the coordinates $y$ and $t$ interchange their timelike and spacelike property, respectively. Hence, a black hole without curvature singularity at the center is obtained. The space-time described by the metric (2.12) then contains closed causal curves inside the event horizon.

We observe that the line-element (2.12) could have been obtained from the line-element of Schwarzschild space-time by setting $(t, r, \theta, \phi)=(t, \zeta, z, x)$ and applying the reparametrization

$$
\begin{equation*}
r \rightarrow \zeta=\sqrt{y^{2}+b^{2}} ; \quad \mathrm{d} r \rightarrow \mathrm{~d} \zeta=\frac{y}{\sqrt{y^{2}+b^{2}}} \mathrm{~d} y \tag{2.13a,b}
\end{equation*}
$$

Since the reparametrization (2.13a) is a diffeomorphism for $y \neq 0$, we conclude that-in accordance with Birkhoff's theorem-away from the defect core the line-element (2.12) is locally equivalent to Schwarzschild space-time. Notice, however, that this is an a posteriori conclusion, since no knowledge of the Schwarzschild solution is necessary to solve the Einstein field equation using the Ansatz (2.10).

As a last remark, we note that, away from the defect, the constant $M$ can be interpreted as a gravitational mass, since for large coordinate $y$ Newton's theory is recovered just as in Schwarzschild space-time. This happens although the energy-momentum tensor used in our calculation vanishes identically. The use of the alternative radial coordinate $\zeta$ effectively cuts out the singular part of the gravitational field of a point mass and hence can be considered as a regularization of the Schwarzschild space-time [46]. Notice that by this regularization not only the metric tensor is changed, but also the global topology of the space-time-the Schwarzschild space-time has topology $\mathbb{R}^{2} \times S^{2}$.

### 2.3.1 Space-time defect with electric charge

It is possible to add an electric charge $Q$ to the black hole, which gives a modified version of the standard Reissner-Nordstöm black hole. Physically, such a defect might be considered as a model for a charged elementary particle such as the electron [24, 46, 80]. Also, contrary to the regulated Schwarzschild black hole, the regulated Reissner-Nordström space-time does not necessarily possess closed causal curves [45]. The metric tensor then takes the form

$$
\begin{array}{r}
\mathrm{d} s^{2}=-\left(1-\frac{2 M}{\sqrt{y^{2}+b^{2}}}+\frac{Q^{2}}{y^{2}+b^{2}}\right) \mathrm{d} t^{2}+\left(1-\frac{2 M}{\sqrt{y^{2}+b^{2}}}+\frac{Q^{2}}{y^{2}+b^{2}}\right)^{-1} \frac{y^{2}}{y^{2}+b^{2}} \mathrm{~d} y^{2}  \tag{2.14}\\
+\left(y^{2}+b^{2}\right)\left(\mathrm{d} z^{2}+\sin ^{2}(z) \mathrm{d} x^{2}\right) .
\end{array}
$$

This metric solves the Einstein field equations for the Coulomb-type energy-momentum tensor

$$
\Theta_{\mu}^{v}=\frac{Q^{2}}{8 \pi\left(y^{2}+b^{2}\right)^{2}} \cdot \begin{cases}-1 & \text { if } \mu=v \in\{t, y\}  \tag{2.15}\\ +1 & \text { if } \mu=v \in\{z, x\} \\ 0 & \text { otherwise }\end{cases}
$$

We restrict ourselves to the case where the space-time described by metric (2.14) does not possess any horizons, i. e. where the defect parameter $b$ satisfies

$$
\begin{equation*}
b>M+\sqrt{M^{2}-Q^{2}} . \tag{2.16}
\end{equation*}
$$

This will also be the case for a regulated naked singularity, i. e. for $Q>M$ and $b \neq 0$. We will consider the charged defect (2.14) again in section 3.3.6.

We see that the regularization has lead to a mass-without-mass and charge-withoutcharge phenomenon: In classical electrodynamics, a combination of Gauß's theorem and Coulomb's law leads to the notion of an electric charge as a $\delta$-like singularity of the charge distribution [34]. This conclusion cannot be drawn here.

### 2.4 Curvature

The Riemann curvature tensor can be easily calculated from the metric (2.12). The independent nonvanishing components are:

$$
\begin{equation*}
R_{y t y}^{t}=\frac{2 M\left(\zeta^{2}-b^{2}\right)}{\zeta^{4}(\zeta-2 M)} ; \quad R_{z t z}^{t}=-\frac{M}{\zeta} ; \quad R_{z y z}^{y}=-\frac{M}{\zeta} ; \quad R_{x z x}^{z}=\frac{2 M \sin ^{2}(z)}{\zeta} \tag{2.17a-d}
\end{equation*}
$$

In the defect case, $b>2 M$, the components of the Riemann curvature tensor are finite over the whole range of the coordinates $(t, y, z, x)$. In the case $M=0$, all components of the Riemann curvature tensor vanish identically, and we deal with flat space-time. Notice that in the case of a coordinate singularity the Riemann tensor may have divergent components, although it remains finite in other coordinate systems. Hence, it is of advantage to consider the coordinate-independent Kretschmann curvature scalar instead of the Riemann tensor given by

$$
\begin{equation*}
K=R_{\mu \nu \rho \sigma} R^{\mu \nu \rho \sigma}=\frac{48 M^{2}}{\zeta^{6}} . \tag{2.18}
\end{equation*}
$$

This quantity is finite for any choice of the parameter $M$, since $\zeta \geq b>0$. Since the Kretschmann scalar is divergent at $r=0$ in Schwarzschild space-time, we know that Schwarzschild space-time and the defect space-time $\mathcal{M}_{b}^{4}$ are not globally equivalent. Also other important curvature invariants remain finite [47]. Since the metric corresponding to the line-element (2.12) is a solution of the vacuum Einstein field equations, its Ricci tensor and Ricci scalar vanish identically,

$$
\begin{equation*}
R_{\mu \nu}=0, \quad R=R_{\mu \nu} g^{\mu \nu}=0 \tag{2.19a,b}
\end{equation*}
$$

in the sense of a limiting procedure to the points $y=0$, as described in section 1.3. Up to symmetry, the non-vanishing components of the affine connection are

$$
\begin{align*}
& \Gamma_{y t}^{t}=\frac{M y}{(\zeta-2 M) \zeta^{2}} ; \quad \Gamma_{t t}^{y}=\frac{M(\zeta-2 M)}{y \zeta^{2}} ; \quad \Gamma_{y y}^{y}=\frac{(\zeta-2 M) b^{2}-M y^{2}}{\zeta^{2}(\zeta-2 M) y} \\
& \Gamma_{z z}^{y}=\frac{\zeta(2 M-\zeta)}{y} ; \quad \Gamma_{x x}^{y}=\frac{\zeta(2 M-\zeta) \sin ^{2}(z)}{y} ; \quad \Gamma_{y z}^{z}=\frac{y}{\zeta^{2}}  \tag{2.20a-i}\\
& \Gamma_{x x}^{z}=-\sin (z) \cos (z) ; \quad \Gamma_{y x}^{x}=\frac{y}{\zeta^{2}} ; \quad \Gamma_{x z}^{x}=\frac{\cos (z)}{\sin (z)} .
\end{align*}
$$

Notice that the affine connection contains elements that diverge for $y \rightarrow 0$, which leads us to expect irregular behaviour of geodesics at the defect core. This will be studied in detail in chapter 3. The affine connection is compatible with the metric tensor,

$$
\begin{equation*}
g_{\alpha \beta ; \gamma}=\frac{\partial g_{\alpha \beta}}{\partial x^{\gamma}}-\Gamma_{\alpha \gamma}^{\mu} g_{\mu \beta}-\Gamma_{\beta \gamma}^{\mu} g_{\alpha \mu}=0 \tag{2.21}
\end{equation*}
$$

and $y=0$ is again a removable singularity of the equation. Condition (2.21) ensures that geodesics are straight lines in locally inertial coordinates [58].

### 2.5 Violation of elementary flatness condition

### 2.5.1 Proof via expansion of metric

The line element (2.12) of the defect space-time $\mathcal{M}_{b}^{4}$ is degenerate at $y=0$ for any choice of the parameters $M$ and $b$, as long as $b \neq 0$. The regularity condition (1.7) cannot be applied to check if the elementary flatness condition is satisfied or violated, since the Killing vector field $\partial_{\mu} x$ does not have vanishing norm on the axis of rotation. Hence, we choose a more elementary method [46].

Consider the space-time point $p$ on the defect core with coordinates $p=\left(0,0, \frac{\pi}{2}, \frac{\pi}{2}\right)$. We define a coordinate system $(t, y, \tilde{z}, \tilde{x})$ related to the old coordintes by

$$
\begin{equation*}
(t, y, z, x) \equiv\left(0,0, \frac{\pi}{2}, \frac{\pi}{2}\right)+\left(t, y, \frac{\tilde{z}}{b}, \frac{\tilde{x}}{b}\right) . \tag{2.22}
\end{equation*}
$$

Expanding the metric around the origin in the new coordinates gives, in first approximation,

$$
\begin{equation*}
\mathrm{d} s^{2} \approx-\left(1-\frac{2 M}{b}\right) \mathrm{d} t^{2}+\left(1-\frac{2 M}{b}\right)^{-1} \frac{y^{2}}{b^{2}} \mathrm{~d} y^{2}+\mathrm{d} \tilde{z}^{2}+\mathrm{d} \tilde{x}^{2} \tag{2.23}
\end{equation*}
$$

By applying the coordinate transformation

$$
\tilde{t}=t \sqrt{1-\frac{2 M}{b}}, \quad \tilde{y}=\left\{\begin{array}{lll}
y^{2} /\left(2 \sqrt{b^{2}-2 M b}\right) & \text { if } & y \geq 0  \tag{2.24a,b}\\
-y^{2} /\left(2 \sqrt{b^{2}-2 M b}\right) & \text { if } & y<0
\end{array},\right.
$$

a patch of space-time is obtained which is, at the space-time point $p$, equal to Minkowski space-time:

$$
\begin{equation*}
\mathrm{d} s^{2} \approx-\mathrm{d} \tilde{t}^{2}+\mathrm{d} \tilde{y}^{2}+\mathrm{d} \tilde{z}^{2}+\mathrm{d} \tilde{x}^{2} . \tag{2.25}
\end{equation*}
$$

Notice that the coordinate transformation (2.24b) is not a diffeomorphism, since its inverse is not differentiable at $y=0$. Also, the transformation (2.24b) is not twice differentiable in $y$. Hence, the coordinate system ( $\tilde{t}, \tilde{y}, \tilde{z}, \tilde{x})$ is not compatible with the differentiable atlas defined in section 2.2. In fact, after also including higher order terms in equation (2.25), one finds that the metric tensor in the coordinate system $(\tilde{t}, \tilde{y}, \tilde{z}, \tilde{x})$ is not even differentiable at the defect core, $\tilde{y}=0$.

In the case of the massless defect it is simpler to convince oneself that the defect spacetime $\mathcal{M}_{b}^{4}$ is not a Lorentzian manifold. Start with the line-element

$$
\begin{equation*}
\mathrm{d} s^{2}=-\mathrm{d} t^{2}+\frac{y^{2}}{y^{2}+b^{2}} \mathrm{~d} y^{2}+\left(y^{2}+b^{2}\right)\left(\mathrm{d} z^{2}+\sin ^{2}(z) \mathrm{d} x^{2}\right) . \tag{2.26}
\end{equation*}
$$

By applying the coordinate transformation

$$
\begin{equation*}
y \rightarrow \zeta=\sqrt{y^{2}+b^{2}} \tag{2.27}
\end{equation*}
$$

one obtains the line-element of Minkowski space-time in spherical coordinates, which certainly is a Lorentzian manifold and clearly can be written locally in the form (1.4), namely by introducing Cartesian coordinates. However, the coordinate transformation (2.27) can only be applied locally and clearly it is not a diffeomorphism.

### 2.5.2 Proof via Gauß-Bonnet theorem

There is a more elegant way to prove the violation of the elementary flatness condition, which is due to [68]. We exploit the fact, that, in the case of Riemannian geometry, there is a connection between the Gaussian curvature $K$ of a compact two-dimensional manifold $\mathcal{M}^{2}$ and its topology, which is given by the famous Gauß-Bonnet theorem [14],

$$
\begin{equation*}
\int_{\mathcal{M}^{2}} K \mathrm{~d} A=2 \pi \chi\left(\mathcal{M}^{2}\right) . \tag{2.28}
\end{equation*}
$$

For a two-dimensional manifold, the Gaussian curvature is simply half of the scalar curvature, $K=R / 2$. The integration is carried out over the surface element $\mathrm{d} A$, and $\chi\left(\mathcal{M}^{2}\right)$ here denotes the Euler characteristic of the manifold.

We now consider the submanifold $\mathcal{M}^{2}$ of $\mathcal{M}_{b}^{3}$, which is shown in figure 2.3. The antipodal points on the dashed line, where the submanifold "intersects" the defect core, are identified and, clearly, $\mathcal{M}^{2}$ is homeomorphic to a hemisphere with antipodal points on the boundary-the equator-identified. This, however, is one possible representation of the real projective plane, $\mathbb{R} P^{2}$, which in turn is a compact two-dimensional manifold and whose Euler characteristic is equal to one [40]. Hence, if our space-time $\mathcal{M}_{b}^{4}$ was a Lorentzian manifold, equation (2.28) should hold on $\mathcal{M}^{2}$. Notice that the non-orientability of $\mathbb{R} P^{2}$ does not prevent us from carrying out a complete surface integral over it and the Gauß-Bonnet theorem can be easily generalized to the non-orientable case [32, section 4.3].

We may now choose the disk $A_{1}$ to lie in the plane $t=0, z=\pi / 2$. The line-element on $A_{1}$ is then given by (neglecting the mass $M$ of the defect),

$$
\begin{equation*}
\left.\mathrm{d} s^{2}\right|_{A_{1}}=\frac{y^{2}}{y^{2}+b^{2}} \mathrm{~d} y^{2}+\left(y^{2}+b^{2}\right) \mathrm{d} x^{2} \tag{2.29}
\end{equation*}
$$

and the Gaussian and scalar curvature associated to (2.29) vanish identically. Hence, the surface $A_{1}$ does not make a contribution to the integral in (2.28). On $A_{2}$, the line-element takes the form

$$
\begin{equation*}
\left.\mathrm{d} s^{2}\right|_{A_{2}}=\left(y^{2}+b^{2}\right) \mathrm{d} z^{2}+\left(y^{2}+b^{2}\right) \sin ^{2}(z) \mathrm{d} x^{2} \tag{2.30}
\end{equation*}
$$

whose Gaussian curvature $K=1 /\left(y^{2}+b^{2}\right)$ is constant for constant radius, which we assume to be much larger than the width of $A_{3}$. The contribution of $A_{2}$ to the integral in (2.28) is easily determined to be $2 \pi$. For the contribution of the edge region $A_{3}$, we argue that the total value of the integral in (2.28) would be $4 \pi$ in the case, where the defect did not exist, as $\mathcal{M}^{2}$ then was homeomorphic to the 2 -sphere, whose Euler characteristic is equal to two. Hence, the surface $A_{3}$ contributes another $2 \pi$ to the integral, and we find

$$
\begin{equation*}
\int_{\mathcal{M}^{2}} K \mathrm{~d} A=4 \pi \neq 2 \pi \tag{2.31}
\end{equation*}
$$

This proves that the geometry cannot be Lorentzian at the defect core. Notice that this proof in principle also works in the case of a massive defect, unless the defect parameter $b$ satisfies a certain condition, where (2.28) happens to be fulfilled.

Figure 2.3. Two dimensional sub-
 manifold $\mathcal{M}^{2}$ in $\mathcal{M}_{b}^{3}$. The disk labeled $A_{1}$ goes through the defect core at the dashed line, where antipodal points are identified. It is joined to the shaded surface $A_{3}$, which we call the edge region. The edge region in turn is joined to the hemisphere $A_{2}$. The $A_{i}$ together form the non-orientable manifold $\mathcal{M}^{2}$.

### 2.6 Stability of defect metric

For a space-time metric to be physically acceptable, we must require that it be stable under small perturbations, i. e. that any perturbation placed on an initial spacelike hypersurface remain small under time development. An instructive analogue for such a perturbation has been given by Regge and Wheeler [66]: Suppose we have a sphere of water, which is held together by its own gravitational forces. If the surface of the sphere of water is disturbed a little bit in one place, the sphere will perform oscillations around its equilibrium and return to its initial shape after a while. The situation is different if the sphere is surrounded by a spherical shell of liquid mercury. This configuration also is in equilibrium, but initially small departures of sphericity of the interface between water and mercury will lead the two layers to interchange their positions. The behaviour is crucially different.

A priori it is not at all clear that a Schwarzschild black hole is in stable equilibrium. It would be conceivable that a small non-spherical perturbation of the Schwarzschild metric would grow large with time, possibly converting the black hole into a naked singularity in the end [81].

For Schwarzschild space-time, a stability analysis has been carried out by Vishveshwara [79] and others. We restrict ourselves to sketching the main ideas. Start from standard Schwarzschild space-time with topology $\mathbb{R}^{2} \times S^{2}$. The perturbation is expressed by adding a small term to the metric,

$$
\begin{equation*}
g_{\mu \nu} \rightarrow g_{\mu \nu}+h_{\mu \nu}, \tag{2.32}
\end{equation*}
$$

where $h_{\mu \nu}$ is chosen such that the perturbed space-time still satisfies the vacuum Einstein field equations

$$
\begin{equation*}
R_{\mu v}(g)+\delta R_{\mu \nu}(h)=\delta R_{\mu v}(h)=0 \tag{2.33}
\end{equation*}
$$

in linear approximation of $h$. Equation (2.33) leads to a radial wave equation of the form [81]

$$
\begin{equation*}
\frac{\partial^{2} f}{\partial t^{2}}=\frac{\partial^{2} f}{\partial r_{*}^{2}}-V_{\text {eff }}\left(r_{*}\right) f, \quad r_{*}=r+2 M \ln \left(\frac{r}{2 M}-1\right), \tag{2.34a,b}
\end{equation*}
$$

where $f$ describes the perturbation, $r_{*}$ is the Regge-Wheeler coordinate and $V_{\text {eff }}$ is an effective potential, which is everywhere positive and goes to zero only at spatial infinity and at the event horizon.

The crucial question is, whether an imaginary frequency of the perturbation's oscillation, signaled by a negative left-hand side in equation (2.34a), is admissible, since then the perturbation would grow exponentially in time. It can be shown [79] that in Schwarzschild space-time any perturbation carrying out an oscillation with purely imaginary frequency must be divergent already on the initial spacelike hypersurface either at spatial infinity or at the event horizon, $r=2 M$, contradicting the assumption that the perturbation be small in the beginning. Hence, it can be concluded that Schwarzschild space-time is inherently stable against small gravitational perturbations.

The existence of an event horizon hence appears to be crucial for the stability of the Schwarzschild line element. We may now apply the transformation $r \rightarrow \zeta=\sqrt{y^{2}+b^{2}}$ to the total metric (2.32). The tensor $h_{\mu \nu}$ still describes a small non-spherical perturbation, and the solutions of equation (2.34a) transform as scalars. In the defect case, $b>2 M$, the event horizon has been cut out by the reparametrization $r \rightarrow \zeta$, and this suggests that gravitational perturbations carrying out oscillations with imaginary frequency are possible. This, in turn, suggests that, within the laws of classical physics, the defect space-time is inherently unstable against small gravitational perturbations. A massive body passing the defect core in some distance could then induce a non-spherical perturbation, which eventually grows large and destroys the geometry.

This supposition will be strengthened by the behaviour of quantum fields near the defect core, which we will study in chapters 4 and 5 . Of course, this does not tell us anything about the final state of such a perturbed defect. Only a detailed analysis can show convincingly whether or not the defect space-time in fact is unstable against small non-spherical perturbations. This is beyond the scope of the present thesis.

## Chapter 3

## Particle motion

The plan for this chapter is as follows: First, we study particle trajectories in the space-time of the massless defect in very much detail. Later, we will add a mass to the defect and obtain explicit solutions for radial trajectories. Finally, we will carry out classical scattering experiments.

In order to calculate geodesics in the defect space-time, we define the Lagrangian

$$
\begin{equation*}
\mathcal{L}=-g_{\mu \nu} \frac{\mathrm{d} x^{\mu}}{\mathrm{d} q} \frac{\mathrm{~d} x^{\nu}}{\mathrm{d} q} \tag{3.1}
\end{equation*}
$$

where $q$ denotes the affine parameter of the geodesic. The Euler-Lagrange equations associated to this Lagrangian are equivalent to the geodesic equations [2]:

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} q} \frac{\partial \mathcal{L}}{\partial\left(\mathrm{~d}_{q} x^{\mu}\right)}-\frac{\partial \mathcal{L}}{\partial x^{\mu}}=0 \quad \Leftrightarrow \quad \frac{\mathrm{~d}^{2} x^{\mu}}{\mathrm{d} q^{2}}+\Gamma_{v \lambda}^{\mu} \frac{\mathrm{d} x^{v}}{\mathrm{~d} q} \frac{\mathrm{~d} x^{\lambda}}{\mathrm{d} q}=0 . \tag{3.2a,b}
\end{equation*}
$$

Using the line-element (2.12), the Lagrangian $\mathcal{L}$ of the space-time $\mathcal{M}_{b}^{4}$ is given by

$$
\begin{equation*}
\mathcal{L}=W(\zeta)\left(\frac{\mathrm{d} t}{\mathrm{~d} q}\right)^{2}-\frac{1}{W(\zeta)} \frac{y^{2}}{y^{2}+b^{2}}\left(\frac{\mathrm{~d} y}{\mathrm{~d} q}\right)^{2}-\left(y^{2}+b^{2}\right)\left[\left(\frac{\mathrm{d} z}{\mathrm{~d} q}\right)^{2}-\sin ^{2}(z)\left(\frac{\mathrm{d} x}{\mathrm{~d} q}\right)^{2}\right] . \tag{3.3}
\end{equation*}
$$

Explicitly, the Euler-Lagrange equations are:

$$
\begin{align*}
& \frac{\mathrm{d}}{\mathrm{~d} q}\left[2\left(\frac{\mathrm{~d} t}{\mathrm{~d} q}\right)\left(1-\frac{2 M}{\sqrt{y^{2}+b^{2}}}\right)\right]=0,  \tag{3.4a}\\
& \frac{\mathrm{~d}}{\mathrm{~d} q}\left[-2\left(\frac{\mathrm{~d} y}{\mathrm{~d} q}\right)\left(1-\frac{2 M}{\sqrt{y^{2}+b^{2}}}\right)^{-1} \frac{y^{2}}{y^{2}+b^{2}}\right]-\frac{\partial \mathcal{L}}{\partial y}=0,  \tag{3.4b}\\
& \frac{\mathrm{~d}}{\mathrm{~d} q}\left[-2\left(\frac{\mathrm{~d} z}{\mathrm{~d} q}\right)\left(y^{2}+b^{2}\right)\right]+2\left(y^{2}+b^{2}\right) \sin (z) \cos (z)\left(\frac{\mathrm{d} x}{\mathrm{~d} q}\right)^{2}=0 .  \tag{3.4c}\\
& \frac{\mathrm{d}}{\mathrm{~d} q}\left[-2\left(\frac{\mathrm{~d} x}{\mathrm{~d} q}\right)\left(y^{2}+b^{2}\right) \sin ^{2}(z)\right]=0, \tag{3.4d}
\end{align*}
$$

Equation (3.4c) is satisfied for the choice of a constant $z$,

$$
\begin{equation*}
z=\frac{\pi}{2}=\text { const. } \tag{3.5}
\end{equation*}
$$

which simplifies the problem of determining geodesics considerably. Notice that, in a spherically symmetric space-time, we can always choose our coordinate system in such a way that the motion is restricted to the equatorial plane, so by the choice (3.5) we do not lose the generality of our consideration. In addition, equations (3.4a) and (3.4d) can be integrated directly and yield two constants of the motion,

$$
\begin{align*}
& \tilde{E}=\left(\frac{\mathrm{d} t}{\mathrm{~d} q}\right)\left(1-\frac{2 M}{\sqrt{y^{2}+b^{2}}}\right)\left[=\frac{E}{m_{0}}\right],  \tag{3.6a}\\
& \tilde{L}=\left(\frac{\mathrm{d} x}{\mathrm{~d} q}\right)\left(y^{2}+b^{2}\right)\left[=\frac{L}{m_{0}}\right] . \tag{3.6b}
\end{align*}
$$

In the case of a massive test particle and if the geodesic parameter $q$ equals proper time $\tau$, the constants $\tilde{E}$ and $\tilde{L}$ can be interpreted as the ratio of energy $E$ and angular momentum $L$ per unit rest mass $m_{0}$, respectively, when measured at infinity [58]. ${ }^{1}$ It would be possible in principle to solve the radial geodesic equation (3.4b) in order to determine the function $y(q)$. However, since we have only one free coordinate left in our calculation, it is simpler to use the Lagrangian (3.3) directly and insert the constants of the motion,

$$
\begin{equation*}
\mathcal{L}=\left(1-\frac{2 M}{\sqrt{y^{2}+b^{2}}}\right)^{-1}\left[\tilde{E}^{2}-\frac{y^{2}}{y^{2}+b^{2}}\left(\frac{\mathrm{~d} y}{\mathrm{~d} q}\right)^{2}\right]-\frac{\tilde{L}^{2}}{y^{2}+b^{2}} . \tag{3.7}
\end{equation*}
$$

For timelike, lightlike and spacelike geodesics, the value of the Lagrangian along the geodesic is constantly $\mathcal{L}=+1,0,-1$, respectively.

### 3.1 Timelike geodesics in massless defect space-time

In order to understand the behaviour of test particles near the defect core better, we investigate the geodesic equation in the case of a massless defect in detail. Besides, we restrict ourselves to timelike geodesics for the moment and choose to parametrize them by proper time, $q=\tau$. Equation (3.7) now takes the form

$$
\begin{equation*}
y^{2}\left(\frac{\mathrm{~d} y}{\mathrm{~d} \tau}\right)^{2}-\left[\left(y^{2}+b^{2}\right)\left(\tilde{E}^{2}-1\right)-\tilde{L}^{2}\right]=0 \tag{3.8}
\end{equation*}
$$

Equation (3.8) is an ordinary differential equation of first order and second degree, since it is of the form [41]

$$
\begin{equation*}
F\left(\tau, y, y^{\prime} \equiv \frac{\mathrm{d} y}{\mathrm{~d} \tau}\right)=L(\tau, y)\left(\frac{\mathrm{d} y}{\mathrm{~d} \tau}\right)^{2}-2 M(\tau, y) \frac{\mathrm{d} y}{\mathrm{~d} \tau}+N(\tau, y)=0 . \tag{3.9}
\end{equation*}
$$

[^2]The nature of the differential equation (3.9) crucially depends on its discriminant

$$
\begin{equation*}
D(\tau, y)=M^{2}(\tau, y)-L(\tau, y) N(\tau, y)=y^{2}\left[\left(y^{2}+b^{2}\right)\left(\tilde{E}^{2}-1\right)-\tilde{L}^{2}\right] . \tag{3.10}
\end{equation*}
$$

There are, roughly speaking, three different cases to be distinguished:

- $D(\tau, y)>0$ : If the discriminant (3.10) is greater than zero, the geodesic equation (3.8) will have two different solutions for every $\tau$.
- $D(\tau, y)<0$ : There are no solutions within the domain of the real numbers.
- $D(\tau, y)=0$ : There is only one solution of equation (3.8). If this domain separates two domains of the first case, the two different solutions will have a contact.

We see that the nature of the geodesic equation (3.8) depends crucially on the constant term

$$
\begin{equation*}
b^{2}\left(\tilde{E}^{2}-1\right)-\tilde{L}^{2} \equiv\left(\tilde{E}^{2}-1\right)\left(b^{2}-d^{2}\right), \tag{3.11}
\end{equation*}
$$

which may be (I) greater, (II) equal to or (III) smaller than zero. For later convenience, we have introduced the quantity $d=\tilde{L} / \sqrt{\tilde{E}^{2}-1}$, which will turn out to be an impact parameter.

Case I: $d>b$
We first assume the expression (3.11) to be smaller than zero, i. e. the impact parameter is greater than $b$. In this case the discriminant (3.10) will vanish in a strip of finite width around $y=0$. The particular solutions of equation (3.8) are obtained by replacing $y^{\prime}$ by a constant real number $C$. It turns out that the resulting equation can only be satisfied for constant $y$, i.e. $C=0$,

$$
\begin{equation*}
y_{\mathrm{I}, \mathrm{part}}(\tau)= \pm \sqrt{d^{2}-b^{2}}=\text { const. } \tag{3.12}
\end{equation*}
$$

The solution (3.12) is an envelope of the singular integral curves. This solution turns out to be unstable in the sense that it depends critically on the initial conditions and does not solve equation (3.4b). The singular integral curves are given by

$$
\begin{equation*}
y_{\mathrm{I}, \operatorname{sing}}(\tau)= \pm \sqrt{\left(\tilde{E}^{2}-1\right)\left(\tau-\tau_{0}\right)^{2}+d^{2}-b^{2}} \tag{3.13}
\end{equation*}
$$

where $\tau_{0}$ is a constant of integration. The corresponding $x$ coordinate of the geodesic is obtained by integrating the conservation equation (3.6b),

$$
\begin{equation*}
x_{\mathrm{I}}(\tau)=\arctan \left(\frac{\sqrt{\tilde{E}^{2}-1}}{d}\left(\tau-\tau_{0}\right)\right)+x_{\mathrm{I}, 0} \tag{3.14}
\end{equation*}
$$

The integration of the conservation equation (3.6a) is trivial in the massless case and we obtain

$$
\begin{equation*}
t(\tau)=\tilde{E} \tau+t_{0} \tag{3.15}
\end{equation*}
$$

A graphical representation of the situation can be found in figure 3.1.


Figure 3.1. Three-dimensional plot of configurations of integral curves of equation (3.8) in case I. Each integral curve (red solid line) can be associated with a curve (red dashed line) on the set of configurations (gray surface). Also plotted is one of the envelopes $y_{1, \text { part }}$ (blue line). Origin of coordinates has been shifted for better visualization. Parameters: $\tilde{E}=$ $\sqrt{2}, b=1, \tilde{L}=\sqrt{1.1}$.

Figure 3.2. As before, but in case II. All integral curves are nondifferentiable at $y=0$. Two half surface curves may be chosen such that they have the same limiting behaviour and together give one geodesic. A second kind of solution of equation (3.8) lies on the lower two surfaces. Parameters: $\tilde{E}=\sqrt{2}, b=1$, $\tilde{L}=d=0$.


Figure 3.3. Three-dimensional plot of configurations of integral curves of equation (3.8) in case III. Each integral curve (red solid line) can be associated with a curve (red dashed line) on the set of configurations (gray surface). Origin of coordinates has been shifted for better visualization. Parameters: $\tilde{E}=\sqrt{2}, b=1, \tilde{L}=1$.

Case II: $d<b$
In this case, the expression (3.11) is positive, and equation (3.8) does not have any particular solutions within the domain of the real numbers. The discriminant $D(\tau, y)$ is greater than zero in the whole $(\tau, y)$-plane, except on the line $(\tau, 0)$. It can be seen in figure 3.2 that any solution runs into a singularity, since its derivative is divergent at $y=0$. Hence, the solution is not of class $C^{1}$. However, we may fit two half solutions together such that they have the same limiting behaviour, which corresponds to a "weakened" kind of differentiability condition for the solution,

$$
\begin{equation*}
\lim _{\epsilon \rightarrow 0^{+}}\left(\frac{y_{\mathrm{II}}^{\prime}(\epsilon)}{y_{\mathrm{II}}^{\prime}(-\epsilon)}\right)=\lim _{\epsilon \rightarrow 0^{+}}\left(\frac{y_{\mathrm{II}}(\epsilon)-y_{\mathrm{II}}(0)}{y_{\mathrm{II}}(0)-y_{\mathrm{II}}(-\epsilon)}\right)=1 . \tag{3.16}
\end{equation*}
$$

Equation (3.16) implies that geodesics through the defect core are antisymmetric under reversal of proper time in a small neighbourhood of $y=0$, as one would intuitively expect. Similar conditions may also be imposed on the other coordinate functions and, by demanding differentiability of the $t$ coordinate, equation (3.6a), we find that this is equivalent to demanding energy conservation. The same argument holds for the $x$ coordinate, equation (3.6b), and angular momentum. Hence, we obtain

$$
y_{\mathrm{II}}(\tau)=\left\{\begin{array}{ll} 
\pm \sqrt{\left(\tilde{E}^{2}-1\right)\left(\tau-\tau_{0}\right)^{2}+2\left(\tau-\tau_{0}\right) \sqrt{\left(\tilde{E}^{2}-1\right)\left(b^{2}-d^{2}\right)}} & \tau \geq \tau_{0}  \tag{3.17}\\
\mp \sqrt{\left(\tilde{E}^{2}-1\right)\left(\tau-\tau_{0}\right)^{2}-2\left(\tau-\tau_{0}\right) \sqrt{\left(\tilde{E}^{2}-1\right)\left(b^{2}-d^{2}\right)}} & \tau<\tau_{0}
\end{array} .\right.
$$

Using equation (3.6b), we obtain

$$
x_{\mathrm{II}}(\tau)=\left\{\begin{array}{ll}
\arctan \left(\frac{\sqrt{\tilde{E}^{2}-1}}{d}\left(\tau-\tau_{0}\right)+\frac{\sqrt{b^{2}-d^{2}}}{d}\right)-\arctan \left(\frac{\sqrt{b^{2}-d^{2}}}{d}\right)+x_{\mathrm{II}, 0} & \tau \geq \tau_{0}  \tag{3.18}\\
\arctan \left(\frac{\sqrt{\tilde{E}^{2}-1}}{d}\left(\tau-\tau_{0}\right)-\frac{\sqrt{b^{2}-d^{2}}}{d}\right)+\arctan \left(\frac{\sqrt{b^{2}-d^{2}}}{d}\right)+x_{\mathrm{II}, 0} & \tau<\tau_{0}
\end{array} .\right.
$$

When transforming back to the original Cartesian coordinates of section 2.1 before the surgery, the situation becomes clearer, see figure 3.4.

The solution of the geodesic equation takes an especially simple form when considering radial geodesics, i. e. when setting the impact parameter $d$ equal to zero. Taking the integration constant $\tau_{0}$ also to be zero, we obtain

$$
y_{\mathrm{II}}(\tau)=\left\{\begin{array}{ll} 
\pm \sqrt{B^{2} \tau^{2}+2 B \tau} b & \tau \geq 0  \tag{3.19}\\
\mp \sqrt{B^{2} \tau^{2}+2 B \tau} b & \tau<0
\end{array}, \quad B=\frac{1}{b} \sqrt{\tilde{E}^{2}-1} .\right.
$$

We now want to take a closer look at the behaviour of geodesics near the defect core. It is clear that any geodesic finally runs into $y=0$. Locally, the derivative of the $y$ coordinate of the geodesic is given by

$$
\begin{equation*}
\frac{\mathrm{d} y}{\mathrm{~d} \tau}= \pm \frac{1}{y} \sqrt{\left(\tilde{E}^{2}-1\right)\left(y^{2}+b^{2}-d^{2}\right)} \tag{3.20}
\end{equation*}
$$

i. e. the geodesics are not differentiable when passing the defect core, $y \rightarrow 0$. In order to understand the nature of the singularity better, we use the concepts of singularity theory in complex analysis. In order to circumvent the singularity at $y=0$, we could try to extend the domain of definition of equation (3.8) to the complex plane. The solution $y(\tau)$ should then be a holomorphic function which coincides with (3.17) when restricted to the real numbers.

However, this is impossible. Clearly, the point $y=0$ is not a removable singularity of a holomorphic function, since the function's derivative is divergent. Further, it cannot be a pole, since multiplying the solution $y(\tau)$ by an arbitrary integral power $\tau^{n}$ does not make the expression regular, since the resulting expression has a diverging $(n+1)$-th derivative,

$$
\begin{equation*}
\lim _{\epsilon \rightarrow 0 \pm}\left(\left.\frac{\mathrm{d}^{n+1}}{\mathrm{~d} \tau^{n+1}} \tau^{n} y(\tau)\right|_{\tau=\epsilon}\right)=\infty \tag{3.21}
\end{equation*}
$$

Hence the point $y=0$ is an essential singularity of the solution. In fact, it is known that a singularity of a solution to a differential equation occuring at a point, where the solution vanishes, is a branch point. It is believed that single-valuedness around a singular point is a necessary condition for a differential equation to be integrable [51], and we conclude that equation (3.8) cannot be solved globally in this case. Also, notice that the second derivative is never matched by condition (3.16) although the geodesic equation is a differential equation of second order. Similarly, we find from equation (3.18) that the $x$ coordinate is only once but not twice differentiable in $\tau$.

## Case III: $d=b$

In this special case, there are only particular solutions. Expression (3.11) vanishes and after inserting the Ansatz $y^{\prime}=C$ into equation (3.8), we obtain

$$
\begin{equation*}
y^{2} C^{2}-y^{2}\left(\tilde{E}^{2}-1\right)=0 \tag{3.22}
\end{equation*}
$$

The geodesics are then linearly dependent on the proper time or constant,

$$
\begin{equation*}
y_{\mathrm{III}}(\tau)= \pm \sqrt{\tilde{E}^{2}-1}\left(\tau-\tau_{0}\right)+y_{\mathrm{III}, 0} . \tag{3.23}
\end{equation*}
$$

It is also possible that the solution vanishes identically. The integral curves are depicted in figure 3.5. By inserting the constant solution for $\tilde{E}=1$ with $y_{\mathrm{II}, 0} \neq 0$ into equation (3.4b), we find that this is only possible for $\tilde{L}=0$. Hence, geodesics which are constant in space are obtained for $\tilde{E}=1$ and $\tilde{L}=0$, in which case the impact parameter $d$ is undefined. The integration of equation (3.6b) is analogous to case I, equation (3.14). For the constantly vanishing solution, we obtain

$$
\begin{equation*}
x_{\mathrm{III}}(\tau)= \pm \tau \frac{\sqrt{\tilde{E}^{2}-1}}{b}+x_{\mathrm{III}, 0} . \tag{3.24}
\end{equation*}
$$

This describes a particle sliding on the surface of the defect core. Another observation is that the solutions (3.23) are valid globally although the geodesics pass $y=0$. It is interesting to view the geodesic in Cartesian-like coordinates, figure 3.5. Intuitively, it is not clear why the test particle follows the calculated geodesic, and one could be lead to speculate that the decision is made by the coordinates themselves.



Figure 3.4. Geodesic (red solid line) through defect core in massless defect space-time, displayed in Cartesianlike, dimensionless coordinates. The impact parameter $d$ is smaller than b, case II. Arrows indicate increasing proper time. Angular momentum is conserved. Values of parameters are $\tilde{E}=2, b=1, d=0.58, \tilde{L}=1$, $\Delta \varphi=70.5^{\circ}$. The deflection angle can be defined by intersecting the asymptotes to the trajectory.

Figure 3.5. Timelike geodesic (red solid line) touching defect core, case III. Arrows indicate increasing proper time. Intuitively, one could have expected the particle to follow the lower, dashed path equally well as it would have done if the calculation had been carried out in Cartesian coordinates. Values of parameters are $\tilde{E}=\sqrt{2}, b=d=1, \tilde{L}=1$, $\Delta \varphi=180^{\circ}$. Also shown is the solution constantly vanishing in y (blue, dashed line).

### 3.1.1 Deflection angle

In a Cartesian-like coordinate system, we may compute the angle of deflection which the test particle undergoes, when passing the defect core. This angle is obtained by subtracting from $\pi$ the sum of the incident and emergent angle of the trajectory relative to the surface of the defect core. The situation is also shown in figure 3.4, where the angle of deflection appears as the intersection angle of the asymptotic trajectories. One finds, for $d<b$,

$$
\begin{equation*}
\Delta \varphi=\pi-2 \arctan \left(\frac{\sqrt{b^{2}-d^{2}}}{d}\right) . \tag{3.25}
\end{equation*}
$$

### 3.1.2 Geodesic deviation

One possibility to define curvature is by geodesic deviation, i. e. by the behaviour of a family of nearby geodesics. We may describe the geodesics by their affine parameter $\tau$ and the impact parameter $d, x^{\mu}=x^{\mu}(\tau, d)$. Further, we may define their tangent vector and deviation vector,

$$
\begin{equation*}
t^{\mu}(\tau, d)=\frac{\partial x^{\mu}}{\partial \tau}, \quad \quad w^{\mu}(\tau, d)=\frac{\partial x^{\mu}}{\partial d} \tag{3.26a,b}
\end{equation*}
$$

The second covariant derivative of the deviation vector along the geodesics is then related to the Riemann curvature tensor [2],

$$
\begin{equation*}
\frac{\mathrm{D}^{2} w^{\mu}}{\mathrm{D}^{2} \tau}=R_{\alpha \beta \gamma}^{\mu} t^{\alpha} w^{\beta} t^{\gamma} \tag{3.27}
\end{equation*}
$$

We may now consider a plane "wave-front" of test particles travelling, say, in the Cartesian $Y$ direction and calculate the geodesic deviation (3.27). The calculation is tedious, and we do not reproduce it here. The outcome is basically the same as in section 1.3: It is found that the expression on the left-hand side of equation (3.27) constantly vanishes until the wave-front hits the defect core, where it is indeterminate, as it contains derivatives of not twice differentiable functions. Hence, the Riemann curvature tensor vanishes globally in the sense of a limiting procedure to the points, where its value is undefined.

### 3.2 Radial Geodesics in massive defect space-time

### 3.2.1 Lightlike radial geodesics

In the case of lightlike particles, the value of the Lagrangian is equal to zero and equation (3.7) simplifies considerably in the case of radial geodesics, $\tilde{L}=0$, since the mass term $M$ drops out. Equation (3.7) then takes the simple form

$$
\begin{equation*}
F\left(q, y, y^{\prime}\right)=y^{2}\left(\frac{\mathrm{~d} y}{\mathrm{~d} q}\right)^{2}-\left(y^{2}+b^{2}\right) \tilde{E}^{2}=0 \tag{3.28}
\end{equation*}
$$

Notice that, in the lightlike case, the parameter $\tilde{E}$ actually becomes infinite, as the particle has zero rest mass, see equation (3.6a). We avoid this problem by rescaling the affine parameter, $\lambda \equiv \tilde{E} q$. Similarly as before, we find that the derivative of the geodesic by the new affine parameter $\lambda$ is divergent at the defect core, and hence the solution possesses a branch point at $y=0$. Nevertheless, equation (3.28) can locally be integrated easily,

$$
y(\lambda)=\left\{\begin{array}{ll}
+\sqrt{\left(\lambda-\lambda_{0}\right)^{2}+2 b\left(\lambda-\lambda_{0}\right)} & \lambda \geq \lambda_{0}  \tag{3.29}\\
-\sqrt{\left(\lambda-\lambda_{0}\right)^{2}-2 b\left(\lambda-\lambda_{0}\right)} & \lambda<\lambda_{0}
\end{array},\right.
$$

where we have used the same boundary conditions as while deriving equation (3.17). We observe that the solution (3.29) shows the same singular behaviour at the defect core as in


Figure 3.6. Coordinates $t$ (blue dashed line) and $y$ (red solid line) of lightlike radial geodesic in massive defect space-time as functions of affine parameter $\lambda$. Coordinates in dimensionless units. The curve $t(\lambda)$ is differentiable at the defect core, the curve $y(\lambda)$ not. Values of parameters are $b=1, M=0.45, \lambda_{0}=0$.
the case of timelike geodesics. In the lightlike case, it is possible to integrate equation (3.6a) analytically, and one obtains

$$
t(\lambda)=\left\{\begin{array}{ll}
\lambda-\lambda_{0}+2 M \ln \left(1+\frac{\lambda-\lambda_{0}}{b-2 M}\right) & \lambda \geq \lambda_{0}  \tag{3.30}\\
\lambda-\lambda_{0}-2 M \ln \left(1-\frac{\lambda-\lambda_{0}}{b-2 M}\right) & \lambda<\lambda_{0}
\end{array} .\right.
$$

See also figure 3.6. Eliminating the parameter $\lambda$ locally, one finds that the derivative of the $y$ coordinate as a function of the coordinate time $t$ is again divergent at the defect core $y=0$. Further, in the case of a massive defect, the time coordinate of the trajectory of a lightlike particle passing the defect core as a function of the affine parameter $\lambda$ is everywhere differentiable in the defect case $b>2 M$, but not twice differentiable at $\lambda=\lambda_{0}$. Since the geodesic equation contains second derivatives, we also cannot call it a global solution. The later kind of singular behaviour vanishes, if the mass $M$ of the defect is set equal to zero. Hence, including the mass term of the line-element (2.12) may change the singularity structure of the geodesic equation at the defect core. This is not too surprising, since after including the mass term in the metric, the affine connection (2.20a-i) contains additional non-vanishing elements.

### 3.2.2 Timelike radial geodesics

When the mass term in equation (3.7) is included, the problem of finding a solution to the geodesic equation in the timelike case is harder to solve, at least if one wishes to have an explicit solution. We first wish to gain a qualitative understanding of the solutions. In the case of radial geodesics, $\tilde{L}=0$, equation (3.7) simplifies to

$$
\begin{equation*}
F\left(\tau, y, y^{\prime}\right)=y^{2}\left(\frac{\mathrm{~d} y}{\mathrm{~d} \tau}\right)^{2}-\left[\left(\tilde{E}^{2}-1\right)\left(y^{2}+b^{2}\right)+2 M \sqrt{y^{2}+b^{2}}\right]=0 . \tag{3.31}
\end{equation*}
$$

We find that, also in this case, the solutions $y(\tau)$ have a branch point when reaching the defect core, since their derivative is divergent at $y=0$,

$$
\begin{equation*}
\frac{\mathrm{d} y}{\mathrm{~d} \tau}= \pm \frac{1}{y} \sqrt{\left(\tilde{E}^{2}-1\right)\left(y^{2}+b^{2}\right)+2 M \sqrt{y^{2}+b^{2}}} \tag{3.32}
\end{equation*}
$$

Equation (3.31) is again of the form (3.9), and its discriminant $D$ is given by

$$
\begin{equation*}
D(\tau, y)=y^{2}\left[\left(\tilde{E}^{2}-1\right)\left(y^{2}+b^{2}\right)+2 M \sqrt{y^{2}+b^{2}}\right] . \tag{3.33}
\end{equation*}
$$

The zeroes of equation (3.33) are given by

$$
\begin{equation*}
y_{0}=0, \quad y_{ \pm}= \pm \frac{\sqrt{4 M^{2}-\left(\tilde{E}^{2}-1\right)^{2} b^{2}}}{\tilde{E}^{2}-1} \tag{3.34a,b}
\end{equation*}
$$

and since we have assumed $b>2 M$, the zeroes $y_{ \pm}$are real-valued iff $\tilde{E}<1$. Differentiating the discriminant (3.33) by $y$ and evaluating at $y_{ \pm}$, we find that the discriminant changes its sign at $y_{ \pm}$. Hence, there will be a strip of width $2 y_{+}$around $y_{\tilde{E}}=0$ in which the test particle oscillates, and $y_{ \pm}$are the two turning points, see figure 3.7. If $\tilde{E}>1$, there will be no turning point, hence the test particle is unbound and goes to infinity with finite asymptotic velocity. The case $\tilde{E}=1$ is a limiting case, in which the turning points lie at infinity,

$$
\begin{equation*}
\lim _{\tilde{E} \rightarrow 1-}\left(y_{ \pm}\right)=\mp \infty . \tag{3.35}
\end{equation*}
$$

In the following, we will solve equation (3.31) in the three cases $\tilde{E}<1$ (I), $\tilde{E}>1$ (II) and $\tilde{E}=1$ (III). In order to integrate equation (3.31), it is easier to go temporarily back to the corresponding equation associated to the Schwarzschild line-element,

$$
\begin{equation*}
\zeta\left(\frac{\mathrm{d} \zeta}{\mathrm{~d} \tau}\right)^{2}-\left[\left(\tilde{E}^{2}-1\right) \zeta+2 M\right]=0 \tag{3.36}
\end{equation*}
$$

Case I: $\tilde{E}<1$
We first consider the case $\tilde{E}<1$, in which the test particle will be bound in the gravitational potential of the defect. Following [58], we introduce the quantity

$$
\begin{equation*}
R=\frac{2 M}{1-\tilde{E}^{2}} . \tag{3.37}
\end{equation*}
$$

We may now integrate equation (3.36), which gives us an implicit solution,

$$
\begin{equation*}
\left(\tau-\tau_{0}\right) \sqrt{1-\tilde{E}^{2}}= \pm \int_{b}^{\zeta(\tau)} \frac{\mathrm{d} \zeta}{\sqrt{\frac{R}{\zeta}-1}}=\left.\mp \frac{R}{2} \arctan \left(\frac{1}{2} \frac{R-2 \zeta}{\sqrt{R \zeta-\zeta^{2}}}\right) \mp \sqrt{R \zeta-\zeta^{2}}\right|_{b} ^{\zeta(\tau)} \tag{3.38}
\end{equation*}
$$



Figure 3.7. Surface of configurations of solutions to geodesic equation in case of bound timelike radial geodesics in massive defect spacetime (gray, semi-transparent surface) and one possible configuration (red, dashed line). Also shown are the projections onto the $(\tau, y)$ and ( $\tau, y^{\prime}$ ) plane. Derivatives $y^{\prime}$ are always divergent at $y=0$. Parameters: $M=$ $0.25, \tilde{E}=\sqrt{0.9}, b=1$. The envelope (blue line) is an unstable solution of (3.36)

This solves the problem of determining timelike geodesics in the massive defect space-time in principle. Equation (3.38) is not very useful in practice, however, since it is a transcendental equation and cannot be solved for $\zeta$ directly. In order to obtain an explicit representation of the coordinate $\zeta$, we use the following Ansatz for a local parametrization of the radial $\zeta$ coordinate:

$$
\begin{equation*}
\zeta_{\mathrm{I}, \mathrm{loc}}(\eta)=\frac{M}{1-\tilde{E}^{2}}\left(1+\cos \left(\eta \pm \eta_{0}\right)\right) . \tag{3.39}
\end{equation*}
$$

After inserting the Ansatz (3.39) into equation (3.38), we obtain

$$
\begin{equation*}
\tau_{\mathrm{I}, \mathrm{loc}}(\eta)=\frac{M}{{\sqrt{1-\tilde{E}^{2}}}^{3}}\left(\eta+\sin \left(\eta \pm \eta_{0}\right) \mp \eta_{1}\right) . \tag{3.40}
\end{equation*}
$$

The parameter $\eta$ is a so called cycloidal parameter due to its geometric interpretation [58]. Glueing together the local solutions at $\left(\tau_{\mathrm{I}}(0), y_{\mathrm{I}}(0)\right)=(0,0)$, we obtain in terms of the $y$ coordinate:

$$
\begin{align*}
& y_{\mathrm{I}}(\eta)=\left\{\begin{array}{ll}
\sqrt{\left(\frac{M}{1-\tilde{E}^{2}}\left(1+\cos \left(\eta-\eta_{0}\right)\right)\right)^{2}-b^{2}} & 2 \eta_{0} \geq \eta \geq 0 \\
\left.-\sqrt{\left(\frac{M}{1-\tilde{E}^{2}}\right.}\left(1+\cos \left(\eta+\eta_{0}\right)\right)\right)^{2}-b^{2} & -2 \eta_{0} \leq \eta<0
\end{array},\right.  \tag{3.41a}\\
& \tau_{\mathrm{I}}(\eta)= \begin{cases}\frac{M}{\sqrt{1-\tilde{E}^{2}}}\left(\eta+\sin \left(\eta-\eta_{0}\right)+\eta_{1}\right) & 2 \eta_{0} \geq \eta \geq 0 \\
\frac{M}{\sqrt{1-\tilde{E}^{2}}}\left(\eta+\sin \left(\eta+\eta_{0}\right)-\eta_{1}\right) & -2 \eta_{0} \leq \eta<0 .\end{cases} \tag{3.41b}
\end{align*}
$$

We adjust the parameters $\eta_{0}$ and $\eta_{1}$ such that $\left(\tau_{\mathrm{I}}(0), y_{\mathrm{I}}(0)\right)=(0,0)$ is satisfied,

$$
\begin{equation*}
\eta_{0}=\arccos \left(\left(1-\tilde{E}^{2}\right) \frac{b}{M}-1\right), \quad \quad \eta_{1}=\sin \left(\eta_{0}\right) \tag{3.42a,b}
\end{equation*}
$$

Notice that the parameters $\eta_{0}$ and $\eta_{1}$ are always real, since the energy $\tilde{E}$ can never take a value smaller than $\tilde{E}_{\text {min }}$,

$$
\begin{equation*}
\tilde{E}>\tilde{E}_{\min }=\sqrt{1-\frac{2 M}{b}}>0 \tag{3.43}
\end{equation*}
$$

This, of course, is due to the reparametrization (2.13a, b), since we have cut out those parts of the gravitational field where the potential energy is arbitrarily low. The solution (3.41) describes one full oscillation of the test particle and could be continued to an infinite range of the parameter $\eta$. The geodesic is depicted in figure 3.8. Notice that there is an additional constant solution $\zeta \equiv R$ of equation (3.36), which is of course unstable.

We observe that the solution $y_{\mathrm{I}}(\eta)$ also possesses a branch point at $\eta=0$, and hence it is not possible to avoid the singularity by going to complex numbers. The solution $\tau_{\mathrm{I}}(\eta)$, in contrast, is once differentiable but still not twice differentiable at $\eta=0$.

## Case II: $\tilde{E}>1$

If the energy satisfies the condition $\tilde{E}>1$, the asymptotic velocity of the test particle is finite, and it will not be bound by the gravitational potential of the defect. We proceed along the same lines as before and find that, locally, the solution is then given by

$$
\begin{equation*}
\tau \sqrt{\tilde{E}^{2}-1}= \pm \int_{b}^{\zeta(\tau)} \frac{\mathrm{d} \zeta}{\sqrt{1+\frac{R}{\zeta}}}= \pm \frac{R}{2} \ln \left(-\frac{R}{2}+\zeta+\sqrt{-R \zeta+\zeta^{2}}\right) \pm\left.\sqrt{-R \zeta+\zeta^{2}}\right|_{b} ^{\zeta(\tau)} \tag{3.44}
\end{equation*}
$$

By inserting the Ansatz [58]

$$
\begin{equation*}
\zeta_{\mathrm{II}, \mathrm{loc}}(\eta)=\frac{M}{\tilde{E}^{2}-1}\left(\cosh \left(\eta \pm \eta_{0}\right)-1\right) \tag{3.45}
\end{equation*}
$$

into equation (3.44), we obtain

$$
\begin{equation*}
\tau_{\mathrm{II}, \mathrm{loc}}(\eta)=\frac{M}{{\sqrt{\tilde{E}^{2}-1}}^{3}}\left(\sinh \left(\eta \pm \eta_{0}\right)-\eta \pm \eta_{1}\right) . \tag{3.46}
\end{equation*}
$$

In terms of the $y$ coordinate, we then obtain

$$
\begin{align*}
& y_{\mathrm{II}}(\eta)=\left\{\begin{array}{ll}
\sqrt{\left(\frac{M}{\tilde{E}^{2}-1}\left(\cosh \left(\eta-\eta_{0}\right)-1\right)\right)^{2}-b^{2}} & \eta \geq 0 \\
-\sqrt{\left(\frac{M}{\tilde{E}^{2}-1}\left(\cosh \left(\eta+\eta_{0}\right)-1\right)\right)^{2}-b^{2}} & \eta<0
\end{array},\right.  \tag{3.47a}\\
& \tau_{\mathrm{II}}(\eta)=\left\{\begin{array}{ll}
\frac{M}{\sqrt{\tilde{E}^{2}-1}} 3 \\
\left.\frac{M}{\sqrt{\tilde{E}^{2}-1}} 3 \sinh \left(\eta-\eta_{0}\right)-\eta+\eta_{1}\right) & \eta \geq 0
\end{array} .\right. \tag{3.47b}
\end{align*}
$$

Imposing the boundary conditions $y(0)=0$ and $\tau(0)=0$ leads to

$$
\begin{equation*}
\eta_{0}=\operatorname{arcosh}\left(\left(\tilde{E}^{2}-1\right) \frac{b}{M}+1\right), \quad \quad \eta_{1}=\sinh \left(\eta_{0}\right) \tag{3.48a,b}
\end{equation*}
$$



Figure 3.8. Timelike radial geodesics through defect core in massive defect space-time as functions of proper time, corresponding to different energies $\tilde{E}$. Coordinates in dimensionless units. Values of parameters are $\tilde{E}=1.2$ (green dashed line), $\tilde{E}=1$ (red solid line), $\tilde{E}=0.9$ (blue dash-dotted line), $b=1, M=0.25$.

Case III: $\tilde{E}=1$
In the limiting case, $\tilde{E}=1$, equation (3.7) simplifies to

$$
\begin{equation*}
y^{2}(\tau)\left(\frac{\mathrm{d} y}{\mathrm{~d} \tau}\right)^{2}=2 M \sqrt{y^{2}(\tau)+b^{2}} \tag{3.49}
\end{equation*}
$$

Equation (3.49) may be integrated directly and after piecing together the two half-geodesics around the defect core, we obtain

$$
y_{\mathrm{III}}(\tau)=\left\{\begin{array}{ll}
\frac{1}{2} \sqrt{\left(36 M\left(\tau+\frac{1}{3} \sqrt{\frac{2 b^{3}}{M}}\right)^{2}\right)^{2 / 3}-4 b^{2}} & \tau \geq 0  \tag{3.50}\\
-\frac{1}{2} \sqrt{\left(36 M\left(\tau-\frac{1}{3} \sqrt{\frac{2 b^{3}}{M}}\right)^{2}\right)^{2 / 3}-4 b^{2}} & \tau<0
\end{array},\right.
$$

where we have adjusted the integration constant $\tau_{0}$ such that the test particle passes the defect core at $\tau=0$.

### 3.2.3 Tidal forces

One possible characterization of a singularity in general relativity is the existence of infinite curvature in a frame parallely propagated along a geodesic path [26,38]. It is known that there are space-times, in which all curvature invariants remain finite, yet observers moving on geodesics may experience arbitrarily large tidal forces. We shall follow [39] and calculate the Riemann tensor in a frame of an observer falling into the massive defect. We rewrite the metric tensor (2.14) in the form

$$
\begin{equation*}
\mathrm{d} s^{2}=-\frac{F(y)}{G(y)} \mathrm{d} t^{2}+\frac{\mathrm{d} y^{2}}{F(y)}+R^{2}(y)\left(\mathrm{d} z^{2}+\sin ^{2}(z) \mathrm{d} x^{2}\right) . \tag{3.51}
\end{equation*}
$$

In order to obtain the Riemann curvature tensor in a parallel-propagated frame, we first calculate its components in a static frame,

$$
\left(e_{0}\right)_{\mu}=-\sqrt{\frac{F}{G}} \partial_{\mu} t, \quad\left(e_{1}\right)_{\mu}=\frac{\partial_{\mu} y}{\sqrt{F}}, \quad\left(e_{2}\right)_{\mu}=R \partial_{\mu} z, \quad\left(e_{3}\right)_{\mu}=R \sin (z) \partial_{\mu} x . \quad(3.52 \mathrm{a}, \mathrm{~b}, \mathrm{c}, \mathrm{~d})
$$

and then apply a Lorentz boost in the radial direction to obtain the parallel-propagated orthonormale frame. We restrict ourselves to the case, where the observer moves on a timelike radial geodesic in the equatorial plane. The boost parameter is given by [39]

$$
\begin{equation*}
\alpha=\operatorname{arcosh}\left(-\frac{\tilde{E}}{\left(e_{0}\right)_{t}}\right)=\operatorname{arcosh}\left(\tilde{E} \sqrt{\frac{G(y)}{F(y)}}\right) . \tag{3.53}
\end{equation*}
$$

In the boosted frame, the relevant components of the curvature tensor are then given by

$$
\begin{equation*}
R_{\hat{0} 2 \hat{0} 2}=R_{\hat{0} 3 \hat{0} 3}=\frac{M \sqrt{y^{2}+b^{2}}-Q^{2}}{\left(y^{2}+b^{2}\right)^{2}} \tag{3.54}
\end{equation*}
$$

These quantities describe the tidal forces in the transverse directions and remain finite. Using the mass and charge of, say, an electron, we find that the curvature only approaches the Planck scale, if the defect parameter $b$ itself is near the Planck length. Hence, we do not have a parallely propagated curvature singularity at the defect core.

### 3.3 General causal geodesics

The geodesics in the general case are hard to determine explicitly. Since the space-time $\mathcal{M}_{b}^{4}$ is locally diffeomorphic to Schwarzschild space-time, geodesics that do not touch the defect core are related to those in Schwarzschild space-time by the same diffeomorphism. At the defect core, we impose boundary conditions, analogously to the previous sections.

By combining the Lagrangian (3.7) with the conservation equation (3.6b) for angular momentum, we obtain the geodesic equation

$$
\begin{equation*}
y^{2}\left(\frac{\mathrm{~d} y}{\mathrm{~d} x}\right)^{2}=\frac{\left(y^{2}+b^{2}\right)^{3}}{\tilde{L}^{2}}\left[\tilde{E}^{2}-\left(1-\frac{2 M}{\sqrt{y^{2}+b^{2}}}\right)\left(\mathcal{L}+\frac{\tilde{L}^{2}}{y^{2}+b^{2}}\right)\right] \tag{3.55}
\end{equation*}
$$

Equation (3.55) is valid both in the lightlike and in the timelike case. In the following, we will restrict ourselves to lightlike and timelike escape orbits, i. e. those orbits which start and end at spatial infinity.

### 3.3.1 Effective potential

In order to understand the motion of test particles qualitatively, it is useful to look at the effective potential $\tilde{V}_{\text {eff }}$ in which the test particle moves. For a massive test particle, moving
on a timelike geodesic, the squared effective potential is given by

$$
\begin{equation*}
\tilde{V}_{\mathrm{eff}}^{2}=\left(1-\frac{2 M}{\sqrt{y^{2}+b^{2}}}\right)\left(1+\frac{\tilde{L}^{2}}{y^{2}+b^{2}}\right) \tag{3.56}
\end{equation*}
$$

If the energy of the test particle intersects the effective potential at some place, the geodesic will have a turning point there, since according to equation (3.55) the derivative of the radial coordinate with respect to the angular coordinate vanishes. The kinds of possible motions also depend on the parameter $b$, see figure 3.9. If the energy of the test particle is big enough to reach the shaded area, it will pass the defect core.

### 3.3.2 Timelike geodesics

Introducing an inverse radial coordinate,

$$
\begin{equation*}
u(x)=\frac{1}{\sqrt{y^{2}(x)+b^{2}}} \tag{3.57}
\end{equation*}
$$

we can rewrite equation (3.55) as

$$
\begin{equation*}
\left(\frac{\mathrm{d} u}{\mathrm{~d} x}\right)^{2}=\frac{\tilde{E}^{2}}{\tilde{L}^{2}}-(1-2 M u(x))\left(\frac{1}{\tilde{L}^{2}}+u^{2}(x)\right)=2 M u^{3}(x)-u^{2}(x)+\frac{2 M}{\tilde{L}^{2}} u(x)+\frac{\tilde{E}^{2}-1}{\tilde{L}^{2}} . \tag{3.58}
\end{equation*}
$$

The right-hand side of equation (3.58) is a cubic polynomial and possesses three real or complex roots,

$$
\begin{equation*}
\left(\frac{\mathrm{d} u}{\mathrm{~d} x}\right)^{2}=2 M\left(u-u_{1}\right)\left(u-u_{2}\right)\left(u-u_{3}\right) . \tag{3.59}
\end{equation*}
$$

The explicit expressions for the three roots $u_{1}, u_{2}$ and $u_{3}$ are too long to be displayed. Any root can be associated to an intersection of the energy $\tilde{E}$ of the test particle and the effective potential $\tilde{V}_{\text {eff }}$ and marks a turning point of the geodesic. According to Descartes' rule of signs, it may possess two or zero real and positive roots if $\tilde{E}>1$ and three or one real and positive root if $\tilde{E}<1$, respectively. There is always one real negative root for $\tilde{E}>1$ and none for $\tilde{E}<1$. The number of positive roots, in turn, determines the nature of the motion of the test particle.

It is possible to give a solution to equation (3.59) in terms of the sinus amplitudinis function with elliptic modulus $k$ [16]. By restricting the motion of the particle into a given sense of rotation, the solution can be given explicitly as

$$
\begin{align*}
u(x) & =u_{3}+\left(u_{2}-u_{3}\right) \operatorname{sn}^{2}\left(\frac{x}{2} \sqrt{2 M\left(u_{1}-u_{3}\right)} ; k\right) \equiv u_{3}+\left(u_{2}-u_{3}\right) \operatorname{sn}^{2}(\eta(x) ; k),  \tag{3.60a}\\
k & =\sqrt{\frac{u_{2}-u_{3}}{u_{1}-u_{3}}} \tag{3.60b}
\end{align*}
$$

The nature of the motion depends on the sign of the smallest root, $u_{3}$. If it is negative, the motion will be hyperbolic, and the particle escapes to infinity. If positive, the motion will be elliptic. This can also be seen qualitatively in figure 3.9. Bound orbits of massive test particles will emerge, if all three roots are real, at least two of them are positive and $u_{1}>u_{2}>u(x)>u_{3}$ [16]. Clearly, there cannot be any bound orbits of test particles not reaching the defect core in the massless case, $M=0$, since then only one positive root exists, as one can easily check.

### 3.3.3 Lightlike geodesics

In this case, equation (3.55) takes the form

$$
\begin{equation*}
y^{2}\left(\frac{\mathrm{~d} y}{\mathrm{~d} x}\right)^{2}=\frac{\left(y^{2}+b^{2}\right)^{3}}{\tilde{L}^{2}}\left[\tilde{E}^{2}-\frac{\tilde{L}^{2}}{y^{2}+b^{2}}\left(1-\frac{2 M}{\sqrt{y^{2}+b^{2}}}\right)\right] . \tag{3.61}
\end{equation*}
$$

We find that, unlike in the case of timelike geodesics, the differential equation (3.61) only has two independent parameters, the mass $M$ of the defect and the ratio $\tilde{E} / \tilde{L}$, which is, in the lightlike case, the inverse of the impact parameter $d$. Hence, the three roots of the cubic polynomial are not algebraically independent. Introducing again the inverse radial coordinate $u$ analogously to equation (3.57), we find

$$
\begin{equation*}
\left(\frac{\mathrm{d} u}{\mathrm{~d} x}\right)^{2}=2 M u^{3}(x)-u^{2}(x)+\frac{1}{d^{2}}=2 M\left(u-u_{1}\right)\left(u-u_{2}\right)\left(u-u_{3}\right) . \tag{3.62}
\end{equation*}
$$

In the lightlike case, the three roots may be given explicitly in terms of the "perihelion" distance, which we call $P$,

$$
\begin{equation*}
u_{1}=\frac{Q+P-2 M}{4 M P}, \quad u_{2}=\frac{1}{P}, \quad u_{3}=\frac{P-2 M-Q}{4 M P} . \tag{3.63a,b,c}
\end{equation*}
$$

Here we have introduced the abbreviation $Q$ and the perihelion distance in turn can be easily related to the impact parameter,

$$
\begin{equation*}
Q^{2}=(P-2 M)(P+6 M), \quad d=\sqrt{\frac{P^{3}}{P-2 M}} . \tag{3.64a,b}
\end{equation*}
$$

The solution $u(x)$ is then given by

$$
\begin{equation*}
u(x)=-\frac{Q-P+2 M}{4 M P}+\frac{Q-P+6 M}{4 M P} \operatorname{sn}^{2}\left(\frac{x}{2} \sqrt{\frac{Q}{P}} ; k\right), k=\sqrt{\frac{Q-P+6 M}{2 Q}} . \tag{3.65a,b}
\end{equation*}
$$

### 3.3.4 Behaviour at defect core

When computing geodesics through the defect core, it is important to keep in mind that we are actually calculating geodesics which solve the geodesic equations (3.2b). Decomposing the solution $u(x)$, we obtain

$$
\begin{equation*}
\frac{\mathrm{d} u}{\mathrm{~d} x}=\frac{\partial u}{\partial \zeta} \frac{\partial \zeta}{\partial y} \frac{\mathrm{~d} y}{\mathrm{~d} x}=-\frac{1}{\zeta^{2}} \frac{y}{\zeta} \frac{\mathrm{~d} y}{\mathrm{~d} x}=-\frac{y}{\zeta^{3}} \frac{\mathrm{~d} y}{\mathrm{~d} q}\left(\frac{\mathrm{~d} x}{\mathrm{~d} q}\right)^{-1} \tag{3.66}
\end{equation*}
$$

This quantity changes its sign at $y=0$, since, according to equations (3.7) and (3.16),

$$
\begin{equation*}
\frac{\mathrm{d} y}{\mathrm{~d} q}= \pm \frac{\sqrt{y^{2}+b^{2}}}{|y|} \sqrt{\tilde{E}^{2}-\left(1-\frac{2 M}{\sqrt{y^{2}+b^{2}}}\right)\left(\mathcal{L}+\frac{\tilde{L}^{2}}{y^{2}+b^{2}}\right)} \tag{3.67}
\end{equation*}
$$

and the derivative of the $x$ coordinate by the affine parameter $\tau$ is continuous. Hence, solutions of the geodesic equation (3.2b) are not differentiable at the defect core when represented in the form $u=u(x)$. Rather, the first derivative changes its sign. The correct boundary condition at $u(x)=1 / b$ analogous to the one used in section 3.1 is then

$$
\begin{equation*}
\lim _{y \rightarrow 0+}\left(\frac{\mathrm{d} u}{\mathrm{~d} x}\right)=-\lim _{y \rightarrow 0-}\left(\frac{\mathrm{d} u}{\mathrm{~d} x}\right) . \tag{3.68}
\end{equation*}
$$

### 3.3.5 Deflection angle

We first consider the case of hyperbolic motion. Far away from the defect core, the two branches of the hyperbola are straight lines, and the associated asymptotes possess a certain angle of intersection. The motion starts at the smallest positive solution of the equation

$$
\begin{equation*}
u_{3}+\left(u_{2}-u_{3}\right) \operatorname{sn}^{2}(\eta ; k)=0 \tag{3.69}
\end{equation*}
$$

which we label $\eta_{1}$ and which may be associated to an angle $\varphi_{1}$. The test particle escapes to infinity at $2 K(k)-\eta_{1}$, where $K(k)$ is the quarter-period of the sinus amplitudinis,

$$
\begin{equation*}
K(k)=\int_{0}^{\frac{\pi}{2}} \frac{\mathrm{~d} \theta}{\sqrt{1-k^{2} \sin ^{2}(\theta)}} . \tag{3.70}
\end{equation*}
$$

The intersection angle of the asymptotes is then given by

$$
\begin{equation*}
\Delta \varphi=\frac{4 K(k)-4 \eta_{1}}{\sqrt{2 M\left(u_{1}-u_{3}\right)}}=\frac{4 K(k)}{\sqrt{2 M\left(u_{1}-u_{3}\right)}}-2 \varphi_{1} . \tag{3.71}
\end{equation*}
$$

The situation changes if the test particle touches the defect core, i. e. if $u_{2}>1 / b$. In this case, the "perihelion" of the orbit lies inside the area that was cut out by the reparametrization $r \rightarrow \zeta$. The intersection angle of the asymptotes is then given by

$$
\begin{equation*}
\Delta \varphi=\pi+2 \varphi_{0}-2 \varphi_{1}, \tag{3.72}
\end{equation*}
$$



Figure 3.9. Squared effective potential as function of radial Schwarzschild coordinate. The orbit of the particle with energy $\tilde{E}_{1}$ will be elliptic, those of 2 and 3 will be hyperbolic and crossing the defect. In case 3, the particle spirals in for an infinite amount of proper time in the unregulated case and leaves the gravitational potential in finite proper time in the regulated case. Parameters: $b=5, \tilde{L}=\sqrt{5}, M=0.5$, $\tilde{E}_{1}^{2}=0.96, \tilde{E}_{2}^{2}=1.014, \tilde{E}_{3}^{2}=1.13$.

Figure 3.10. Quasi-hyperbolic motion of a massive test particle in massive defect space-time (red, solid line). Also depicted is the motion in Schwarzschild space-time (red, dashed line). The modification of the space-time effectively changes the trajectory by an additional angle of the asymptotic motion (blue lines). Parameters: $M=1 / 2, b=9 / 4$, $\tilde{E}=\sqrt{7 / 5}, \tilde{L}=6 / \sqrt{5}, \varphi_{0}=3.581$, $\varphi_{1}=1.191, \Delta \varphi=1.638 \approx 93.9^{\circ}$.
where $\varphi_{0}$ is the smallest positive solution of the equation

$$
\begin{equation*}
u_{3}+\left(u_{2}-u_{3}\right) \operatorname{sn}^{2}\left(\frac{x}{2} \sqrt{2 M\left(u_{1}-u_{3}\right)} ; k\right)=\frac{1}{b} . \tag{3.73}
\end{equation*}
$$

It is found that the modification of the space-time effectively changes the deflection angle of a test particle in the gravitational potential, see figure 3.10. This would make it possible to detect a macroscopic defect by scattering experiments in the asymptotic region.

### 3.3.6 Charged test particles in modified Reissner-Nordström space-time

We now deviate a little from the track of the rest of the work and return to the charged defect space-time, whose metric is given by (2.14). A test particle with charge per rest mass $\tilde{q}$ moves in the electric potential [35]

$$
\begin{equation*}
A_{\mu}=\frac{Q}{\sqrt{y^{2}+b^{2}}} \partial_{\mu} t \tag{3.74}
\end{equation*}
$$

The geodesic equation is then modified by an additional Coulomb term containing the electromagnetic tensor $\boldsymbol{F}$ associated to the electric potential (3.74) [13],

$$
\begin{equation*}
\frac{\mathrm{d}^{2} x^{\mu}}{\mathrm{d} q^{2}}+\Gamma_{v \lambda}^{\mu} \frac{\mathrm{d} x^{v}}{\mathrm{~d} q} \frac{\mathrm{~d} x^{\lambda}}{\mathrm{d} q}=-\tilde{q} F_{v}^{\mu} \frac{\mathrm{d} x^{v}}{\mathrm{~d} q} \equiv-\tilde{q} g^{\mu \rho}\left(A_{\rho ; v}-A_{v ; \rho}\right) \frac{\mathrm{d} x^{\nu}}{\mathrm{d} q} \tag{3.75}
\end{equation*}
$$

This leads to a first-order radial equation analogous to (3.7) [12, 35],

$$
\begin{equation*}
\frac{y^{2}}{y^{2}+b^{2}}\left(\frac{\mathrm{~d} y}{\mathrm{~d} q}\right)^{2}=\left(\tilde{E}-\frac{Q \tilde{q}}{\sqrt{y^{2}+b^{2}}}\right)^{2}-\left(1-\frac{2 M}{\sqrt{y^{2}+b^{2}}}+\frac{Q^{2}}{y^{2}+b^{2}}\right)\left(\mathcal{L}+\frac{\tilde{L}^{2}}{y^{2}+b^{2}}\right), \tag{3.76}
\end{equation*}
$$

whereas the conservation equation (3.6b) remains unchanged. [In principle, it is possible to add also a magnetic charge to the defect, i.e. to construct a regulated magnetic monopole. The motion of test particles will then no longer be restricted to the equatorial plane. We do not consider this possibility in the following, however.] Using again the $\zeta$ coordinate and parametrizing the trajectory locally by the angular coordinate $x$, we find

$$
\begin{equation*}
\left(\frac{\mathrm{d} \zeta}{\mathrm{~d} x}\right)^{2}=\frac{\tilde{E}^{2}-\mathcal{L}}{\tilde{L}^{2}} \zeta^{4}+\frac{2 M \mathcal{L}-2 \tilde{E} Q \tilde{q}}{\tilde{L}^{2}} \zeta^{3}+\left(\frac{Q^{2}\left(\tilde{q}^{2}-\mathcal{L}\right)}{\tilde{L}^{2}}-1\right) \zeta^{2}+2 M \zeta-Q^{2} \tag{3.77}
\end{equation*}
$$

Equation (3.77) can be simplified by making the $\operatorname{Ansatz} \zeta(x) \equiv 1 / v(x)+\zeta_{\mathrm{R}}$, where $\zeta_{\mathrm{R}}$ denotes the smallest positive root of the quartic on the right-hand side of equation (3.77). This leads to a third order polynomial

$$
\begin{equation*}
\left(\frac{\mathrm{d} v}{\mathrm{~d} x}\right)^{2} \equiv b_{3} v^{3}(x)+b_{2} v^{2}(x)+b_{1} v(x)+b_{0} \tag{3.78}
\end{equation*}
$$

where

$$
\begin{align*}
& b_{0}=\frac{\tilde{E}^{2}-\mathcal{L}}{\tilde{L}^{2}}, \quad b_{1}=4 \frac{\tilde{E}^{2}-\mathcal{L}}{\tilde{L}^{2}} \zeta_{\mathrm{R}}+2 \frac{M \mathcal{L}-\tilde{E} Q \tilde{q}}{\tilde{L}^{2}},  \tag{3.79a,b}\\
& b_{2}=6 \frac{\tilde{E}^{2}-\mathcal{L}}{\tilde{L}^{2}} \zeta_{\mathrm{R}}^{2}+6 \frac{M \mathcal{L}-\tilde{E} Q \tilde{q}}{\tilde{L}^{2}} \zeta_{\mathrm{R}}+\frac{Q^{2}\left(\tilde{q}^{2}-\mathcal{L}\right)}{\tilde{L}^{2}}-1,  \tag{3.79c}\\
& b_{3}=4 \frac{\tilde{E}^{2}-\mathcal{L}}{\tilde{L}^{2}} \zeta_{\mathrm{R}}^{3}+6 \frac{M \mathcal{L}-\tilde{E} Q \tilde{q}}{\tilde{L}^{2}} \zeta_{\mathrm{R}}^{2}+2\left(\frac{Q^{2}\left(\tilde{q}^{2}-\mathcal{L}\right)}{\tilde{L}^{2}}-1\right) \zeta_{\mathrm{R}}+2 M . \tag{3.79d}
\end{align*}
$$

By making a further substitution, we arrive at an even simpler differential equation,

$$
\begin{equation*}
w(x) \equiv \frac{1}{4}\left(b_{3} v(x)+\frac{b_{2}}{3}\right), \quad \Rightarrow\left(\frac{\mathrm{d} w}{\mathrm{~d} x}\right)^{2}=4 w^{3}(x)-g_{2} w(x)-g_{3}, \tag{3.80a,b}
\end{equation*}
$$

where

$$
\begin{equation*}
g_{2}=\frac{b_{2}^{2}}{12}-\frac{b_{1} b_{3}}{4}, \quad \quad g_{3}=\frac{b_{1} b_{2} b_{3}}{48}-\frac{b_{0} b_{3}^{2}}{16}-\frac{b_{2}^{3}}{216} \tag{3.81a,b}
\end{equation*}
$$

Equation (3.80b) is solved by the Weierstraß $\wp$ elliptic function with invariants $g_{2}$ and $g_{3}$. Hence, we obtain as solution of equation (3.77)

$$
\begin{equation*}
\zeta(x)=\frac{b_{3}}{4 \wp\left(x ; g_{2}, g_{3}\right)-\frac{b_{2}}{3}}+\zeta_{\mathrm{R}} . \tag{3.82}
\end{equation*}
$$

Notice that although we cannot give the solution (3.82) in explicit form for reasons of space, it is in principle completely analytic.

Analogously to subsection 3.3.4, the local solutions (3.82) are glued together to form complete trajectories, if the defect core is hit. The angle of deflection can be defined similarly as in subsection 3.3 .5 by use of inverse Weierstraß functions $\wp^{-1}$. If the test particle does not pass the defect, we obtain

$$
\begin{equation*}
\Delta \varphi=2 \wp^{-1}\left(\frac{b_{2}}{12} ; g_{2}, g_{3}\right), \tag{3.83}
\end{equation*}
$$

and if it does, we have

$$
\begin{equation*}
\Delta \varphi=2\left[\wp^{-1}\left(\frac{b_{2}}{12} ; g_{2}, g_{3}\right)-\wp^{-1}\left(\frac{b b_{2}+6 m b_{3}-b_{2} \zeta_{\mathrm{R}}}{12\left(b-\zeta_{\mathrm{R}}\right)}\right)\right] . \tag{3.84}
\end{equation*}
$$

We have used the fact that the Weierstra $\beta \wp$ function is even, $\wp\left(-z ; g_{2}, g_{3}\right)=\wp\left(z ; g_{2}, g_{3}\right)$ [1].

### 3.4 Scattering

In this section, we investigate the scattering of test particles off the various defects in the classical limit. We have to keep in mind that geodesics with different impact parameters may have equal scattering angles, corresponding to different branches of the classical deflection function $\Delta \varphi(d)$. The contributions of different impact parameters then sum up to a total differential cross-section [58],

$$
\begin{equation*}
\left(\frac{\mathrm{d} \sigma}{\mathrm{~d} \Omega}\right)_{\text {classical }}=\sum_{\text {branches } i}\left|\frac{d_{i}(\Delta \varphi)}{\sin (\Delta \varphi)} \frac{\mathrm{d} d_{i}}{\mathrm{~d} \Delta \varphi}\right| . \tag{3.85}
\end{equation*}
$$

### 3.4.1 Massless defect

In the case of a massless defect, the deflection function is clearly injective, as long as the impact parameter $d$ is not greater than $b$. Differentiating equation (3.25) by the impact parameter $d$, we obtain

$$
\begin{equation*}
\frac{\mathrm{d} \Delta \varphi}{\mathrm{~d} d}=\frac{2}{\sqrt{b^{2}-d^{2}}} . \tag{3.86}
\end{equation*}
$$




Figure 3.11. Lightlike geodesics in massive defect space-time with impact parameter varying from $d=0$ to $d=8$. Polar plot of $\zeta$ and $\varphi$ Schwarzschild coordinates. The phenomenon of spiral scattering [28] occurs at a certain critical impact parameter $d_{\text {crit }}$. The light rays then enter the photon sphere at $\zeta=3 \mathrm{M}$. Parameters: $M=1, b=2.5$.

Figure 3.12. Deflection angle $\Delta \varphi$ of timelike geodesics as function of impact parameter d. The functions have two different branches and are not injective. Red dashed lines indicate discontinuities. For non-vanishing mass of the defect, the deflection angle passes smoothly through $\pi$ and hence there is a backward glory at the glory impact parameter $d_{g}$. Parameters: $\tilde{E}=\sqrt{7 / 5}, b=5$, mass $M$ as indicated.

The probability of a test particle to be deflected into a solid angle element of infinitesimal width $\mathrm{d} \Omega$ after having passed the defect core is given by

$$
\begin{equation*}
\frac{\mathrm{d} \sigma}{\mathrm{~d} \Omega}=\frac{d(\Delta \varphi)}{\sin (\Delta \varphi)}\left(\frac{\mathrm{d} d}{\mathrm{~d} \Delta \varphi}\right)=\frac{d(\Delta \varphi)}{\sin (\Delta \varphi)}\left(\frac{\mathrm{d} \Delta \varphi}{\mathrm{~d} d}\right)^{-1}=\frac{b^{2}}{4}=\text { const. } \tag{3.87}
\end{equation*}
$$

We find that the differential cross-section is constant. By integrating the differential crosssection over the complete solid angle $\Omega$, one finds $\sigma=\pi b^{2}$ for the total cross-section, as one could have intuitively expected.

### 3.4.2 Massive defect

Figure 3.12 shows the situation in the case of regulated Schwarzschild space-time. One finds that the deflection function is not continuous with respect to the impact parameter $d$. The differential cross-section for the scattering of massive test particles is shown in figure 3.14. The discontinuity in the deflection function leads to a discontinuity of the differential
cross-section. For $b \gg M$, this occurs when the second root $r_{2}$ is equal to $1 / b$ and the corresponding impact parameter $d=d_{0}$ is given by

$$
\begin{equation*}
d_{0}=b \sqrt{\frac{\left(\tilde{E}^{2}-1\right) b+2 M}{\left(\tilde{E}^{2}-1\right)(b-2 M)}} \xrightarrow{\tilde{E} \rightarrow \infty} b \sqrt{\frac{b}{b-2 M}} . \tag{3.88}
\end{equation*}
$$

Expression (3.88) is divergent for non-vanishing mass $M$ and $\tilde{E} \rightarrow 1$, as one would have expected-a test particle, which is at rest at spatial infinity cannot avoid falling into the defect. For the ratio

$$
\frac{b}{M}<\frac{3 \tilde{E}^{2}+\tilde{E} \sqrt{9 \tilde{E}^{2}-8}-4}{2\left(\tilde{E}^{2}-1\right)} \longrightarrow\left\{\begin{array}{ll}
3 & \tilde{E} \rightarrow \infty  \tag{3.89}\\
4 & \tilde{E} \rightarrow 1
\end{array},\right.
$$

all geodesics with impact parameter $d$ smaller than

$$
\begin{equation*}
d_{0}=\frac{\sqrt{2} M}{2\left(\tilde{E}^{2}-1\right)} \sqrt{27 \tilde{E}^{4}-36 \tilde{E}^{2}+\tilde{E} \sqrt{\left(9 \tilde{E}^{2}-8\right)^{3}}+8} \xrightarrow{\tilde{E} \rightarrow \infty} 3 \sqrt{3} M \tag{3.90}
\end{equation*}
$$

hit the defect core. We find that the scattering is qualitatively different, depending on whether the defect radius $b$ is greater or smaller than the expression in equation (3.89), see also figure 3.11. The corresponding formulae for the lightlike case are obtained by considering the limit $\tilde{E} \rightarrow \infty$.

In the latter case, there is the phenomenon of spiral scattering. The deflection function is divergent, when the impact parameter $d_{0}$, equation (3.90), is approached from the left. There are then infinitely many contributions to the differential cross-section. Nonetheless, the sum in equation (3.85) is convergent, since also the derivative of the deflection function is divergent.

A qualitative difference between the scattering off a massive defect from the one off a massless defect is the existence of a backward glory. Since the deflection function passes smoothly through $\pi$, the corresponding summand in equation (3.85) will be divergent and the differential cross-section behaves near the backward direction as [28]

$$
\begin{equation*}
\left.\frac{\mathrm{d} \sigma}{\mathrm{~d} \Omega}\right|_{\delta \varphi \approx \pi} \approx \frac{a_{1}}{(\pi-\delta \varphi)}, \tag{3.91}
\end{equation*}
$$

where $a_{1}$ is a positive constant. [Notice that, in our notation, $\Delta \varphi$ denotes the classical deflection function and $\delta \varphi$ the angle of observation, which may be different from the former.] This behaviour vanishes, if we assign a negative mass to the defect, see figures 3.15 and 3.16.


Figure 3.13. Timelike geodesics in defect space-time. Impact parameterd ranging from 0 to 10. Parameters: $\tilde{E}=\sqrt{7 / 5}, b=5, M=1$.


Figure 3.14. Differential cross-section of scattering of massive test particles off massive space-time defect $\mathcal{M}_{b}^{4}$ (red, solid line) and Schwarzschild spacetime (blue, dashed line) as function of scattering angle.

### 3.4.3 Modified Reissner-Nordström space-time

When adding a charge to the defect and the massive test particle, the situation does not seem to change substantially, see figures 3.17 and 3.18. The scattering of light rays off a regulated naked Reissner-Nordström singularity is, however, different from the one off the massive defect without charge, see figures 3.19 and 3.20. In this case, no backward glory exists. Instead, we find the new phenomenon of rainbow scattering, in analogy to the phenomenon in optics. This occurs if the classical deflection function has an extremum,

$$
\begin{equation*}
\frac{\mathrm{d} \Delta \varphi}{\mathrm{~d} d_{\text {rainbow }}}=0, \quad \Delta \varphi\left(d_{\text {rainbow }}\right) \equiv \delta \varphi_{\text {rainbow }} \tag{3.92a,b}
\end{equation*}
$$

This leads to a one-sided divergence in the differential cross-section, which is near the rainbow scattering angle approximately of the form [28]

$$
\begin{equation*}
\left.\frac{\mathrm{d} \sigma}{\mathrm{~d} \Omega}\right|_{\delta \varphi \approx \delta \varphi_{\text {rainbow }}} \approx \frac{a_{2}}{\sqrt{\delta \varphi_{\text {rainbow }}-\delta \varphi}} \tag{3.93}
\end{equation*}
$$

where $a_{2}$ is another positive constant, and it has been assumed that the second derivative of the deflection function $\Delta \varphi$ is non-zero at the extremum. On the right of the rainbow scattering angle $\delta \varphi$, there is no contribution to the differential cross-section.


Figure 3.15. Timelike geodesics in regulated defect space-time in case of antigravity. Impact parameter d ranging from 0 to 10. Parameters: $\tilde{E}=\sqrt{12 / 5}, b=5, M=-1$, $d_{0}=3.57, \Delta \varphi_{1}\left(d_{0}\right)=0.84, \Delta \varphi_{2}\left(d_{0}\right)=3.98$.


Figure 3.17. Timelike trajectories of charged test particles in regulated Reissner-Nordström space-time. Impact parameterd ranging from 0 to 10. Parameters: $\tilde{E}=2.5, b=5, M=1$, $Q=1, \tilde{q}=-3$.


Figure 3.16. Differential cross-section of scattering of massive test particles off massive space-time defect $\mathcal{M}_{b}^{4}$ (red, solid line) and unregulated Schwarzschild space-time $\mathcal{M}_{0}^{4}$ (blue, dashed line) as function of scattering angle. The differential cross-section is discontinuous at $\delta \varphi_{1}\left(d_{0}\right)$.


Figure 3.18. Differential cross-section of scattering of massive test particles off charged space-time defect. Parameters: $\tilde{E}=2.5, b=5$, $M=1, Q=1, \tilde{q}=-3$ (red, solid line) and $Q=\tilde{q}=0$ (blue, dashed line).


Figure 3.19. Lightlike geodesics in modified Reissner-Nordström space-time, case of regulated naked singularity. Impact parameter $d$ ranging from 0.2 to 5 . Parameters: $Q=1$, $b=0.5, M=0.5$.


Figure 3.20. Differential cross-section of scattering of photons off regulated naked singularity (red, solid line). There exists a rainbow scattering angle at $\delta \varphi_{\text {rainbow }}=0.536$. Also plotted is the differential cross-section in the unregulated case (blue, dashed line).

### 3.5 Discussion

One of the important results of this section is that geodesics, going through the defect core at non-vanishing incidence angle, necessarily possess an essential singularity and hence cannot be considered as global solutions of the geodesic equation in the strict mathematical sense. Geodesic incompleteness is one possible characterization of singular space-times [30, 38] and it is usually defined by inextendibility of $C^{1}$ curves. ${ }^{2}$ Hence, in this sense, the defect space-time $\mathcal{M}_{b}^{4}$ is singular at the defect core.

Nevertheless, we see that the construction of a continuous function $y(\tau)$ is always possible, which solves the geodesic equation (3.7) locally and which satisfies the "weakened" differentiability condition (3.16). Imposing that condition appears to be a natural working hypothesis although, as already mentioned, the second derivative of the geodesics crossing the defect core with respect to their affine parameter is never well-behaved, and the defining equation of a geodesic does contain second derivatives with respect to the affine parameter [38]. It is only due to the spherical symmetry of the space-time that we can reduce the second-order geodesic equations to first order differential equations.

As figured out in section 3.1, angular momentum is conserved in the motion of test particles, as we would have expected due to the spherical symmetry of the line-element,

[^3]

Figure 3.21. In this situation, a massive test particle is trapped by the defect, although its energy exceeds the gravitational potential. In principle, energy and angular momentum of the test particle can be scaled to arbitrarily high values. Parameters: $b=1.0001, \tilde{L}=\sqrt{10}, M=0.5$, $\tilde{E}^{2}=1.6$.
see figure 3.4. The linear momentum of the test particle, when evaluated far away from the defect, changes, however, even in the case of a massless defect. It is an interesting question, where the missing momentum goes. One could imagine that the missing momentum is transferred to the defect itself. Since the defect remains stationary, this would correspond to an infinite inertial mass of the defect-although we can choose the gravitational mass of the defect to be zero. It should be kept in mind that, when calculating the motion of particles using the geodesic equation, the back-reaction of the particle on the gravitational field is generally neglected. It is not clear if the negligence of the back-reaction is possible in this case. Somewhat counterintuitive is the fact that there are geodesics describing massive particles trapped on the defect core although their energy exceeds the zero-point at infinity and there is no gravitational potential.

Another insight into the geometrical properties of this space-time is gained by the geodesic depicted in figure 3.5. Intuitively, there is no reason why the geodesic should follow one of the both possible paths after touching the defect core and, in fact, the correct course of the geodesic was found by imposing the condition of differentiability of the coordinate function. One could interpret this such that the coordinates in this space-time $d o$ have a physical meaning in the sense that they influence the course of geodesics. When interpreting geodesics as the trajectories of test particles, the resulting physics would then not be coordinate-independent, contrary to the fundamental idea of general relativity.

The most striking difference between the defect space-time and the standard Schwarzschild space-time is of course the fact that test particles may pass the defect core and even oscillate through it. There are even bound oscillatory solutions for energies $\tilde{E}>1$, i. e. particles whose potential energy is above the zero-point at infinity, see figure 3.21. In this way, one could construct bound orbits oscillating through the defect with arbitrarily high energy and angular momentum. These solutions do not seem to be physically realistic. One would expect such an energetic particle to disturb the defect metric and possibly leave the gravitational potential. Notice that these geodesics also exist in the standard Schwarzschild space-time, but in that case they necessarily start at the event horizon and end in the gravitational singularity at the center.

We now want to discuss the scattering experiments of section 3.4 more generally. We notice that defects with negative mass behave differently than those with positive mass as the Schwarzschild solution with negative mass does not possess an event horizon, and hence the defect parameter $b$ may be arbitrarily small. A space-time defect with negative mass $M$ and infinitely small defect parameter $b$ cannot be distinguished by scattering experiments from Schwarzschild space-time with negative mass, as long as massive test particles are used, since infinitely high energies would be necessary to probe the defect core.

Similarly, massive test particles will not hit the core of a regulated naked ReissnerNordström singularity, when the defect parameter $b$ goes to zero, as the effective potential [35]

$$
\begin{equation*}
\tilde{V}_{\mathrm{eff}}^{ \pm}(y)=\frac{Q \tilde{q}}{\sqrt{y^{2}+b^{2}}} \pm \sqrt{\left(1-\frac{2 M}{\sqrt{y^{2}+b^{2}}}+\frac{Q^{2}}{y^{2}+b^{2}}\right)\left(\mathcal{L}+\frac{\tilde{L}^{2}}{y^{2}+b^{2}}\right)} \tag{3.94}
\end{equation*}
$$

is divergent at small radii. This does not hold true in the lightlike case. Notice, however, that the Reissner-Nordström solution describes the space-time geometry for a prescribed energy-momentum tensor $\Theta_{\mu \nu}$ without coupling of the Einstein and Maxwell equations. This is unphysical near the central singularity [58], since a distribution of charged matter, which gives rise to the Coulomb-type field (2.15), necessarily modifies those fields outside its boundary for

$$
\begin{equation*}
r<r_{0}=\frac{Q^{2}}{2 M} \tag{3.95}
\end{equation*}
$$

and hence in order to obtain physically consistent results, we should not choose the defect parameter $b$ to be much smaller than (3.95). Plugging in the charge and mass of the electron, we find

$$
\begin{equation*}
r_{0}=1.4 \cdot 10^{-15} \mathrm{~m} \tag{3.96}
\end{equation*}
$$

which is half the classical radius of the electron. It is known experimentally, however, that the electron does not possess a substructure of that size. In order to account for this effect, one should deal with the coupled Einstein-Maxwell equations.

It is important to know, however, that the naked singularity can in principle be cut out by a regularization, and hence the laws of general relativity alone could be enough to describe, say, an elementary particle. A more realistic model for an electron could possibly be obtained by a modification of the Kerr-Newman metric, which describes a spinning, charged and massive object with gyromagnetic ratio equal to 2 . The Kerr-Newman metric is stable for $|e|>M$, and the fact that it predicts the gyromagnetic ratio for an electron properly might be a sign that classical general relativity and electrodynamics alone are sufficient to describe important aspects of particle physics [11, 17, 43].

Summarizing section 3.4, we have found that it is-at least in principle-possible to distinguish the defects associated to Schwarzschild and Reissner-Nordström space-time by scattering experiments from the unregulated versions. Needless to say, it is questionable if
macroscopic versions of the defects can exist, which may be probed by test particles. There are no objects known other than black holes, which are so compact that they allow for $180^{\circ}$ turns of test particles [58].

Finally, we want to note that it is conceivable to formulate a theory, in which all particles are represented by singularities of the metric tensor and where the particle motion itself is described by the Einstein field equations and not the geodesic equation. In fact, it has been pointed out by Einstein, Infeld and Hoffmann that [22]
"the only equations of gravitation which follow without ambiguity from the fundamental assumptions of the general theory of relativity are the equations for empty space."

It was shown that the vacuum Einstein field equations are sufficient in order to determine the motion of singularities. However, it does not seem to be possible to incorporate electromagnetic effects into this theory [13].

## Chapter 4

## Scalar field

In this chapter we solve the Klein-Gordon equation for a scalar quantum field in the defect space-time $\mathcal{M}_{b}^{4}$ and investigate the behaviour of the solutions. Additionally, we compare the solutions to the situation in Minkowski space-time $\mathcal{M}_{0}^{4}$ in order to find possible effects of the non-Lorentzian geometry and non-trivial topology.

Although we mostly deal with quantum fields of higher spin in nature, the advantage of considering a scalar field is that the important aspects of quantum field theory are already present at this lower level of mathematical complexity.

### 4.1 Klein-Gordon equation in curved space-time

In a general curved space-time, the Klein-Gordon equation for a scalar field $\Phi(t, y, z, x)$ is given by [8],

$$
\begin{equation*}
\frac{1}{\sqrt{-g}} \partial_{\mu}\left(\sqrt{-g} g^{\mu v} \partial_{\nu} \Phi\right)-m_{0}^{2} \Phi=0 \tag{4.1}
\end{equation*}
$$

where $g \equiv \operatorname{det}\left(g_{\mu \nu}\right)$ and $m_{0}$ denotes the mass of the field. Explicitly, the Klein-Gordon equation is given by

$$
\begin{align*}
-\frac{1}{W(\zeta)} \frac{\partial^{2} \Phi}{\partial t^{2}}+W(\zeta) \frac{\zeta^{2}}{y^{2}} \frac{\partial^{2} \Phi}{\partial y^{2}} & +\frac{y \zeta^{2} W^{\prime}(\zeta)+\left(2 y^{2}-b^{2}\right) W(\zeta)}{y^{3}} \frac{\partial \Phi}{\partial y} \\
& +\frac{1}{\zeta^{2}} \frac{\partial^{2} \Phi}{\partial z^{2}}+\frac{\cot (z)}{\zeta^{2}} \frac{\partial \Phi}{\partial z}+\frac{1}{\zeta^{2} \sin ^{2}(z)} \frac{\partial^{2} \Phi}{\partial x^{2}}-m_{0}^{2} \Phi=0, \tag{4.2}
\end{align*}
$$

where $W(\zeta)$ was defined in equation (2.11a). In order to derive an analytic solution to the Klein-Gordon equation, we set the mass $M$ of the defect equal to zero, i. e. $W(\zeta) \equiv 1$,

$$
\begin{equation*}
-\frac{\partial^{2} \Phi}{\partial t^{2}}+\frac{\zeta^{2}}{y^{2}} \frac{\partial^{2} \Phi}{\partial y^{2}}+\frac{2 y^{2}-b^{2}}{y^{3}} \frac{\partial \Phi}{\partial y}+\frac{1}{\zeta^{2}} \frac{\partial^{2} \Phi}{\partial z^{2}}+\frac{\cot (z)}{\zeta^{2}} \frac{\partial \Phi}{\partial z}+\frac{1}{\zeta^{2} \sin ^{2}(z)} \frac{\partial^{2} \Phi}{\partial x^{2}}-m_{0}^{2} \Phi=0 . \tag{4.3}
\end{equation*}
$$

Equation (4.3) can be split into four linear ordinary differential equations by using the following Ansatz,

$$
\begin{equation*}
\Phi(t, y, z, x)=T(t) R(y) Z(z) X(x) \tag{4.4}
\end{equation*}
$$

Using this Ansatz it is possible to separate the variables in the Klein-Gordon equation, and for each separation we obtain one free parameter. We call the three parameters $k, l$ and $m$. It will turn out a posteriori that one has to place restrictions on the values of these parameters. In the defect space-time $\mathcal{M}_{b}^{4}$ the functions $T, R, Z, X$ are solutions of the four ordinary differential equations

$$
\begin{align*}
& \frac{\partial^{2} T_{k}}{\partial t^{2}}+\left(k^{2}+m_{0}^{2}\right) T_{k}(t)=0,  \tag{4.5a}\\
& \frac{y^{2}+b^{2}}{y^{2}} \frac{\partial^{2} R_{k l}}{\partial y^{2}}+\frac{2 y^{2}-b^{2}}{y^{3}} \frac{\partial R_{k l}}{\partial y}+\left(k^{2}-\frac{l(l+1)}{y^{2}+b^{2}}\right) R_{k l}(y)=0,  \tag{4.5b}\\
& \frac{\partial^{2} Z_{l m}}{\partial z^{2}}+\cot (z) \frac{\partial Z_{l m}}{\partial z}+\left(l(l+1)-\frac{m^{2}}{\sin ^{2}(z)}\right) Z_{l m}(z)=0,  \tag{4.5c}\\
& \frac{\partial^{2} X_{m}}{\partial x^{2}}+m^{2} X_{m}(x)=0, \tag{4.5d}
\end{align*}
$$

where the subscripts remind us on which parameters the functions depend. As we are interested only in solutions propagating forward in time, the solution to equation (4.5a) is given by

$$
\begin{equation*}
T_{k}(t) \propto \mathrm{e}^{-i \omega_{k} t}, \quad \quad \omega_{k}=\sqrt{k^{2}+m_{0}^{2}} \tag{4.6a,b}
\end{equation*}
$$

The parameter $k$ will be assumed to be a real number in the following. Equation (4.5c) is the defining equation of the associated Legendre polynomials, and hence we know that $Z_{l m}(z) \propto P_{l m}(\cos (z))$, where the $P_{l m}$ are the associated Legendre polynomials [1]. Equation (4.5d) has the general solution

$$
\begin{equation*}
X_{m}(x)=a_{1} \mathrm{e}^{i m x}+a_{2} \mathrm{e}^{-i m x} \tag{4.7}
\end{equation*}
$$

and by requiring that the field $X_{m}(x)$ be an eigenfunction of the angular momentum operator, we can set the constant $a_{2}$ equal to zero. Introducing additional normalization factors, the angular factors together give then the spherical harmonics,

$$
\begin{equation*}
X_{m}(x) Z_{l m}(z) \equiv N_{l m} P_{l m}(\cos (z)) \mathrm{e}^{i m x}=Y_{l m}(z, x) . \tag{4.8}
\end{equation*}
$$

Notice that equation (4.5c) has only solutions which are nonsingular for $z \in[0 ; \pi]$ if $l$ and $m$ are integers with $0 \leq|m| \leq l$. There are additional half-integer valued solutions, which are still square-integrable [62], but which can be ruled out by the condition mentioned in section 4.3. We now investigate the radial equation (4.5b) in more detail.

### 4.2 Power series solution of radial equation

### 4.2.1 Defect space-time

In this section we investigate equation (4.5b) in detail due to its singular behaviour at $y=0$. In the theory of ordinary differential equations an important distinction is made between irregular singular points and regular singular points. We bring equation (4.5b) now into the form

$$
\begin{equation*}
y^{2} \frac{\partial^{2} R_{k l}}{\partial y^{2}}+y \frac{2 y^{2}-b^{2}}{y^{2}+b^{2}} \frac{\partial R_{k l}}{\partial y}+y^{4}\left(\frac{k^{2}}{y^{2}+b^{2}}-\frac{l(l+1)}{\left(y^{2}+b^{2}\right)^{2}}\right) R_{k l}(y)=0 . \tag{4.9}
\end{equation*}
$$

This is an ordinary differential equation of the form

$$
\begin{equation*}
y^{2} \frac{\partial^{2} R_{k l}}{\partial y^{2}}+y p(y) \frac{\partial R_{k l}}{\partial y}+q(y) R_{k l}(y)=0 \tag{4.10}
\end{equation*}
$$

The functions $p(y)$ and $q(y)$, defined by

$$
\begin{align*}
& p(y)=\frac{2 y^{2}-b^{2}}{y^{2}+b^{2}}=-1+\frac{3}{b^{2}} y^{2}-\frac{3}{b^{4}} y^{4}+\mathcal{O}\left(y^{6}\right) \equiv \sum_{n=0}^{\infty} p_{n} y^{n},  \tag{4.11a}\\
& q(y)=y^{4}\left(\frac{k^{2}}{y^{2}+b^{2}}-\frac{l(l+1)}{\left(y^{2}+b^{2}\right)^{2}}\right)=y^{4}\left(\frac{k^{2}}{b^{2}}-\frac{l(l+1)}{b^{4}}\right)+O\left(y^{6}\right) \equiv \sum_{n=0}^{\infty} q_{n} y^{n}, \tag{4.11b}
\end{align*}
$$

are analytic at $y=0$ and, according to Fuchs's theorem, $y=0$ is then a regular singular point of the differential equation [7]. The radius of convergence of the power series (4.11a, 4.11 b ) is equal to $b$ and thus finite, if $b>0$ :

$$
\begin{align*}
& r_{p}=\frac{1}{\limsup _{n \rightarrow \infty}\left(\sqrt[n]{\left|p_{n}\right|}\right)}=\frac{1}{\lim _{n \rightarrow \infty}\left(\sqrt[n]{\frac{3}{b^{n}}}\right)}=b,  \tag{4.12a}\\
& r_{q}=\frac{1}{\limsup _{n \rightarrow \infty}\left(\sqrt[n]{\left|q_{n}\right|}\right)}=\frac{1}{\lim _{n \rightarrow \infty}\left(\sqrt[n]{\frac{l(l+1)(n-2)}{2 b^{n}}}\right)}=\frac{1}{\lim _{n \rightarrow \infty}\left(\sqrt[n]{\frac{1}{b^{n}}}\right) \lim _{n \rightarrow \infty}\left(\sqrt[n]{\frac{l(l+1)(n-2)}{2}}\right)}=b . \tag{4.12b}
\end{align*}
$$

The point $y=\infty$ is an irregular singular point of equation (4.9), however, as can be seen by making the following change of variables:

$$
\begin{align*}
& w \equiv \frac{1}{y}  \tag{4.13}\\
& \Rightarrow \frac{\partial^{2} \tilde{R}_{k l}}{\partial w^{2}}+\frac{3 b^{2} w}{b^{2} w^{2}+1} \frac{\partial \tilde{R}_{k l}}{\partial w}+\left(\frac{k^{2}}{w^{4}\left(b^{2} w^{2}+1\right)}-\frac{l(l+1)}{w^{2}\left(b^{2} w^{2}+1\right)^{2}}\right) \tilde{R}_{k l}(w)=0 . \tag{4.14}
\end{align*}
$$

The function $\tilde{q}(w)$ corresponding to $q(y)$ in equation (4.10), defined by

$$
\begin{equation*}
\tilde{q}(w)=\frac{k^{2}}{w^{2}\left(b^{2} w^{2}+1\right)}-\frac{l(l+1)}{\left(b^{2} w^{2}+1\right)^{2}}, \tag{4.15}
\end{equation*}
$$

possesses a pole of second order in $w$ and thus cannot be analytic at $w=0$. The points $y=0$ and $y=\infty$ are the only singular points of equation (4.9) for $y \in \mathbb{R}$. Since $y=0$ is a regular singular point, at least one solution to the differential equation can be found by assuming the form of a Frobenius power series around $y=0$ [76]:

$$
\begin{equation*}
R_{k l}^{(1)}(y)=\sum_{n=0}^{\infty} c_{n} y^{n+s}, \tag{4.16}
\end{equation*}
$$

where $s$ and the coefficients $c_{n}$ are to be determined. By inserting this Ansatz into (4.10) one can obtain a recursion formula for the coefficients $c_{n}$,

$$
\begin{equation*}
c_{n}=-\frac{\sum_{i=1}^{n}\left[(n+s-i) p_{i}+q_{i}\right] c_{n-i}}{(n+s)(n+s-1)+(n+s) p_{0}+q_{0}}, \tag{4.17}
\end{equation*}
$$

where the $p_{i}$ and $q_{i}$ are the Taylor coefficients of the functions $p$ and $q$, respectively, when expanded around $y=0$. The coefficient $c_{0}$ can be chosen arbitrarily, which corresponds to the freedom of multiplying the solution (4.16) with a constant factor. For convenience, we set it equal to one. The number $s$ is determined by the so called indicial equation

$$
\begin{equation*}
s^{2}+\left(p_{0}-1\right) s+q_{0}=0 \tag{4.18}
\end{equation*}
$$

This equation will in general have two solutions $s_{1}$ and $s_{2}$. We assume that both $s_{i}$ are real and without loss of generality $s_{1} \geq s_{2}$. We set the parameter $s$ in equations (4.16) and (4.17) equal to $s_{1}$ and define the difference of the two solutions, $\Delta=s_{1}-s_{2}$. A second solution to equation (4.9), which is linearly independent from the first one, is given by

$$
\begin{equation*}
R_{k l}^{(2)}(y)=\alpha R_{k l}^{(1)}(y) \log (y)+\sum_{n=0}^{\infty} d_{n} y^{n+s_{2}} \tag{4.19}
\end{equation*}
$$

In principle, it is possible to define the solution (4.19) a priori also for real negative arguments $y$ by making the replacement $\log (y) \rightarrow \log (|y|)$. However, the point $y=0$ is an essential singularity of the logarithm, and hence one can never include it as part of a solution of equation (4.9). The coefficients $d_{n}$ satisfy the recursion formula

$$
\begin{equation*}
d_{n}=-\frac{\sum_{i=1}^{n}\left[\left(n+s_{2}-i\right) p_{i}+q_{i}\right] d_{n-i}+\alpha\left[\left(2\left(n+s_{2}\right)-1\right) c_{n-\Delta}+\sum_{i=0}^{n-\Delta} c_{n-\Delta-i} p_{i}\right]}{\left(n+s_{2}\right)\left(n+s_{2}-1\right)+\left(n+s_{2}\right) p_{0}+q_{0}} \tag{4.20}
\end{equation*}
$$

Again, the coefficient $d_{0}$ can be chosen arbitrarily, and we set it equal to one for convenience. If the two solutions of the indicial equation differ by an integer, the denominator on the righthand side of equation (4.20) vanishes for $n=\Delta$, i. e. the coefficient $d_{\Delta}$ is undetermined and
can be set to zero ${ }^{1}$. The parameter $\alpha$ is determined by the requirement that also the numerator on the right-hand side of (4.20) vanish for $n=\Delta$. Explicitly, the indicial equation (4.18) is given by

$$
\begin{equation*}
s^{2}-2 s=s(s-2)=0 \tag{4.21}
\end{equation*}
$$

and has the two solutions $s_{1}=2, s_{2}=0$, i. e. their difference is an integer, $\Delta=2$. Using the expansions (4.11a) and (4.11b) it is easy to show that the parameter $\alpha$ in equation (4.19) vanishes generally, i. e. there is no logarithmic term in $R_{k l}^{(2)}(y)$ for any choice of the parameters $k$ and $l$ :

$$
\begin{equation*}
\alpha=-\frac{\sum_{i=1}^{\Delta}\left[\left(s_{1}-i\right) p_{i}+q_{i}\right] d_{\Delta-i}}{\left(2 s_{1}-1\right) c_{0}+p_{0} c_{0}}=-\frac{\left(p_{1}+q_{1}\right) d_{1}+q_{2} d_{0}}{2 c_{0}}=0 . \tag{4.22}
\end{equation*}
$$

The recursion formula (4.20) can therefore be simplified considerably,

$$
\begin{equation*}
d_{n}=-\frac{\sum_{i=1}^{n}\left[\left(n+s_{2}-i\right) p_{i}+q_{i}\right] d_{n-i}}{\left(n+s_{2}\right)\left(n+s_{2}-1\right)+\left(n+s_{2}\right) p_{0}+q_{0}} . \tag{4.23}
\end{equation*}
$$

Since both solutions of the indicial equation are non-negative, and the logarithmic term in equation (4.19) vanishes, all solutions to equation (4.9) are nonsingular in a neighbourhood of $y=0$. Explicitly, the solutions $R_{k l}^{(1)}(y)$ and $R_{k l}^{(2)}(y)$ take the form

$$
\begin{align*}
& R_{k l}^{(1)}(y)=y^{2}\left(1-\frac{3}{4 b^{2}} y^{2}+\frac{l(l+1)-b^{2} k^{2}+15}{24 b^{4}} y^{4}+O\left(y^{6}\right)\right),  \tag{4.24a}\\
& R_{k l}^{(2)}(y)=1+y^{2}+\frac{l(l+1)-b^{2}\left(6+k^{2}\right)}{8 b^{4}} y^{4}+O\left(y^{6}\right) . \tag{4.24b}
\end{align*}
$$

Since the functions $p(y)$ and $q(y)$ have even parity under the replacement $y \rightarrow-y$, their Taylor coefficients $p_{i}$ and $q_{i}$ in the expansion around the point $y=0$ vanish for odd numbers $i[49]$. Since also the first two coefficients $c_{1}$ and $d_{1}$ vanish for all combinations of the parameters $k$ and $l$, it can be seen easily from the recursion formulae (4.17) and (4.23), that also the coefficients $c_{n}$ and $d_{n}$ vanish for odd $n$ :

$$
\begin{equation*}
c_{1}=c_{3}=\cdots=c_{2 n^{\prime}+1}=0, \quad d_{1}=d_{3}=\cdots=c_{2 n^{\prime}+1}=0, \tag{4.25a,b}
\end{equation*}
$$

where $n^{\prime}$ denotes a natural number. As an important outcome of this calculation, we find that all solutions of equation (4.9) and hence also of equation (4.1) have even parity under the transformation $y \rightarrow-y$.

[^4]

Figure 4.1. Radial wave functions in defect space-time $\mathcal{M}_{b}^{4}$ (blue, dashed line) and in Minkowski space-time $\mathcal{M}_{0}^{4}$ (red, solid line). Any radial wave function in $\mathcal{M}_{b}^{4}$ is an even function in $y$. Dimensionless units. Values of parameters: $b=1, l=3, k=1$.

It can be shown that the general solution to equation (4.9) takes the form

$$
\begin{equation*}
R_{k l}(y)=C_{1} R_{k l}^{(1)}(y)+C_{2} R_{k l}^{(2)}(y)=C_{1} j_{l}\left(k \sqrt{y^{2}+b^{2}}\right)+C_{2} y_{l}\left(k \sqrt{y^{2}+b^{2}}\right), \tag{4.26}
\end{equation*}
$$

where $j_{l}$ and $y_{l}$ are the spherical Bessel functions of the first and second kind, respectively. The constant $C_{2}$ can be set to zero by arguing that, far away from the defect, the solution (4.26) should be arbitrarily close to the solution in Minkowski space-time, which will turn out to be given by

$$
\begin{equation*}
R_{k l}^{\left(1, \mathcal{M}_{0}^{4}\right)}(y)=j_{l}(k y) . \tag{4.27}
\end{equation*}
$$

Thus, for given values of the quantum numbers $k, l$ and $m$, the total wave function $\Phi_{k l m}$ can be written in the form

$$
\begin{equation*}
\Phi_{k l m}(t, y, z, x)=C_{k l m} j_{l}\left(k \sqrt{y^{2}+b^{2}}\right) Y_{l m}(z, x) \mathrm{e}^{-i \omega_{k} t} \tag{4.28}
\end{equation*}
$$

where $C_{k l m}$ is an arbitrary normalization constant.

## Imaginary wavenumbers

Notice that equation (4.1) does also possess solutions, which are exponentially growing with time. Suppose we have a massless field, $m_{0}=0$, and choose the wavenumber $k$ to be $k=i \beta$, where $\beta$ is a real positive number. The radial wave function (4.26) then takes the form

$$
\begin{equation*}
R_{\beta l}(y)=\tilde{C}_{1} i_{l}\left(\beta \sqrt{y^{2}+b^{2}}\right)+\tilde{C}_{2} k_{l}\left(\beta \sqrt{y^{2}+b^{2}}\right), \tag{4.29}
\end{equation*}
$$

where $i_{l}$ and $k_{l}$ denote the modified spherical Bessel functions of the first and second kind, respectively, and the integration constants have been redefined. The constant $\tilde{C}_{1}$ has to be set to zero, as we demand that the solution vanish at infinity. Hence, we obtain

$$
\begin{equation*}
\Phi_{\beta l m}(t, y, z, x)=\tilde{C}_{\beta l m} k_{l}\left(\beta \sqrt{y^{2}+b^{2}}\right) Y_{l m}(z, x) \mathrm{e}^{\beta t} . \tag{4.30}
\end{equation*}
$$

On the space-like hypersurface $t=0$, solution (4.30) is a completely regular solution, which can be chosen to be initially arbitrarily small.

### 4.2.2 Minkowski space-time

In order to be able to compare our results of the previous section, we carry out an analogous calculation for Minkowski space-time $\mathcal{M}_{0}^{4}$, written in coordinates adapted to those of the defect space-time. We find that the differential equations for the factors $T_{k}, X_{m}$ and $Z_{l m}$ are not changed in comparison with those for the defect space-time $\mathcal{M}_{b}^{4}$, and we can simply adopt the previous results. Equation (4.9) reads in Minkowski space-time

$$
\begin{equation*}
y^{2} \frac{\mathrm{~d}^{2} R(y)}{\mathrm{d} y^{2}}+2 y \frac{\mathrm{~d} R(y)}{\mathrm{d} y}+\left(k^{2} y^{2}-l(l+1)\right) R(y)=0 . \tag{4.31}
\end{equation*}
$$

Again, equation (4.31) possesses two singular points, a regular singular point at $y=0$ and an irregular singular point at $y=\infty$. The basic nature of the differential equation satisfied by the radial wave function of the scalar field $\Phi$ is hence not altered by introducing the non-trivial topology. Again, we solve equation (4.31) by using the Ansatz of a Frobenius power series. The indicial equation

$$
\begin{equation*}
s^{2}+\left(p_{0}-1\right) s+q_{0}=0 \tag{4.32}
\end{equation*}
$$

possesses the two roots $s_{1}=l$ and $s_{2}=-l-1$ and since, as already mentioned, the quantum numbers $l$ are integers, the difference of the roots is again an integer, $\Delta=2 l+1$. Since $l \geq 0$, the condition $s_{1}>s_{2}$ will always be satisfied. The second solution will become singular at $y=0$, since $-l-1<0$. The functions $p(y)$ and $q(y)$ are their own expansions,

$$
\begin{equation*}
p(y)=2, \quad q(y)=-l(l+1)+k^{2} y^{2}, \tag{4.33a,b}
\end{equation*}
$$

so clearly the convergence radii of these expansions are infinite. The recursion formulae for $c_{n}$ and $d_{n}$ take the simpler forms

$$
\begin{align*}
c_{n} & =-\frac{\sum_{i=1}^{n} q_{i} c_{n-i}}{(n+l)(n+l-1)+(n+l) p_{0}+q_{0}},  \tag{4.34a}\\
d_{n} & =-\frac{\sum_{i=1}^{n} q_{i} d_{n-i}+\alpha\left[(2 n-2 l-3) c_{n-\Delta}+c_{n-\Delta} p_{0}\right]}{(n-l-1)(n-l-2)+(n-l-1) p_{0}+q_{0}} . \tag{4.34b}
\end{align*}
$$

Again, the parameter $\alpha$ vanishes,

$$
\begin{equation*}
\alpha=\frac{\sum_{i=1}^{\perp} q_{i} d_{\Delta-i}}{2 l+1}=\frac{k^{2} d_{\Delta-2}}{2 l+1} \propto d_{\Delta-4} \propto \cdots \propto d_{1}=0, \tag{4.35}
\end{equation*}
$$

since the quantity $\Delta=2 l+1$ is an odd integer for integer $l$. This proves that the solution $R_{k l}^{(2)}(y)$ does not contain a logarithmic term. Hence, the solutions $R_{k l}^{(2)}(y)$ of equation (4.31) possess only poles as singularities and do not possess any essential singularities. Since $l \geq 0$, the solution $R_{k l}^{(1)}(y)$ is nonsingular in a neighbourhood of $y=0$. Explicitly, the solutions are given by,

$$
\begin{align*}
& R_{k l}^{(1)}(y)=y^{l}\left(1-\frac{k^{2}}{4 l+6} y^{2}+\frac{k^{4}}{8(2 l+5)(2 l+3)} y^{4}+O\left(y^{5}\right)\right) \propto j_{l}(k y)  \tag{4.36a}\\
& R_{k l}^{(2)}(y)=y^{-l-1}\left(1+\frac{k^{2}}{4 l-2} y^{2}+\frac{k^{4}}{8(2 l-3)(2 l-1)} y^{4}+O\left(y^{5}\right)\right) \propto y_{l}(k y) . \tag{4.36b}
\end{align*}
$$

The series in equations (4.36) turn out to be proportional to the expansions around the origin of the spherical Bessel functions of the first and second kind, respectively [1].

Regular solutions with imaginary wavenumbers $k$ in analogy to (4.30) do not exist in Minkowski space-time, as any linear combination of modified spherical Bessel functions is divergent either at the origin or at infinity.

### 4.3 Single and multi-valuedness of solutions

In the following sections we want to study the wave functions $\Phi_{k l m}$ of the scalar field in the defect space-time $\mathcal{M}_{b}^{4}$ in more detail. A first observation is that there are solutions of the Klein-Gordon equation, which are multi-valued, i. e. their numerical value at the same space-time point may be different when approaching that point from different directions. It turns out that this is the case for the solutions $\Phi_{k l m}$ with odd $l$. Their numerical value is only defined up to a sign. On the contrary, the solutions $\Phi_{k l m}$ with even $l$ are single-valued.

The possibility of multi-valuedness of a quantum wave function has been investigated by Schrödinger and Pauli already in the early years after the discovery of quantum mechanics. Generally speaking, it is not possible to discard double-valued wave functions a priori. In order to decide, whether or not a multi-valued wave function is acceptable, Pauli proposed the following condition [62]: Let $\left\{u_{l m}\right\}$ be a system of regular or only square-integrable eigenfunctions of the square of the angular momentum operator, $\hat{L}^{2}$, with the same eigenvalue $l$. Applying one of the angular momentum operators $\hat{L}_{i}$ on one of these eigenfunctions must not yield a function, which is linearly independent of the $u_{l m}$. Stated mathematically,

$$
\begin{equation*}
\hat{L}_{i} u_{l m}=\sum_{m^{\prime}} c_{m m^{\prime}}^{i} u_{l m^{\prime}} . \tag{4.37}
\end{equation*}
$$

Since the defect space-time is spherically symmetric, it allows for the Killing vector fields

$$
\begin{align*}
\left(\eta_{1}\right)_{\mu} & =\sin (x) \partial_{\mu} z+\cos (x) \cot (z) \partial_{\mu} x,  \tag{4.38a}\\
\left(\eta_{2}\right)_{\mu} & =\cos (x) \partial_{\mu} z-\sin (x) \cot (z) \partial_{\mu} x,  \tag{4.38b}\\
\left(\eta_{3}\right)_{\mu} & =\partial_{\mu} x . \tag{4.38c}
\end{align*}
$$

As these Killing vector fields are generators of rotations, this leads to the definition of the following angular momentum operators:

$$
\begin{align*}
& \hat{L}_{1}=i \sin (x) \frac{\partial}{\partial z}+i \cos (x) \cot (z) \frac{\partial}{\partial x}=\mathrm{e}^{i x}\left[\frac{\partial}{\partial z}+i \cot (z) \frac{\partial}{\partial x}\right]  \tag{4.39a}\\
& \hat{L}_{2}=-i \cos (x) \frac{\partial}{\partial z}+i \sin (x) \cot (z) \frac{\partial}{\partial x}=\mathrm{e}^{-i x}\left[-\frac{\partial}{\partial z}+i \cot (z) \frac{\partial}{\partial x}\right],  \tag{4.39b}\\
& \hat{L}_{3}=-i \frac{\partial}{\partial x} \tag{4.39c}
\end{align*}
$$

which satisfy the commutation relations $\left[\hat{L}_{a}, \hat{L}_{b}\right]=i \epsilon_{a b c} \hat{L}_{c}$. As it turns out, it is more convenient to work with the operators $\hat{L}_{+}=\hat{L}_{1}+i \hat{L}_{2}, \hat{L}_{-}=\hat{L}_{1}-i \hat{L}_{2}$ and $\hat{L}_{3}$. Using the general recursion relations [1] between associated Legendre polynomials,

$$
\begin{align*}
& \frac{\partial}{\partial z} P_{l m}(\cos (z))=P_{l, m+1}(\cos (z))+m \cot (z) P_{l m}(\cos (z))  \tag{4.40a}\\
& \frac{\partial}{\partial z} P_{l m}(\cos (z))=-(l+m)(l-m+1) P_{l, m-1}(\cos (z))-m \cot (z) P_{l m}(\cos (z)) \tag{4.40b}
\end{align*}
$$

it can be shown that

$$
\begin{equation*}
\hat{L}_{+} Y_{l m}(z, x) \propto Y_{l, m+1}(z, x), \quad \hat{L}_{-} Y_{l m}(z, x) \propto Y_{l, m-1}(z, x) . \tag{4.41a,b}
\end{equation*}
$$

The proportionalities (4.41a, b) are trivial, if $m+1>l$ or $m-1<-l$. For $\hat{L}_{3}$ the proportionality is obvious, when looking at the defining equation (4.8) of the spherical harmonics. Thus,

$$
\begin{align*}
& \hat{L}_{1} Y_{l m}(z, x)=\frac{1}{2}\left(\hat{L}_{+}+\hat{L}_{-}\right) Y_{l m}(z, x)=c_{l, m+1}^{1} Y_{l, m+1}(z, x)+c_{l, m-1}^{1} Y_{l, m-1}(z, x)  \tag{4.42a}\\
& \hat{L}_{2} Y_{l m}(z, x)=\frac{1}{2 i}\left(\hat{L}_{+}-\hat{L}_{-}\right) Y_{l m}(z, x)=c_{l, m+1}^{2} Y_{l, m+1}(z, x)+c_{l, m-1}^{2} Y_{l, m-1}(z, x)  \tag{4.42b}\\
& \hat{L}_{3} Y_{l m}(z, x)=c_{l m}^{3} Y_{l m}(z, x) . \tag{4.42c}
\end{align*}
$$

This shows that equation (4.37) will always be satisfied for the angular wave functions (4.8). Pauli's condition is fulfilled, and we cannot discard any of the solutions found in section 4.2 using his criterion.

### 4.3.1 Superpositions of solutions

As equation (4.1) is a linear differential equation, it should be possible to add two solutions and obtain another solution. Thus, we suppose for the moment that the most general solution of equation (4.1) is given by

$$
\begin{equation*}
\Phi(t, y, z, x)=\int \mathrm{d} k \sum_{l, m} a_{k l m} \Phi_{k l m}(t, y, z, x) . \tag{4.43}
\end{equation*}
$$



Figure 4.2. Sketch of defect core. The points inside the ball were removed by surgery and antipodal points on the surface of the ball are identified. Points on plane $P$ are not covered by the coordinate chart. Red dashed arrows indicate the ranges of the coordinates. For $\epsilon \rightarrow 0$, the two points $A$ and $B$ describe the same physical space-time point, and hence we require all observable quantities to agree at these points.

We now consider a superposition of two wave functions with different eigenvalues of the angular momentum operators,

$$
\begin{equation*}
\Phi=\Phi_{k l_{1} m_{1}}+\Phi_{k l_{2} m_{2}} \tag{4.44}
\end{equation*}
$$

and calculate the norm of the so obtained state,

$$
\begin{equation*}
|\Phi|^{2}=\left|\Phi_{k l_{1} m_{1}}+\Phi_{k l_{2} m_{2}}\right|^{2}=\left|\Phi_{k l_{1} m_{1}}\right|^{2}+\left|\Phi_{k l_{2} m_{2}}\right|^{2}+2 \mathfrak{R}\left(\Phi_{k l_{1} m_{1}}^{*} \Phi_{k l_{2} m_{2}}\right) . \tag{4.45}
\end{equation*}
$$

Neglecting the time dependence of the wave functions for the moment, we evaluate the norm (4.45) at two points $A=\left(y=y_{0}, z=z_{0}, x=\epsilon\right)$ and $B=\left(y=-y_{0}, z=\pi-z_{0}, x=\pi-\epsilon\right)$ on the spatial hypersurface $\mathcal{M}_{b}^{3}$, which are located close to the boundary of the coordinate system, see figure 4.2. For vanishing $\epsilon$, both points converge to the same space-time point. We define the quantity $\delta$ as the difference of the two norms,

$$
\begin{equation*}
\left.\delta \equiv|\Phi|^{2}\right|_{A}-\left.|\Phi|^{2}\right|_{B} . \tag{4.46}
\end{equation*}
$$

The terms of the form $\left|\Phi_{k l_{i} m_{i}}\right|^{2}$ in equation (4.46) cancel, so we are left with

$$
\begin{equation*}
\delta=\left.2 \Re\left(\Phi_{k l_{1} m_{1}}^{*} \Phi_{k l_{2} m_{2}}\right)\right|_{A}-\left.2 \Re\left(\Phi_{k l_{1} m_{1}}^{*} \Phi_{k l_{2} m_{2}}\right)\right|_{B} \tag{4.47}
\end{equation*}
$$

According to the results of section 4.2, the scalar wave functions (4.28) in the defect spacetime $\mathcal{M}_{b}^{4}$ do not change under the parity transformation $y_{0} \rightarrow-y_{0}$. Also, the radial part of the wave function is purely real. Hence, we obtain:

$$
\begin{align*}
\delta & =2 R_{k l_{1}}\left(y_{0}\right) R_{k l_{2}}\left(y_{0}\right) \mathfrak{R}\left[Y_{l_{1} m_{1}}^{*} Y_{l_{2} m_{2}}\left(z_{0}, \epsilon\right)-Y_{l_{1} m_{1}}^{*} Y_{l_{2} m_{2}}\left(\pi-z_{0}, \pi-\epsilon\right)\right]  \tag{4.48}\\
& =2 R_{k l_{1}}\left(y_{0}\right) R_{k l_{2}}\left(y_{0}\right) \mathfrak{R}\left[Y_{l_{1} m_{1}}^{*} Y_{l_{2} m_{2}}\left(z_{0}, \epsilon\right)-Y_{l_{1} m_{1}}^{*} Y_{l_{2} m_{2}}\left(\pi-z_{0}, \pi+\epsilon\right) \mathrm{e}^{2 i\left(m_{1}-m_{2}\right) \epsilon}\right] .
\end{align*}
$$

We can now use the general transformation property of spherical harmonics under parity transformations in standard spherical coordinates,

$$
\begin{equation*}
Y_{l m}(\pi-\theta, \pi+\phi)=(-1)^{l} Y_{l m}(\theta, \phi) \tag{4.49}
\end{equation*}
$$

which yields

$$
\begin{align*}
& \delta=2 R_{k l_{1}}\left(y_{0}\right) R_{k l_{2}}\left(y_{0}\right) \mathfrak{R}\left[Y_{l_{1} m_{1}}^{*}\left(z_{0}, \epsilon\right) Y_{l_{2} m_{2}}\left(z_{0}, \epsilon\right)\right. \\
&\left.-Y_{l_{1} m_{1}}^{*}\left(z_{0}, \epsilon\right) Y_{l_{2} m_{2}}\left(z_{0}, \epsilon\right)(-1)^{l_{1}+l_{2}} \mathrm{e}^{2 i\left(m_{1}-m_{2}\right) \epsilon}\right] . \tag{4.50}
\end{align*}
$$

In the limit of $\epsilon \rightarrow 0$, the spherical harmonics are like the exponential function purely real and we can write

$$
\begin{equation*}
\lim _{\epsilon \rightarrow 0}(\delta)=R_{k l_{1}}\left(y_{0}\right) R_{k l_{2}}\left(y_{0}\right) Y_{l_{1} m_{1}}\left(z_{0}, 0\right) Y_{l_{2} m_{2}}\left(z_{0}, 0\right) \cdot\left[1-(-1)^{l_{1}+l_{2}}\right] . \tag{4.51}
\end{equation*}
$$

This expression does in general not vanish, if $l_{1}$ and $l_{2}$ differ by an odd integer. We find that, in this case, the square of the norm of the wave function (4.44) is not uniquely defined on the boundary of the coordinate system.

Notice that equation (4.1) is purely real, and a priori there does not appear to be any need to introduce complex scalar fields. However, since the spherical harmonics are purely real on the boundary of the coordinate chart, the argument would not have been changed if we had dealt with real scalar fields from the beginning.

Similar results can be obtained for the stress-energy-momentum tensor of the scalar field, which is given by [38]

$$
\begin{equation*}
T_{\mu \nu}=\frac{1}{2}\left(\left(\partial_{\mu} \Phi^{*}\right)\left(\partial_{v} \Phi\right)+\left(\partial_{\mu} \Phi\right)\left(\partial_{\nu} \Phi^{*}\right)\right)-\frac{1}{2} g_{\mu v}\left(g^{\rho \sigma}\left(\partial_{\rho} \Phi^{*}\right)\left(\partial_{\sigma} \Phi\right)+m_{0}^{2} \Phi^{*} \Phi\right) . \tag{4.52}
\end{equation*}
$$

Here it should be kept in mind that the $y$ - and $z$-axes have opposite directions on both sides of the boundary of the coordinate system. An analogous calculation can be carried out for the scalar field wave functions in Minkowski space-time, $\mathcal{M}_{0}^{4}$. The differences always vanish in this case.

### 4.3.2 Orthogonality of solutions

In this section we want to study the orthogonality of the wave functions $\Phi_{k l m}$ obtained in section 4.2 for the defect space-time, $\mathcal{M}_{b}^{4}$. Consider two wave functions $\Phi_{1}=\Phi_{k_{1} l_{1} m_{1}}$ and $\Phi_{2}=\Phi_{k_{2} l_{2} m_{2}}$. In curved space-time, the scalar product of two scalar-field wave functions is defined as [8,82]:

$$
\begin{equation*}
\left(\Phi_{1}, \Phi_{2}\right)=-i \int_{\Sigma} \mathrm{d} \Sigma n^{\mu} \sqrt{g_{\Sigma}}\left(\Phi_{1} \stackrel{\leftrightarrow}{\partial_{\mu}} \Phi_{2}^{*}\right), \tag{4.53}
\end{equation*}
$$

where $n^{\mu}$ is a future-directed unit vector orthogonal to a spacelike hypersurface $\Sigma$ and $\mathrm{d} \Sigma n^{\mu} \sqrt{g_{\Sigma}}$ a positive definite volume measure. It can be shown that this integral is independent
of the choice of the spatial hypersurface as long as one uses local oriented coordinates. In our case, the integral (4.53) simplifies to

$$
\begin{equation*}
\left(\Phi_{1}, \Phi_{2}\right)=-i \int_{\Sigma} \mathrm{d} x \mathrm{~d} y \mathrm{~d} z \sqrt{g_{\Sigma}}\left(\Phi_{1} \stackrel{\leftrightarrow}{\partial_{t}} \Phi_{2}^{*}\right), \tag{4.54}
\end{equation*}
$$

where

$$
\begin{equation*}
\sqrt{g_{\Sigma}}=|y| \sqrt{y^{2}+b^{2}} \sin (z) \tag{4.55}
\end{equation*}
$$

Inserting the solutions $\Phi_{1}$ and $\Phi_{2}$ and setting $t=0$ for convenience, we obtain for (4.53)

$$
\begin{equation*}
\left(\omega_{k_{1}}+\omega_{k_{2}}\right) \int_{-\infty}^{\infty} \mathrm{d} y|y| \zeta(y) R_{k_{1} l_{1}}(y) R_{k_{2} l_{2}}(y) \int_{0}^{\pi} \int_{0}^{\pi} \mathrm{d} x \mathrm{~d} z \sin (z) Y_{l_{1} m_{1}}(z, x) Y_{l_{2} m_{2}}^{*}(z, x) \tag{4.56}
\end{equation*}
$$

We first concentrate on the single integral over $y$ in equation (4.56). It is a symmetric integral over a function with even parity, since the radial wave functions $R_{k l}$ defined in equation (4.26) always have even parity. Hence, it does not vanish in general. Now we investigate the double integral, which we abbreviate as $I_{x z}$ for convenience. If the integration over $x$ ran from 0 to $2 \pi$, we would obtain the standard orthogonality relation between the spherical harmonics. Here we conduct a case-by-case analysis.

## Case $m_{1} \neq m_{2}$, difference an odd integer

We insert the expression (4.8) for the spherical harmonics and carry out the integration over $x$. This yields

$$
\begin{array}{rlrl}
I_{x z} & =N_{l_{1} m_{1}} N_{l_{2} m_{2}}\left[\frac{(-1)^{m_{2}-m_{1}}-1}{i\left(m_{2}-m_{1}\right)}\right] \cdot \int_{0}^{\pi} \mathrm{d} z P_{l_{1} m_{1}}(\cos (z)) P_{l_{2} m_{2}}(\cos (z)) \sin (z) \\
& \propto \int_{-1}^{1} \mathrm{~d} u P_{l_{1} m_{1}}(u) P_{l_{2} m_{2}}(u), & & {[u=\cos (z)] .} \tag{4.57}
\end{array}
$$

The last expression is a symmetric integral over a function with parity $(-1)^{l_{1}+l_{2}+m_{1}+m_{2}}$ and will in general not vanish if the parity is even. We assumed $m_{1}$ and $m_{2}$ to differ by an odd integer, which means that $I_{x z}$ will not vanish for the $l_{i}$ differing by an odd integer.

## Case $m_{1} \neq m_{2}$, difference an even integer

In this case the expression

$$
\begin{equation*}
\left[\frac{(-1)^{m_{2}-m_{1}}-1}{i\left(m_{2}-m_{1}\right)}\right] \tag{4.58}
\end{equation*}
$$

in equation (4.57) will vanish, which means that the wave functions $\Phi_{1}$ and $\Phi_{2}$ are orthogonal for any $l_{i}$.

## Case $m_{1}=m_{2}$

If $m_{1}$ and $m_{2}$ are equal, the integrand will not depend on $x$, and the integration over $x$ will be trivial. We can then rely on the standard orthogonality relation between spherical harmonics [3], which gives in this case

$$
\begin{equation*}
\int_{0}^{\pi} \mathrm{d} x \int_{0}^{\pi} \mathrm{d} z \sin (z) Y_{l_{1} m_{1}}^{*}(z, x) Y_{l_{2} m_{1}}(z, x)=\frac{1}{2} \delta_{l_{1} l_{2}} . \tag{4.59}
\end{equation*}
$$

An analogous calculation can be carried out for the wave functions in Minkowski spacetime, $\mathcal{M}_{0}^{4}$. The integral $I_{x z}$ is identical, but the integration over the radial coordinate in equation (4.56) changes. If the quantum numbers $l_{1}$ and $l_{2}$ differ by an odd integer, the integrand of the integral over $y$ in equation (4.56) will have odd parity. Hence, we find that the scalar product vanishes for any two wave functions with different quantum numbers $l$ and $m$.

### 4.4 Discussion

One of the most important results of the analysis in this chapter is the fact that the singular point in the radial Klein-Gordon equation (4.5b) at the defect core, $y=0$, is a regular singular point and that it is possible to construct smooth solutions by using the Ansatz of a Frobenius power series. This implies that the Klein-Gordon equation can be solved globally in spite of the singular behaviour of the space-time geometry at the defect core. The global solvability of the Klein-Gordon equation is necessary if one wants to study the propagation of scalar fields in the defect space-time [72].

Another important aspect is the fact that all wave functions of scalar fields in the defect space-time $\mathcal{M}_{b}^{4}$ necessarily have even parity, i. e. do not change under a transformation $y \rightarrow-y$. This implies that all parity-odd observables in the space-time $\mathcal{M}_{b}^{4}$ have vanishing matrix elements [48] and can be interpreted as a selection rule for transitions between quantum states. When using the space-time defect as a model for an elementary particle such as the electron, presumably there should be parity-odd observables with non-vanishing matrix elements.

Notice, however, that this behaviour of the solutions is partly due to the coordinate system, which we introduced in section 2.2. In a coordinate system, where the ranges of the coordinates are $y \in[0 ; \infty), z \in(0 ; \pi)$ and $x \in[0 ; 2 \pi]$, the parity transformation would be $z \rightarrow \pi-z$ and $x \rightarrow \pi+x$. The parity of the wave function $\Phi_{k l m}$ would then be determined completely by the spherical harmonics, i. e. would be $(-1)^{l}$. This coordinate system has not been used, since it is not suited to study the behaviour of the wave functions near the defect core.

Also, it turned out that the solutions (4.28) have some further strange properties, which appeared to be related to their single- and multi-valuedness, respectively. It was found in section 4.3.1 that it is not possible to define an inner product of the wave functions by
the modulus consistently throughout the whole space-time. This phenomenon also affects current densities and the energy-momentum tensor (4.52) and implies that scalar quantities such as the Lagrangian

$$
\begin{equation*}
\mathcal{L}=\frac{1}{2}\left(g^{\mu \nu}\left(\partial_{\mu} \Phi^{*}\right)\left(\partial_{\nu} \Phi\right)+m_{0}^{2} \Phi^{*} \Phi\right) \tag{4.60}
\end{equation*}
$$

are not single-valued. Notice that analogous results also hold in the case of a real scalar field with Lagrangian

$$
\begin{equation*}
\mathcal{L}=\frac{1}{2}\left(g^{\mu v}\left(\partial_{\mu} \Phi\right)\left(\partial_{\nu} \Phi\right)+m_{0}^{2} \Phi^{2}\right) \tag{4.61}
\end{equation*}
$$

Another surprise was the fact that the solutions (4.28) were not orthogonal for different quantum numbers $l$ and $m$, when the scalar product (4.53) was computed in the coordinate system $(t, y, z, x)$ as defined in section 2.2. In curved space-time, the physical interpretation of the scalar product of two wave functions may not be as clear as in non-relativistic quantum mechanics. However, the scalar product (4.53) is an invariant in Lorentzian geometry [8], and the wave functions certainly are orthogonal for different angular momentum quantum numbers in a coordinate system based on standard spherical coordinates and with boundary at the defect core as described above. Presumably it is necessary that we have a well-behaved scalar-product in order to be able to quantize the scalar field. Another important observation is that none of these problems occur, when dealing with the corresponding solutions in Minkowski space-time in coordinates based on the coordinate system $(t, y, z, x)$.

These observations lead us to conclude that the solutions (4.28) have to be split into two distinct sets. One set contains all solutions $\Phi_{k l m}$ with even quantum number $l$ and the other set contains all solutions $\Phi_{k l m}$ with odd quantum number $l$. This splitting restores the invariance of the scalar product (4.53) and the single-valuedness of quantities like the energy-momentum tensor or the Lagrangian itself, from which the Klein-Gordon equation is derived. It will turn out later in chapter 6 that this effect can be interpreted as the existence of topologically inequivalent twisted quantum fields.

Mathematically, the difference between both sets is that the solutions $\Phi_{k l m}$ with odd quantum number $l$ change their sign on the boundary of the coordinate system and are hence $t$ wisted, while those with even quantum number $l$ do not change sign and are hence strictly single-valued. It should be kept in mind that, mathematically, the information about the topology of the space-time enters the calculation by imposing boundary conditions at the boundary of the coordinate system. This splitting of the solutions (4.28) into two sets also sheds some light on the selection rule described before, which can then be interpreted rather as a superselection rule. The superselection rule does not forbid certain transitions between quantum states, but rather places restrictions on the quantum mechanical measurements that are possible in principle. Had we used a coordinate system based on standard spherical coordinates, we had obtained one set of wave functions with even parity and one set with odd parity, which again implies that parity-odd observables have vanishing matrix elementsthe reason rather being that it is not possible in principle to measure transitions between quantum states with even and with odd parity.

The described splitting of the wave functions has the consequence, that the angular factors no longer form a complete set in the sense that any function on a sphere can be expanded in terms of the solutions of the Klein-Gordon equation. This is of importance if we interpret the defect space-time as part of a classical space-time foam and want to investigate the influence of these structures on the propagation of matter fields. When there are only odd or only even $l$ modes, we are unable to make a plane wave expansion of the form

$$
\begin{equation*}
\mathrm{e}^{i k r \cos (\theta)}=\sum_{l} i^{l}(2 l+1) j_{l}(k r) P_{l}(\cos (\theta))=\sum_{l} i^{l} \sqrt{4 \pi(2 l+1)} j_{l}(k r) Y_{l 0}(\theta), \tag{4.62}
\end{equation*}
$$

which makes it impossible to speak of "scattering" of an incident plane scalar wave off a defect.

One more insight into the nature of the defect is gained by the solutions with imaginary wavenumbers described on page 54. We find that there are exponentially growing scalar fields, which are perfectly regular solutions to the Klein-Gordon equation (4.1) and which can be defined to be initially arbitrarily small. This unphysical behaviour suggests the inherent instability of the defect background metric against scalar perturbations [81]. Notice that these solutions do not exist in Minkowski space-time, as in that case they are not well-behaved at the origin of the coordinate system and hence cannot represent a valid perturbation. This instability does not seem to have anything to do with the non-trivial topology and hence must rather be a consequence of the non-Lorentzian geometry of the defect.

## Chapter 5

## Spin-1/2 field

After having investigated the simple case of a scalar field in very much detail, we now turn our attention to the technically more complicated spin- $1 / 2$ field. The first section of this chapter will be devoted to deriving the Dirac equation, making some assumptions about its nature near the defect core. In the following two sections we will solve the Dirac equation in the both cases of defect and Minkowski space-time and investigate the nature of the solutions.

### 5.1 Dirac equation in curved space-time

The Dirac equation for a spin- $1 / 2$ particle in a general curved space-time is given by [8, 65]:

$$
\begin{equation*}
i \gamma^{\mu} \nabla_{\mu} \psi-m_{0} \psi=0, \tag{5.1}
\end{equation*}
$$

where $m_{0}$ denotes the rest mass of the particle and the precise definition of the Dirac $\gamma$ matrices will be given later. The ordinary partial derivative of flat space-time has been replaced by a covariant derivative,

$$
\begin{equation*}
\nabla_{\mu}=\partial_{\mu}+\Gamma_{\mu}, \quad \Gamma_{\mu}=\frac{1}{8} e_{\nu A}\left[\left(\partial_{\mu} e_{B}^{v}\right)+\Gamma_{\mu \alpha}^{\nu} e_{B}^{\alpha}\right]\left[\gamma^{A}, \gamma^{B}\right] \tag{5.2a,b}
\end{equation*}
$$

and $\Gamma_{\mu}$ denotes the spinorial affine connection. In order to write down an explicit expression for the Dirac equation (5.1), we first have to choose a Vierbein $e$. In our convention for the rest of the section, greek letters denote the indices of the underlying coordinates $(t, y, z, x)$, while capital roman letters denote the local Lorentz indices ( $0,1,2,3$ ). We also use the sign convention (,,,+--- ) temporarily, which is more common in quantum theory.

## Vierbeine

In local coordinates, the Vierbein $e_{\mu}^{A}$ and its inverse $e_{A}^{\mu}$ satisfy the relations

$$
\begin{equation*}
g_{\mu \nu}(x)=e_{\mu}^{A}(x) e_{v}^{B}(x) \eta_{A B}, \quad \quad g^{\mu \nu}(x)=e_{A}^{\mu}(x) e_{B}^{v}(x) \eta^{A B} \tag{5.3a,b}
\end{equation*}
$$

respectively, where $\eta_{A B}$ and $\eta^{A B}$ denote the Minkowski metric and its inverse, respectively. In a space-time with Lorentzian geometry, the Vierbeine can be defined by [8]

$$
\begin{equation*}
e_{\mu}^{A}(p)=\left(\frac{\partial y_{p}^{A}}{\partial x^{\mu}}\right)_{x=p} \tag{5.4}
\end{equation*}
$$

where the $y_{p}^{A}$ are normal coordinates at a space-time point $p$, as introduced in section 1.1.4.

### 5.1.1 Defect space-time

Since the defect space-time violates the elementary flatness condition, there are points $p$ at which no normal coordinates exist, and the proper definition of the Vierbeine is less clear. We follow [83] and treat the Vierbeine as a set of four smooth vector fields satisfying equations ( $5.3 \mathrm{a}, \mathrm{b}$ ). Demanding at least differentiability is natural, since the spinorial affine connection (5.2b) contains derivatives of the Vierbeine. Applying the determinant map to equations (5.3a, b) and inserting the line-element, equation (2.12), we obtain

$$
\begin{equation*}
-y^{2}\left(y^{2}+b^{2}\right) \sin ^{2}(z)=-\operatorname{det}^{2}\left(e_{\mu}^{A}\right) . \tag{5.5}
\end{equation*}
$$

The determinant mapping is smooth, and so the Vierbeine can only be smooth if their determinant is a smooth function of the coordinates. Thus, we see that equation (5.5) has only two proper solutions,

$$
\begin{equation*}
\operatorname{det}\left(e_{\mu}^{A}\right)= \pm y \sqrt{y^{2}+b^{2}} \sin (z) . \tag{5.6}
\end{equation*}
$$

It is possible to assign an orientation to a Vierbein, which corresponds to the sign of its determinant [64]. Since the range of the coordinate $y$ is $(-\infty, \infty)$, we see that it is not possible to cover the whole coordinate chart with a smooth Vierbein of everywhere identical orientation. Notice that for reasons of continuity this can never happen, if the metric tensor of the space-time is nowhere degenerate.

For our calculations, we will stick to Vierbein axes parallel to a rectangular Cartesian coordinate system [9,69]. The inverse Vierbein is explicitly given by

$$
e_{A}^{\mu}=\left(\begin{array}{cccc}
1 / R(\zeta) & 0 & 0 & 0  \tag{5.7}\\
0 & R(\zeta) \frac{\zeta \sin (z) \cos (x)}{y} & R(\zeta) \frac{\zeta \sin (z) \sin (x)}{y} & R(\zeta) \frac{\zeta \cos (z)}{(y)} \\
0 & \frac{\cos (z) \cos (x)}{\zeta} & \frac{\cos (z) \sin (x)}{\zeta} & -\frac{\sin (z)}{\zeta} \\
0 & -\frac{\sin (x)}{\sin (z) \zeta} & \frac{\cos (x)}{\sin (z) \zeta} & 0
\end{array}\right),
$$

where the abbreviation

$$
\begin{equation*}
R(\zeta):=\sqrt{1-\frac{2 M}{\zeta}}=\sqrt{W(\zeta)}, \quad \zeta=\sqrt{y^{2}+b^{2}} \tag{5.8a,b}
\end{equation*}
$$

has been introduced. The following calculations have also been carried out using a Vierbein with local axes along the coordinate axes,

$$
\tilde{e}_{A}^{\mu}=\left(\begin{array}{cccc}
1 / R(\zeta) & 0 & 0 & 0  \tag{5.9}\\
0 & R(\zeta) \frac{\zeta}{y} & 0 & 0 \\
0 & 0 & \frac{1}{\zeta} & 0 \\
0 & 0 & 0 & \frac{1}{\sin (z) \zeta}
\end{array}\right) \text {, }
$$

with identical results. The Dirac matrices $\gamma^{\mu}$, which satisfy the anticommutation relations of curved space-time analogous to those of flat space-time,

$$
\begin{equation*}
\left\{\gamma^{\mu}, \gamma^{\nu}\right\}=2 g^{\mu \nu} 1, \quad\left\{\gamma^{A}, \gamma^{B}\right\}=2 \eta^{A B} \mathbb{1}, \tag{5.10a,b}
\end{equation*}
$$

are now obtained by contracting the Dirac matrices of flat Minkowski space-time onto the Vierbeine, $\gamma^{\mu}=e_{A}^{\mu} \gamma^{A}$. With the Vierbein (5.7) and the affine connection (2.20) given, we may compute the components of the spinorial affine connection. After making use of equation (5.10b) several times, we find

$$
\begin{align*}
& \Gamma_{t}=\frac{M \sin (z) \cos (x)}{\zeta^{2}} \gamma^{0} \gamma^{1}+\frac{M \sin (z) \sin (x)}{\zeta^{2}} \gamma^{0} \gamma^{2}+\frac{M \cos (z)}{\zeta^{2}} \gamma^{0} \gamma^{3}  \tag{5.10a}\\
& \Gamma_{y}=0, \quad \Gamma_{z}=-\cos (x)(R(\zeta)-1) \gamma^{1} \gamma^{3}-\sin (x)(R(\zeta)-1) \gamma^{2} \gamma^{3}  \tag{5.10b,c}\\
& \begin{aligned}
\Gamma_{x} & =\sin ^{2}(z)(R(\zeta)-1) \gamma^{1} \gamma^{2}+\sin (x) \sin (z) \cos (z)(R(\zeta)-1) \gamma^{1} \gamma^{3}
\end{aligned}  \tag{5.10d}\\
& \quad-\cos (x) \sin (z) \cos (z)(R(\zeta)-1) \gamma^{2} \gamma^{3} .
\end{align*}
$$

The Dirac equation in the defect space-time is then explicitly given by

$$
\begin{equation*}
\left\{i \gamma^{t} \partial_{t}+i \gamma^{y}\left[\partial_{y}+\frac{y}{\zeta}\left(\frac{R(\zeta)-1}{R(\zeta) \zeta}+\frac{M}{2 \zeta^{2} R^{2}(\zeta)}\right)\right]+i \gamma^{z} \partial_{z}+i \gamma^{x} \partial_{x}-m_{0}\right\} \psi=0 \tag{5.11}
\end{equation*}
$$

Since we are interested in stationary solutions of the Dirac equation (5.11), we may split off the time dependence and define a Hamiltonian operator $\hat{H}_{b}$ as in the case of non-relativistic quantum mechanics, given by

$$
\begin{equation*}
\hat{H}_{b}=\gamma^{0}\left\{\gamma^{y} \frac{R(\zeta)}{i}\left[\partial_{y}+\frac{y}{\zeta}\left(\frac{R(\zeta)-1}{\zeta R(\zeta)}+\frac{M}{2 \zeta^{2} R^{2}(\zeta)}\right)\right]+\frac{R(\zeta)}{i} \gamma^{z} \partial_{z}+\frac{R(\zeta)}{i} \gamma^{x} \partial_{x}+m_{0} R(\zeta)\right\} . \tag{5.12}
\end{equation*}
$$

Equation (5.12) does have the downside that, as in the case of Schwarzschild space-time, it cannot be solved analytically in general, i. e. for non-vanishing mass. The massless case, $M=0$, corresponds to $R(\zeta)=1$ and hence will be of special interest for us due to its simplicity. The Hamiltonian (5.12) then takes the simple form

$$
\begin{equation*}
\hat{H}_{b}=\gamma^{0}\left\{\gamma^{y} \frac{1}{i} \partial_{y}+\frac{1}{i} \gamma^{z} \partial_{z}+\frac{1}{i} \gamma^{x} \partial_{x}+m_{0}\right\} . \tag{5.13}
\end{equation*}
$$

### 5.1.2 Minkowski space-time

In order to be able to compare the behaviour of the spin- $1 / 2$ field in the defect space-time $\mathcal{M}_{b}^{4}$ with non-trivial topology, we introduce an analogous Vierbein on Minkowski space-time $\mathcal{M}_{0}^{4}$,

$$
\bar{e}_{A}^{\mu}=\left(\begin{array}{cccc}
1 & 0 & 0 & 0  \tag{5.14}\\
0 & \sin (z) \cos (x) & \sin (z) \sin (x) & \cos (z) \\
0 & \frac{\cos (z) \cos (x)}{y} & \frac{\cos (z) \sin (x)}{y} & -\frac{\sin (z)}{y} \\
0 & -\frac{\sin (x)}{y \sin (z)} & \frac{\cos (x)}{y \sin (z)} & 0
\end{array}\right) .
$$

The determinant of $\bar{e}_{A}^{\mu}$ is strictly positive,

$$
\begin{equation*}
\operatorname{det}\left(\bar{e}_{A}^{\mu}\right)=\frac{1}{y^{2} \sin (z)}>0 \tag{5.15}
\end{equation*}
$$

and hence the Vierbein (5.14) has an identical orientation throughout the whole coordinate chart. As before, we deduce the Hamiltonian

$$
\begin{equation*}
\hat{H}_{0}=\gamma^{0}\left\{\bar{\gamma}^{y} \frac{1}{i} \partial_{y}+\frac{1}{i} \bar{\gamma}^{z} \partial_{z}+\frac{1}{i} \bar{\gamma}^{x} \partial_{x}+m_{0}\right\} . \tag{5.16}
\end{equation*}
$$

The difference between the Hamiltonian (5.16) and the Hamiltonian (5.12) is hidden in the Dirac matrices $\bar{\gamma}^{\mu}$.

### 5.2 Solution of Dirac equation

Our plan in this section is to decompose the Hamiltonian (5.12) into commuting operators, which allows us to construct the solution as a product of simultaneous eigenfunctions. In order to determine the eigenfunctions explicitly, we first have to choose a representation of the Dirac matrices in locally flat space,

$$
\gamma^{0}=\left(\begin{array}{cc}
\mathbb{1} & 0  \tag{5.17a-d}\\
0 & -\mathbb{1}
\end{array}\right), \quad \gamma^{i}=\left(\begin{array}{cc}
0 & \sigma^{i} \\
-\sigma^{i} & 0
\end{array}\right), \quad i \in\{1,2,3\},
$$

where the $\sigma^{i}$ are the Pauli matrices. It turns out that, for solving the Dirac equation, it is convenient also to define matrices $\alpha^{m}$ and $\beta$ adapted to the underlying curved coordinates by

$$
\alpha^{m}=\left(\begin{array}{cc}
0 & \sigma^{m}  \tag{5.18a-d}\\
\sigma^{m} & 0
\end{array}\right), \quad \beta=\gamma^{0}=\left(\begin{array}{cc}
\mathbb{1} & 0 \\
0 & -\mathbb{1}
\end{array}\right),
$$

where $m \in\{y, z, x\}$ and

$$
\begin{align*}
& \sigma^{y}=\sigma^{1} \sin (z) \cos (x)+\sigma^{2} \sin (z) \sin (x)+\sigma^{3} \cos (z),  \tag{5.19a}\\
& \sigma^{z}=\sigma^{1} \cos (z) \cos (x)+\sigma^{2} \cos (z) \sin (x)-\sigma^{3} \sin (z),  \tag{5.19b}\\
& \sigma^{x}=-\sigma^{1} \frac{\sin (x)}{\sin (z)}+\sigma^{2} \frac{\cos (x)}{\sin (z)} \tag{5.19c}
\end{align*}
$$

### 5.2.1 Defect space-time

Inserting these matrices, we obtain for the Hamiltonian in the defect space-time $\mathcal{M}_{b}^{4}$ :

$$
\begin{equation*}
\hat{H}_{b}=\frac{1}{i}\left[\alpha^{y} \frac{\zeta}{y} \frac{\partial}{\partial y}+\frac{1}{\zeta} \alpha^{z} \frac{\partial}{\partial z}+\frac{1}{\zeta} \alpha^{x} \frac{\partial}{\partial x}\right]+m_{0} \beta . \tag{5.20}
\end{equation*}
$$

One can now introduce the operator $\hat{K}$, which can be seen to be connected to the matrices ( $5.18 \mathrm{a}-\mathrm{d}$ ),

$$
\hat{K} \equiv \beta(\hat{\Sigma} \cdot \hat{L}+\mathbb{1})=\beta\left(-\alpha^{y} \alpha^{z} \frac{\partial}{\partial z}-\alpha^{y} \alpha^{x} \frac{\partial}{\partial x}+\mathbb{1}\right) \equiv\left(\begin{array}{cc}
\hat{\kappa} & 0  \tag{5.21}\\
0 & -\hat{\kappa}
\end{array}\right),
$$

where we have used the spin operator

$$
\hat{\Sigma}=\left(\begin{array}{cc}
\sigma^{i} & 0  \tag{5.22}\\
0 & \sigma^{i}
\end{array}\right), \quad i \in\{1,2,3\}
$$

and the angular momentum operator $\hat{L}$ as defined in equation (4.39). The operator $\hat{K}$ commutes with the Hamiltonian $\hat{H}_{b}$ and has a spectrum of integer eigenvalues, where zero is excluded [19,67]. An analysis similar to the one of section 4.3 shows that additional, half-integer eigenvalues are not associated to regular eigenfunctions [62]. The Hamiltonian can now be brought into the form

$$
\begin{equation*}
\hat{H}_{b}=\frac{1}{i} \alpha^{y}\left(\frac{\zeta}{y} \frac{\partial}{\partial y}+\frac{1}{\zeta}\right)+\frac{i}{\zeta} \alpha^{y} \beta \hat{K}+m_{0} \beta . \tag{5.23}
\end{equation*}
$$

We have now separated the radial dependence from the angular dependence. In order to determine the eigenspinors of the Hamiltonian $\hat{H}$, we make two different Ansätze for eigenspinors of $\hat{K}$ with positive eigenvalue $k$ and with negative eigenvalue $k$, which we label ( $\pm$ ) for clarity,

$$
\begin{equation*}
\psi_{j m k}^{(+)}=\binom{f_{k}(y) \xi_{j m}^{+}(z, x)}{g_{k}(y) \xi_{j m}^{-}(z, x)}, \quad \psi_{j m k}^{(-)}=\binom{f_{k}(y) \xi_{j m}^{-}(z, x)}{g_{k}(y) \xi_{j m}^{+}(z, x)} \tag{5.24a,b}
\end{equation*}
$$

where the $\xi_{j m}^{ \pm}$are the spinor spherical harmonics, which are eigenspinors to the operator $\hat{\kappa}$ defined in equation (5.21) with eigenvalue $\pm k$, respectively. They further satisfy the relation $\sigma^{y} \xi_{j m}^{ \pm}=\xi_{j m}^{\mp}$ [33]. After inserting this Ansatz into the eigenvalue equation $\hat{H} \psi=E \psi$, we obtain a set of two coupled linear differential equations,

$$
\begin{align*}
& \left(m_{0}-E\right) f_{k}(y)-i\left(\frac{\zeta}{y} \partial_{y}+\frac{1}{\zeta}\right) g_{k}(y)-\frac{i k}{\zeta} g_{k}(y)=0  \tag{5.25a}\\
& \left(m_{0}+E\right) g_{k}(y)+i\left(\frac{\zeta}{y} \partial_{y}+\frac{1}{\zeta}\right) f_{k}(y)-\frac{i k}{\zeta} f_{k}(y)=0 \tag{5.25b}
\end{align*}
$$

By means of a substitution, these equations can be decoupled,

$$
\begin{align*}
& \frac{\partial^{2} f_{k}(y)}{\partial y^{2}}+\frac{2 y^{2}-b^{2}}{y\left(y^{2}+b^{2}\right)} \frac{\partial f_{k}(y)}{\partial y}+y^{2}\left(\frac{E^{2}-m_{0}^{2}}{y^{2}+b^{2}}-\frac{k(k-1)}{\left(y^{2}+b^{2}\right)^{2}}\right) f_{k}(y)=0,  \tag{5.26a}\\
& \frac{\partial^{2} g_{k}(y)}{\partial y^{2}}+\frac{2 y^{2}-b^{2}}{y\left(y^{2}+b^{2}\right)} \frac{\partial g_{k}(y)}{\partial y}+y^{2}\left(\frac{E^{2}-m_{0}^{2}}{y^{2}+b^{2}}-\frac{k(k+1)}{\left(y^{2}+b^{2}\right)^{2}}\right) g_{k}(y)=0 . \tag{5.26b}
\end{align*}
$$

These equations are in direct correspondence with equation (4.9), i. e. with the differential equation, which the radial wave function of the scalar field satisfies. Analagously to the case of the scalar field it is possible to show by means of the Frobenius method, that only parity-even radial wave functions $f$ and $g$ can be constructed in a small interval around the defect core. Hence, in order to solve equations (5.26), we only have to make the replacements

$$
k \rightarrow k_{E}:=\sqrt{E^{2}-m_{0}^{2}}, \quad l \rightarrow\left\{\begin{array}{lll}
k-1 & \text { for } & f_{k}(y)  \tag{5.27a,b}\\
k & \text { for } & g_{k}(y)
\end{array} .\right.
$$

The general solutions to equations (5.26) are given by

$$
\begin{align*}
& f_{k}(y)=C_{1} j_{k-1}\left(k_{E} \sqrt{y^{2}+b^{2}}\right)+C_{2} y_{k-1}\left(k_{E} \sqrt{y^{2}+b^{2}}\right),  \tag{5.28a}\\
& g_{k}(y)=C_{3} j_{k}\left(k_{E} \sqrt{y^{2}+b^{2}}\right)+C_{4} y_{k}\left(k_{E} \sqrt{y^{2}+b^{2}}\right) . \tag{5.28b}
\end{align*}
$$

In order to obtain relations between the constants $C_{i}$, we insert the solution (5.28b) into equation (5.25a). Using the recursion relations [1]

$$
\begin{align*}
& \frac{\partial}{\partial x} b_{n}(x)=\frac{n}{x} b_{n}(x)-b_{n+1}(x),  \tag{5.29}\\
& b_{n-1}(x)-\frac{2 n+1}{x} b_{n}(x)+b_{n+1}(x)=0, \tag{5.30}
\end{align*}
$$

between spherical Bessel functions $b_{n}$ of either first or second kind, we obtain the following two relations by equating coefficients

$$
\begin{equation*}
C_{1}=-\frac{i k_{E}}{E-m_{0}} C_{3}, \quad C_{2}=-\frac{i k_{E}}{E-m_{0}} C_{4} . \tag{5.31a,b}
\end{equation*}
$$

Again, we could require that the spinors (5.24a, b) approach the solution in Minkowski spacetime far away from the defect, but this is not necessary. We now look at the angular factors in more detail, in order to understand the solution (5.24a, b) better. Since the square of the operator $\hat{K}$ depends linearly on the square of the total angular momentum operator [19, 67],

$$
\begin{equation*}
\hat{K}^{2}=\hat{J}^{2}+\frac{1}{4}, \tag{5.32}
\end{equation*}
$$

the eigenvalues $j$ and $k$ are related,

$$
\begin{equation*}
k^{2}=j(j+1)+\frac{1}{4}=\left(j+\frac{1}{2}\right)^{2} \Rightarrow k= \pm\left(j+\frac{1}{2}\right), \tag{5.33}
\end{equation*}
$$

where the upper sign corresponds to positive $k$ and the lower sign to negative $k$. The total angular momentum quantum number $j$ may have any half-integer value greater than zero,

$$
\begin{equation*}
j=|k|-\frac{1}{2} \in\left\{\left.n-\frac{1}{2} \right\rvert\, n \in \mathbb{N}\right\} . \tag{5.34}
\end{equation*}
$$

The quantum number $m$ is necessarily half-integer valued [62] and lies in the range

$$
\begin{equation*}
m \in\{-j,-j+1, \ldots, j-1, j\} . \tag{5.35}
\end{equation*}
$$

Explicitly, the spinor spherical harmonics $\xi_{j m}^{ \pm}(z, x)$ are defined as [33]

$$
\begin{equation*}
\xi_{j m}^{+}(z, x)=\binom{c_{j m} Y_{j-1 / 2, m-1 / 2}(z, x)}{c_{j,-m} Y_{j-1 / 2, m+1 / 2}(z, x)} ; \xi_{j m}^{-}(z, x)=\binom{c_{j+1,-m} Y_{j+1 / 2, m-1 / 2}(z, x)}{-c_{j+1, m} Y_{j+1 / 2, m+1 / 2}(z, x)}, \tag{5.36a,b}
\end{equation*}
$$

where $Y$ are the standard spherical harmonics and the Clebsch-Gordan coefficients $c_{j m}$ are given explicitly by

$$
\begin{equation*}
c_{j m}=\sqrt{\frac{j+m}{2 j}} . \tag{5.37}
\end{equation*}
$$

In standard spherical coordinates $\{r, \theta, \phi\}$, the spinor spherical harmonics transform under parity transformations $\theta \rightarrow \pi-\theta, \phi \rightarrow \phi+\pi$ as

$$
\begin{align*}
& \xi_{j m}^{+}(\pi-\theta, \phi+\pi)=(-1)^{j-\frac{1}{2}} \xi_{j m}^{+}(\theta, \phi)=(-1)^{|k|-1} \xi_{j m}^{+}(\theta, \phi)=(-1)^{k-1} \xi_{j m}^{+}(\theta, \phi),  \tag{5.38a}\\
& \xi_{j m}^{-}(\pi-\theta, \phi+\pi)=(-1)^{j+\frac{1}{2}} \xi_{j m}^{-}(\theta, \phi)=(-1)^{|k|} \xi_{j m}^{-}(\theta, \phi)=(-1)^{k} \xi_{j m}^{-}(\theta, \phi), \tag{5.38b}
\end{align*}
$$

where we have used equations (5.34) and (4.49).

### 5.2.2 Minkowski space-time

The procedure of solving the Dirac equation in Minkowski space-time is completely analogous, and the Hamiltonian takes the form

$$
\begin{equation*}
\hat{H}_{0}=\frac{1}{i} \alpha^{y}\left(\frac{\partial}{\partial y}+\frac{1}{y}\right)+\frac{i}{y} \alpha^{y} \beta \hat{K}+m_{0} \beta . \tag{5.39}
\end{equation*}
$$

The only difference regarding the solutions is the different behaviour of the radial wave functions under parity transformations. If the quantum number $k$ is positive, the spherical Bessel functions of the second kind will be singular at $y=0$ and by requiring that the spinor $\psi$ be normalizable at the origin, we may set the integration constants $C_{2}$ and $C_{4}$ to zero [33]. If in turn $k$ is negative, the spherical Bessel functions of the second kind will be regular at $y=0$,

$$
\begin{equation*}
y_{k}(y)=(-1)^{k+1} j_{-k-1}(y) \tag{5.40}
\end{equation*}
$$

and the spherical Bessel functions of the first kind will be singular at $y=0$,

$$
\begin{equation*}
j_{k}(y)=(-1)^{k} y_{-k-1}(y) . \tag{5.41}
\end{equation*}
$$

Hence, for $-k$ a natural number, we set the coefficients $C_{1}$ and $C_{3}$ equal to zero. Notice that the behaviour of the radial wave functions $f_{k}$ and $g_{k}$ under parity transformations $y \rightarrow-y$ depends on the sign of the quantum number $k$. For positive $k$, we have

$$
\begin{align*}
& f_{k}(-y) \propto j_{k-1}(-y)=(-1)^{k-1} j_{k-1}(y) \propto(-1)^{k+1} f_{k}(y),  \tag{5.42a}\\
& g_{k}(-y) \propto j_{k}(-y)=(-1)^{k} j_{k}(y) \propto(-1)^{k} g_{k}(y), \tag{5.42b}
\end{align*}
$$

while for negative $k$ :

$$
\begin{align*}
& f_{k}(-y) \propto y_{k-1}(-y)=(-1)^{k} y_{k-1}(y) \propto(-1)^{k} f_{k}(y),  \tag{5.43a}\\
& g_{k}(-y) \propto y_{k}(-y)=(-1)^{k+1} y_{k}(y) \propto(-1)^{k+1} g_{k}(y) . \tag{5.43b}
\end{align*}
$$

The relation between the constants $C_{1}, C_{3}$ and $C_{2}, C_{4}$ is the same as stated before for the defect case, equation (5.31a, b).

### 5.3 Single- and multi-valuedness

### 5.3.1 Defect space-time

In the defect space-time $\mathcal{M}_{b}^{4}$, the solutions of the Dirac equation are multi-valued. For positive $k$ and neglecting the time dependence for the moment, they satisfy the following relation on the boundary of the coordinate chart, see figure 4.2:

$$
\begin{equation*}
\left.\psi_{j m k}^{(+)}\right|_{B}=\binom{f_{k}\left(-y_{0}\right) \xi_{j m}^{+}\left(\pi-z_{0}, \pi\right)}{g_{k}\left(-y_{0}\right) \xi_{j m}^{-}\left(\pi-z_{0}, \pi\right)}=\binom{(-1)^{k+1} f_{k}\left(y_{0}\right) \xi_{j m}^{+}\left(z_{0}, 0\right)}{(-1)^{k} g_{k}\left(y_{0}\right) \xi_{j m}^{-}\left(z_{0}, 0\right)}=\left.(-1)^{k+1} \gamma^{0} \psi_{j m k}^{(+)}\right|_{A}, \tag{5.44}
\end{equation*}
$$

where we have already taken the limit $\epsilon \rightarrow 0$. For negative $k$ they satisfy the relation

$$
\begin{equation*}
\left.\psi_{j m k}^{(-)}\right|_{B}=\binom{f_{k}\left(-y_{0}\right) \xi_{j m}^{-}\left(\pi-z_{0}, \pi\right)}{g_{k}\left(-y_{0}\right) \xi_{j m}^{+}\left(\pi-z_{0}, \pi\right)}=\binom{(-1)^{k} f_{k}\left(y_{0}\right) \xi_{j m}^{-}\left(z_{0}, 0\right)}{(-1)^{k+1} g_{k}\left(y_{0}\right) \xi_{j m}^{+}\left(z_{0}, 0\right)}=\left.(-1)^{k} \gamma^{0} \psi_{j m k}^{(-)}\right|_{A} . \tag{5.45}
\end{equation*}
$$

Notice that the prefactors $\pm \gamma^{0}$ correspond to two possible choices of the parity operator $\hat{P}$ acting on a spinor [5]. That these additional factors arise is not surprising, since the Vierbein (5.7) changes its spatial orientation on the boundary of the coordinate chart.

In analogy to section 4.3.1, we now consider the superposition

$$
\begin{equation*}
\Psi \equiv \psi_{j_{1} m_{1} k_{1}}^{(+)}+\psi_{j_{2} m_{2} k_{2}}^{(+)} \equiv \psi_{1}+\psi_{2} \tag{5.46}
\end{equation*}
$$

and construct a scalar [8] by multiplying the hermitian conjugate spinor ${ }^{1} \bar{\Psi}=\Psi^{\dagger} \gamma^{0}$ with $\Psi$,

$$
\begin{equation*}
\bar{\Psi} \Psi=\left(\psi_{1}^{\dagger}+\psi_{2}^{\dagger}\right) \gamma^{0}\left(\psi_{1}+\psi_{2}\right)=\bar{\psi}_{1} \psi_{1}+\bar{\psi}_{2} \psi_{2}+\bar{\psi}_{2} \psi_{1}+\bar{\psi}_{1} \psi_{2} . \tag{5.47}
\end{equation*}
$$

[^5]Next we compute the difference of this scalar when evaluated at two points $A$ and $B$ on both sides of the boundary of the coordinate chart, see figure 4.2,

$$
\begin{align*}
\left.\bar{\Psi} \Psi\right|_{A}-\left.\bar{\Psi} \Psi\right|_{B} & =\left[1-(-1)^{k_{1}-k_{2}}\right]\left(\left.\bar{\psi}_{1} \psi_{2}\right|_{A}+\left.\bar{\psi}_{2} \psi_{1}\right|_{A}\right)  \tag{5.48}\\
& =\left.\left[1-(-1)^{k_{1}-k_{2}}\right] \cdot 2 \Re\left(\bar{\psi}_{1} \psi_{2}\right)\right|_{A} .
\end{align*}
$$

This quantity only vanishes identically if the quantum numbers $k_{i}$ differ by an even integer. Analogous results hold for the components of the four-current density $j^{\mu}=\bar{\Psi} \gamma^{\mu} \Psi[8,65]$. Similarly, we can consider the behaviour of pseudoscalars constructed out of a spinor and its hermitian conjugate at the boundary of the coordinate chart:

$$
\begin{equation*}
\left.\left.\bar{\psi}_{j m k}^{(+)}\right|_{B} \gamma^{5} \psi_{j m k}^{(+)}\right|_{B}=\left.\left(\left.\psi_{j m k}^{(+)}\right|_{A}\right)^{\dagger}\left(\gamma^{0}\right)^{\dagger}(-1)^{k+1} \gamma^{0} \gamma^{5}(-1)^{k+1} \gamma^{0} \psi_{j m k}^{(+)}\right|_{A}=-\left.\left.\bar{\psi}_{j m k}^{(+)}\right|_{A} \gamma^{5} \psi_{j m k}^{(+)}\right|_{A}, \tag{5.49}
\end{equation*}
$$

where we have used the fifth Dirac matrix $\gamma^{5}=i \gamma^{0} \gamma^{1} \gamma^{2} \gamma^{3}$. Clearly, pseudoscalars constructed out of arbitrary spinors will change their sign on the boundary of the coordinate system. At this point the question arises, in which frame observable quantities are actually to be determined if the Vierbeine change their orientation. Should there be an additional minus sign if the orientation of the Vierbein is reversed? We see that this does not affect the argumentation, however: The sign change of a pseudoscalar will then occur at the defect core. The situation is completely analogous for the polarization vector, given by [27]

$$
\begin{equation*}
S^{\mu}=i \bar{\Psi} \gamma^{\mu} \gamma^{5} \Psi \tag{5.50}
\end{equation*}
$$

### 5.3.2 Minkowski space-time

On the contrary, the solutions of the Dirac equation in Minkowski space-time $\mathcal{M}_{0}^{4}$ are strictly single-valued. For positive $k$, we obtain

$$
\begin{equation*}
\left.\psi_{j m k}^{(+)}\right|_{B}=\binom{f_{k}\left(-y_{0}\right) \zeta_{j m}^{+}\left(\pi-z_{0}, \pi\right)}{g_{k}\left(-y_{0}\right) \zeta_{j m}^{-}\left(\pi-z_{0}, \pi\right)}=\binom{(-1)^{k+1} f_{k}\left(y_{0}\right)(-1)^{k-1} \zeta_{j m}^{+}\left(z_{0}, 0\right)}{(-1)^{k} g_{k}\left(y_{0}\right)(-1)^{k} \zeta_{j m}^{-}\left(z_{0}, 0\right)}=\left.\psi_{j m k}^{(+)}\right|_{A} \tag{5.51}
\end{equation*}
$$

and, analogously, for negative $k$ :

$$
\begin{equation*}
\left.\psi_{j m k}^{(-)}\right|_{B}=\binom{f_{k}\left(-y_{0}\right) \zeta_{j m}^{-}\left(\pi-z_{0}, \pi\right)}{g_{k}\left(-y_{0}\right) \zeta_{j m}^{+}\left(\pi-z_{0}, \pi\right)}=\binom{(-1)^{k} f_{k}\left(y_{0}\right)(-1)^{k} \zeta_{j m}^{-}\left(z_{0}, 0\right)}{(-1)^{k+1} g_{k}\left(y_{0}\right)(-1)^{k+1} \zeta_{j m}^{+}\left(z_{0}, 0\right)}=\left.\psi_{j m k}^{(-)}\right|_{A} \tag{5.52}
\end{equation*}
$$

Clearly, scalars and pseudoscalars constructed out of these spinors will always be singlevalued throughout the space-time $\mathcal{M}_{0}^{4}$.

### 5.3.3 Orthogonality of solutions

The scalar product (4.53) can be generalized to spinors by [36]

$$
\begin{equation*}
\left(\psi_{1}, \psi_{2}\right)=-i \int_{\Sigma} \mathrm{d} \Sigma \psi_{1}^{\dagger} \hbar \psi_{2} \tag{5.53}
\end{equation*}
$$

where $\Sigma$ is a spacelike hypersurface and $\not \hbar$ the contraction $\gamma_{\mu} n^{\mu}$ of the unit normal to it. The results are essentially the same as in section 4.3.2-in $\mathcal{M}_{b}^{4}$ spinors with angular momentum quantum numbers $j_{i}$ differing by an odd integer are not necessarily orthogonal.

### 5.4 Discussion

An interesting outcome of this chapter is that it is possible to construct spin- $1 / 2$ fields globally in this space-time although the geometry is non-Lorentzian and the metric tensor becomes degenerate at the defect core. In Lorentzian geometry, it is a necessary and sufficient condition for a spinor structure to exist on a space-time that there is a global field of orthonormal Vierbeine, i. e. a set of four smooth, linearly independent and non-vanishing vector fields [30]. It seems to be possible to weaken this demand in the non-Lorentzian setting.

We have found in section 5.3.1 that the solutions (5.24a, b) to the Dirac equation (5.1) in the defect space-time $\mathcal{M}_{b}^{4}$ are generally multi-valued functions of position, i. e. it is not possible to assign to any solution one unique value per space-time point globally. What is more, there are two kinds of multi-valued solutions, which have a different behaviour at the boundary of the coordinate system. Hence, in order to keep quantities like scalars constructed out of the spinors ( $5.24 \mathrm{a}, \mathrm{b}$ ) single-valued, it appears natural to split up the solutions into two different sets of spinor fields, in analogy to the scalar field. This also ensures that we have a single-valued Dirac Lagrangian

$$
\begin{equation*}
\mathcal{L}=i \bar{\Psi} \gamma^{\mu} \nabla_{\mu} \Psi-m_{0} \bar{\Psi} \Psi . \tag{5.54}
\end{equation*}
$$

This splitting into two different fields also appears to be a consequence of the non-trivial topology of the space-time $\mathcal{M}_{b}^{4}$, see also chapter 6. No such effect is observed in the topologically trivial Minkowski space-time $\mathcal{M}_{0}^{4}$.

Since in the defect space-time $\mathcal{M}_{b}^{4}$, according to equations (5.44) and (5.45), arbitrary solutions evaluated on both sides of the boundary of the coordinate chart are related by a multiplication with the Dirac matrix $\gamma^{0}$ (the parity operator) and the Dirac matrix $\gamma^{5}$ (the chirality operator) does not commute with the $\gamma^{0}$ matrix,

$$
\begin{equation*}
\left[\gamma^{5}, \hat{P}\right]=\left[\gamma^{5}, \pm \gamma^{0}\right] \neq 0, \tag{5.55}
\end{equation*}
$$

it does not appear to be possible to reduce the four-component Dirac spinors to twocomponent spinors with fixed eigenvalue of the chirality operator in this space-time. This must be a consequence of the non-Lorentzian geometry of the space-time, since on an orientable Lorentzian manifold, massless fermions can be reduced to chiral particles [5]. Finally, notice that the Dirac Hamiltonian (5.23) in the defect space-time does not commute with a parity operation,

$$
\begin{equation*}
\left[\gamma^{0}(y \rightarrow-y), \hat{H}_{b}\right] \neq 0, \tag{5.56}
\end{equation*}
$$

whereas the Hamiltonian (5.39) in Minkowski space-time does. As a consequence, the solutions to the Dirac equation in the defect space-time are also not invariant under this operation.

It has been pointed out by R. Geroch and G. Horowitz [31] that in the case of Lorentzian manifolds one can draw the conclusion from the experimental observation of symmetry
violations in particle physics such as parity violation that the space-time manifold must be total orientable, i.e. only the topological orientability of the underlying manifold is of relevance. According to the results of chapter 2, the underlying manifold of this space-time is orientable, yet there seem to be severe problems when constructing chiral particles and considering interaction terms like $\overline{v_{e}} \mathbb{Z}\left(\mathbb{1}+\gamma^{5}\right) v_{e}$ in the standard model.

We want to note that the meaning of the particle concept is not clear if a space-time does not possess a Cauchy surface [8], and, as mentioned in section 1.5, the nature of the Cauchy problem is obscure in the case of non-Lorentzian geometry.

Completely analogously to the case of a scalar field, also the spinor field in the defect space-time does possess solutions with imaginary wavenumbers, signifying that the background metric is actually unstable against the perturbation by a spinor field. These instabilities should be respected when the propagation of a spinor field in a space-time foam made out of the presented defects is considered. This does not hold true in the case of Minkowski space-time $\mathcal{M}_{0}^{4}$.

## Chapter 6

## Twisted quantum fields

It has been noted by Isham [42] and others that the global topology of a space-time may influence the propagation of matter fields. In particular, a fixed non-trivial topology of a pseudo-Riemannian space-time may lead to the existence of inequivalent twisted quantum fields which, mathematically, are sections of non-isomorphic vector bundles, see figure 6.1. The set of inequivalent twisted quantum fields is then determined by topological invariants of the space-time manifold alone. In the following, we compare the results of chapters 4 and 5 with these theoretical concepts.

### 6.1 Homotopy type

### 6.1.1 Defect space-time

The first observation concerning our defect space-time is the topological splitting into a spatial hypersurface and the timelike real line, $\mathcal{M}_{b}^{4}=\mathbb{R} \times \mathcal{M}_{b}^{3}$. The latter can be continuously contracted onto a point, i. e. the spatial hypersurface $\mathcal{M}_{b}^{3} \times\{0\}$ is a deformation retract of the whole space-time $\mathcal{M}_{b}^{4}$. Explicitly, the deformation retraction can be written as

$$
\begin{equation*}
f: \mathcal{M}_{b}^{4} \times[0,1] \rightarrow \mathcal{M}_{b}^{4} ; \quad((t, \boldsymbol{x}), s) \mapsto(t(1-s), \boldsymbol{x}) \tag{6.1}
\end{equation*}
$$



Figure 6.1. Two inequivalent vector bundles over the circle, $S^{1}$, and sections of them, which may be interpreted as two inequivalent real scalar fields [42].
where $t$ denotes the coordinate on the timelike axis and $\boldsymbol{x}$ denotes a point on the spacelike hypersurface. The map $f$ is continuous and satisfies the defining conditions of a deformation retraction [37]:

$$
\begin{align*}
& f((t, \boldsymbol{x}), 0)=\mathbb{1}_{\mathcal{M}_{b}^{4}} \\
& f((t, \boldsymbol{x}), 1) \in \mathcal{M}_{b}^{3} \times\{0\}  \tag{6.2}\\
& f((0, \boldsymbol{x}), 1)=\mathbb{1}_{\mathcal{M}_{b}^{3} \times\{0\}}
\end{align*}
$$

where $\mathbb{1}$ denotes the identity mapping on the respective manifold. Now we consider the mapping

$$
\begin{equation*}
g: \mathcal{M}_{b}^{3} \times[0,1] \rightarrow \mathcal{M}_{b}^{3} ; \quad\left(\left(y_{i}, z_{i}, x_{i}\right), s\right) \mapsto\left(y_{i}(1-s), z_{i}, x_{i}\right) \tag{6.3}
\end{equation*}
$$

where the $\left(y_{i}, z_{i}, x_{i}\right)$ are the coordinates in each of the three coordinate charts introduced in section 2.2. Clearly, this mapping is again continuous and satisfies the conditions

$$
\begin{align*}
g\left(\left(y_{i}, z_{i}, x_{i}\right), 0\right) & =\mathbb{1}_{\mathcal{M}_{b}^{3}} \\
g\left(\left(y_{i}, z_{i}, x_{i}\right), 1\right) & \in \mathbb{R} P^{2} \times\{0\}  \tag{6.4}\\
g\left(\left(0, z_{i}, x_{i}\right), 1\right) & =\mathbb{1}_{\mathbb{R} P^{2} \times\{0\}} .
\end{align*}
$$

The mapping $g$ defines a deformation retraction, and it is a standard result of homotopy theory that deformation retractions from one manifold onto a submanifold are special cases of homotopy equivalences [37]. Since homotopy equivalence is an equivalence relation, we thus see that our defect space-time is homotopy equivalent to the real projective plane,

$$
\begin{equation*}
\mathcal{M}_{b}^{4} \simeq \mathbb{R} P^{2} \tag{6.5}
\end{equation*}
$$

As a byproduct, we find that the fundamental group of the space-time $\mathcal{M}_{b}^{4}$ is isomorphic to the one of $\mathbb{R} P^{2}$ [37, Proposition 1.18],

$$
\begin{equation*}
\pi_{1}\left(\mathcal{M}_{b}^{4}\right) \cong \pi_{1}\left(\mathbb{R} P^{2}\right) \tag{6.6}
\end{equation*}
$$

### 6.1.2 Minkowski space-time

Similarly, it is easy to see that Minkowski space-time $\mathcal{M}_{0}^{4}$ is homotopy equivalent to a point,

$$
\begin{equation*}
\mathcal{M}_{0}^{4} \simeq\{p t\} \tag{6.7}
\end{equation*}
$$

and hence topologically trivial. Explicitly, the deformation retraction is given by

$$
\begin{equation*}
h: \mathcal{M}_{0}^{4} \times[0,1] \rightarrow \mathcal{M}_{0}^{4} ; \quad((t, x), s) \mapsto((1-s) t,(1-s) \boldsymbol{x}) . \tag{6.8}
\end{equation*}
$$

Thus, according to the dimension axiom [37], all homology and cohomology groups of nonzero order are trivial for Minkowski space-time.


Figure 6.2. Construction of real projective plane from cellular complexes [37]. Equally denominated points are identified, similarly for edges.

### 6.2 Homology and cohomology groups

In this section, we compute certain homology and cohomology groups of the defect spacetime $\mathcal{M}_{b}^{4}$. It is a fundamental property of a homology theory that homotopy equivalent manifolds have isomorphic homology and cohomology groups [37, Proposition 4.21]. We thus conclude that the homology groups of our space-time $\mathcal{M}_{b}^{4}$ are isomorphic to the homology groups of the spatial hypersurface $\mathcal{M}_{b}^{3}$, which in turn are isomorphic to the homology groups of the real projective plane,

$$
\begin{equation*}
H_{p}\left(\mathcal{M}_{b}^{4}, R\right) \cong H_{p}\left(\mathcal{M}_{b}^{3}, R\right)=H_{p}\left(\mathbb{R} P^{3}-\{\mathrm{pt}\}, R\right) \cong H_{p}\left(\mathbb{R} P^{2}, R\right), \tag{6.9}
\end{equation*}
$$

where $p$ denotes a natural number and $R$ a coefficient ring. The homology groups can be related to cohomology groups, and, for a smooth manifold, the cohomology groups are isomorphic to those of its cellular decomposition. The cellular decomposition of the real projective plane is depicted in figure 6.2.

### 6.2.1 Homology

In order to determine homology groups of our space-time explicitly, we rely on the explicit construction via the boundary map,

$$
\begin{equation*}
H_{n}=\operatorname{Ker}\left(\partial_{n}\right) / \operatorname{Im}\left(\partial_{n+1}\right), \tag{6.10}
\end{equation*}
$$

where the boundary map $\partial_{n}$ maps $n$-complexes to their boundaries. For example, edges are mapped to the abstract difference of their endpoints, and faces are mapped to the abstract sum of their edges, where a sign represents the direction of the edge.

In the stated case, the edge $c$ is an element of $\operatorname{Ker}\left(\partial_{1}\right)$, since its endpoints coincide. A further element of $\operatorname{Ker}\left(\partial_{1}\right)$ is the abstract difference of the two edges $a$ and $b$, since $\partial_{1} a=w-v$ and $\partial_{1} b=v-w$. The image of the boundary map $\partial_{2}$ is given by the oriented edges of the two faces $U$ and $L$. When multiplying the mentioned elements of $\operatorname{Ker}\left(\partial_{1}\right)$ and $\operatorname{Im}\left(\partial_{2}\right)$ with elements of the coefficient ring $\mathbb{Z}_{2}$, the result will still be an element of that set.

The sets $\operatorname{Ker}\left(\partial_{1}\right)$ and $\operatorname{Im}\left(\partial_{2}\right)$ do not contain any further elements. The first homology group can now be computed as follows,

$$
\begin{align*}
H_{1}\left(\mathbb{R} P^{2}, \mathbb{Z}_{2}\right) & =\left\{u_{1} c+u_{2}(a-b) \mid u_{1}, u_{2} \in \mathbb{Z}_{2}\right\} /\left\{u_{3}(a-b+c)+u_{4}(-a+b+c) \mid u_{3}, u_{4} \in \mathbb{Z}_{2}\right\} \\
& =\left\{u_{1} c+u_{2}(a-b+c) \mid u_{1}, u_{2} \in \mathbb{Z}_{2}\right\} /\left\{u_{3}(a-b+c)+2 u_{4} c \mid u_{3}, u_{4} \in \mathbb{Z}_{2}\right\} \\
& =\left\{u_{1} c \mid u_{1} \in \mathbb{Z}_{2}\right\} /\left\{2 u_{4} c \mid u_{4} \in \mathbb{Z}_{2}\right\} \cong \mathbb{Z}_{2} \tag{6.11}
\end{align*}
$$

Similarly, we find

$$
\begin{align*}
H_{1}\left(\mathbb{R} P^{2}, \mathbb{Z}\right) & =\left\{u_{1} c+u_{2}(a-b) \mid u_{1}, u_{2} \in \mathbb{Z}\right\} /\left\{u_{3}(a-b+c)+u_{4}(-a+b+c) \mid u_{3}, u_{4} \in \mathbb{Z}\right\} \\
& =\left\{u_{1} c \mid u_{1} \in \mathbb{Z}\right\} /\left\{2 u_{4} c \mid u_{4} \in \mathbb{Z}\right\} \cong \mathbb{Z}_{2} \tag{6.12}
\end{align*}
$$

The second homology group is trivial, which can be seen as follows: No non-trivial linear combination of the two faces $U$ and $L$ will be mapped to zero by the boundary map $\partial_{2}$, i. e. its kernel is trivial. Besides, the cellular decomposition of $\mathbb{R} P^{2}$ does not contain threedimensional complexes, i. e. the image of the boundary map $\partial_{3}$ is also trivial. Hence,

$$
\begin{equation*}
H_{2}\left(\mathbb{R} P^{2}, \mathbb{Z}\right)=\{0\} /\{0\} \cong\{0\} . \tag{6.13}
\end{equation*}
$$

### 6.2.2 Cohomology

In order to obtain the first cohomology group $H^{1}\left(\mathbb{R} P^{2}, \mathbb{Z}_{2}\right)$ from the just determined first homology groups, we can use a weakened version of the Poincaré duality theorem [37, Theorem 3.30]: Let $\mathcal{M}$ be a closed, $R$-orientable $n$-manifold, where $R$ denotes a coefficient ring and $n$ an integer. Then the $k$-th cohomology group is isomorphic to the ( $n-k$ )-th homology group,

$$
\begin{equation*}
H^{k}(\mathcal{M}, R) \cong H_{n-k}(\mathcal{M}, R) \tag{6.14}
\end{equation*}
$$

Notably, every manifold is $\mathbb{Z}_{2}$-orientable [37]. Thus,

$$
\begin{equation*}
H^{1}\left(\mathbb{R} P^{2}, \mathbb{Z}_{2}\right) \cong H_{1}\left(\mathbb{R} P^{2}, \mathbb{Z}_{2}\right) \cong \mathbb{Z}_{2} \tag{6.15}
\end{equation*}
$$

Determining the second cohomology group $H^{2}\left(\mathbb{R} P^{2}, \mathbb{Z}\right)$ is somewhat more difficult. Since the real projective plane is not orientable, we cannot use Poincare duality to determine the second cohomology group from the zeroth homology group as before. However, due to the dual universal coefficient theorem, there is an isomorphism

$$
\begin{align*}
H^{2}\left(\mathbb{R} P^{2}, \mathbb{Z}\right) & \cong \operatorname{Hom}\left(H_{2}\left(\mathbb{R} P^{2}, \mathbb{Z}\right), \mathbb{Z}\right) \oplus \operatorname{Ext}\left(H_{1}\left(\mathbb{R} P^{2}, \mathbb{Z}\right), \mathbb{Z}\right)  \tag{6.16}\\
& =\operatorname{Hom}(\{0\}, \mathbb{Z}) \oplus \operatorname{Ext}\left(\mathbb{Z}_{2}, \mathbb{Z}\right)
\end{align*}
$$

where we have used $H_{1}\left(\mathbb{R} P^{2}, \mathbb{Z}\right) \cong \mathbb{Z}_{2}$ and $H_{2}\left(\mathbb{R} P^{2}, \mathbb{Z}\right) \cong\{0\}$, equations (6.12) and (6.13). The expression $\operatorname{Hom}(\{0\}, \mathbb{Z})$ denotes the set of homomorphisms from the trivial group to the integers, which only contains one element. It is a standard property of the Ext functor, that

$$
\begin{equation*}
\operatorname{Ext}\left(\mathbb{Z}_{n}, G\right) \cong G_{n}, \tag{6.17}
\end{equation*}
$$

where $G$ denotes an abelian group [37]. Thus,

$$
\begin{equation*}
H^{2}\left(\mathcal{M}_{4}^{b}, \mathbb{Z}\right) \cong H^{2}\left(\mathbb{R} P^{2}, \mathbb{Z}\right) \cong\{0\} \oplus \mathbb{Z}_{2} \cong \mathbb{Z}_{2} \tag{6.18}
\end{equation*}
$$

We observe that our results for the cohomology of the real projective plane conform to [55].

### 6.3 Inequivalent quantum fields

### 6.3.1 Scalar fields

The number of inequivalent real scalar fields is equal to the number of elements of the first cohomology group with coefficients in the integers modulo two [42],

$$
\begin{equation*}
H^{1}\left(\mathcal{M}, \mathbb{Z}_{2}\right) \cong H^{1}\left(\mathbb{R} P^{2}, \mathbb{Z}_{2}\right) \cong \mathbb{Z}_{2} \tag{6.19}
\end{equation*}
$$

and in accordance with our results of chapter 4, this group possesses two elements. For complex scalar fields, the set of inequivalent twisted quantum fields is in one-to-onecorrespondence to the second cohomology group of the space-time with coefficients in the integers [42],

$$
\begin{equation*}
H^{2}(\mathcal{M}, \mathbb{Z}) \cong H^{2}\left(\mathbb{R} P^{2}, \mathbb{Z}\right) \cong \mathbb{Z}_{2} \tag{6.20}
\end{equation*}
$$

Again, this is in accordance with our previous results. Hence, the non-Lorentzian geometry of the space-time $\mathcal{M}_{b}^{4}$ does not seem to affect the twisted fields defined over the manifold. After all, this is not surprising, since cohomology is an algebraic concept.

Intuitively, it is clear that every real scalar field, which can be described as a section of a twisted, i. e. non-orientable, vector bundle takes on the value zero at some place. The case, where the spatial hypersurface of the space-time is a circle, is very descriptive and shown in figure 6.1. That this is really true in general can also be shown rigorously [57] and has physically interesting implications. When adding to the Lagrangian of the scalar field a quartic potential term,

$$
\begin{equation*}
\mathcal{L}=g^{\mu v}\left(\partial_{\mu} \phi\right)\left(\partial_{\nu} \phi\right)+\frac{\mu^{2}}{2}\left(\phi^{2}-a^{2}\right)^{2} \tag{6.21}
\end{equation*}
$$

the symmetry of the Lagrangian under the $\mathrm{O}(1)$ transformation

$$
\begin{equation*}
\phi \rightarrow-\phi \tag{6.22}
\end{equation*}
$$

will be broken, since the potential term has two minima, $\phi_{1}=a$ and $\phi_{2}=-a$. Hence, for twisted real scalar fields, these equations can never be satisfied globally. This is the case for the wave functions $\Phi_{k l m}$ of chapter 4 with odd $l$, and it can be concluded [42] that the process of spontaneous symmetry breaking is not possible or at least suppressed for such fields.

### 6.3.2 Spin-1/2 fields

For spin- $1 / 2$ fields, the situation is more complicated, and no twisted spinor fields exist in the sense as in the simple case of the real and complex scalar fields. However, a topologically non-trivial space-time (or rather its tangent bundle) may possess a spin structure. That a vector bundle with structure group $\mathrm{SO}(1,3)$ possesses a spin structure is equivalent to saying that its structure group may be replaced by the two-fold covering group $\operatorname{Spin}(1,3)$ [56]. It is a necessary but not sufficient condition that a spin structure exist in order for a globally defined spinor field to exist [64].

If the tangent bundle of a space-time does possess a spin structure, the number of inequivalent spin structures for a given topology is, like the number of inequivalent complex scalar fields, equal to the number of elements in the cohomology group [42]

$$
\begin{equation*}
H^{1}\left(\mathcal{M}, \mathbb{Z}_{2}\right) \tag{6.23}
\end{equation*}
$$

However, we have seen in section 5.1 on page 65 that the determinant of any smooth Vierbein changes its sign on the defect core. The Vierbein (5.7) hence is not a section in the $\mathrm{SO}(1,3)$ principal bundle associated to the tangent bundle of the manifold $\mathcal{M}_{b}^{4}$, since at the boundary of the coordinate chart it changes by an $\mathrm{O}(1,3)$ transformation, as can also be seen by direct calculation or by simply considering the determinants of the Vierbein. The treatment in reference [42] does not apply to our situation for this reason. Notice that there is a generalization of the concept of a spin structure to non-orientable vector bundles with structure group $\mathrm{O}(1,3)$ or $\mathrm{O}(3,1)$, called pin structure [44]. We do not consider this any further, however.

## Chapter 7

## Summary

In this thesis we investigated an isolated nonsingular space-time defect, which could be used to describe a classical space-time foam. We now summarize the results of this thesis and try to answer some of the questions, which appeared after the space-time defect was discovered [46].

### 7.1 Discussion

One interesting observation was made in chapter 3, see also figure 3.5. We want to stress once more that this geodesic is a perfectly regular solution of the geodesic equation. We remember that, according to our understanding, we are in flat space, since the components of the Riemann curvature tensor vanish identically. The notion of a straight line makes sense in flat space, and there are no forces acting on the test particle. However, according to Newton's first law of motion ${ }^{1}$,
"[e]very body perseveres in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed thereon."

Now we could imagine an observer holding a ruler with negligible mass along the dashed path, just touching the defect core. The trajectory will then not be a straight line in this sense, in contradiction to Newtonian physics. A possible resolution of this paradox would be that the defect core can in fact not be seen as a patch of flat empty space and hence the notion of a straight line does not make sense in the way we think about it. We remember that also in the case of the Silberstein solution, which was presented in chapter 1, curvature invariants like the Kretschmann scalar remain finite near the symmetry axis. The violation of the regularity condition (1.7) is usually interpreted as a consequence of the existence of a thin material strut on the axis. Such an interpretation of course would mean that the space-time defect can in fact not be seen as a physically acceptable solution of the vacuum Einstein field equations.

[^6]When we described the motion of test particles passing the defect core at non-vanishing incidence angle in chapter 3, we had to make additional assumptions, whose eligibility might not be clear. We want to stress once again that the local conservation of energy and (angular) momentum is ensured by the Lorentzian geometry in standard general relativity, see section 1.4. Nothing seems to prevent us from describing these curves through the defect core in a consistent way. According to the definition of geodesic completeness given by [38], the space-time defect is geodesically incomplete and hence singular, however.

Another important observation is that quantum fields and also gravitational perturbations are globally regular solutions of their respective field equations. This, in turn, also implies the instability of the geometry of the defect, since we can construct exponentially growing perturbations of the space-time geometry by quantum fields, for example-keeping in mind that our arguments are only valid in a first approximation. The instability under perturbations by quantum fields is important if we wish to determine the implications of a classical space-time foam built out of nonsingular space-time defects on the propagation of particles. When taking the absolute value of the imaginary wavenumber $\beta$ of section 4.2.1 as the inverse of the wave-length of, say, visible light, the scalar perturbations double in size after roughly $10^{-14} \mathrm{~s}$ in the limit of negligible mass. This time should be much longer than the time it takes a wave package to pass the defect if the back-reaction of the quantum field on the geometry of the space-time defect is neglected. Of course, these naive stability arguments need not be valid within the framework of a quantum theory of gravitation and there is no understanding of what determines the topology of a space-time on large and small scales [6].

Further, we have found in chapters 4-6 that for an isolated defect it is necessary to split the quantum fields into topologically inequivalent configurations, i.e. into several twisted quantum fieds if observable quantities like the energy-momentum tensor should be single-valued. This effect is a consequence of the global topological structure of the space-time and there does not seem to be any way to avoid these problems. According to our calculations in chapter 6, this is in complete analogy to the case of Lorentzian manifolds for scalar fields, whereas the situation is different when adding a half-integer spin to the quantum field. It is hence not possible to describe matter fields that behave like in ordinary Minkowski space-time far away from the defect core in a mathematically self-consistent way. However, this does not tell us anything about the situation, where we have a classical space-time foam and the global topology of space-time is more complicated.

In chapter 5 we described a field with half-integer spin in the defect space time. The nonLorentzian geometry implies that we always have to deal with degenerate, non-invertible Vierbeine. Yet, it turned out that it is possible to obtain globally valid solutions of the Dirac equation although in Lorentzian geometry the non-vanishing of the Vierbein field is a necessary and sufficient condition for a globally defined spinor field to exist [29]. The obtained solutions have very peculiar properties, however, and do not seem to be physically realistic.

### 7.2 Conclusion

With regards to the question whether the space-time defect is mathematically acceptable, we conclude from this work that the answer is yes. This space-time clearly is a smooth manifold and nothing prevents us from calling it a mathematically valid solution of the vacuum Einstein field equations. Also from a purely geometrical standpoint, there is no reason to discard degenerate covariant metrics, as the components of the (singular) contravariant metric tensor do not have a direct geometrical meaning.

The question of whether it is a physically acceptable solution of the vacuum Einstein field equations is not so easily answered. The pathological behaviour of classical particle trajectories suggest that the space-time defect consists of more than empty space. Further, there are strong indications that the presented solution is inherently unstable against small perturbations within the framework of classical physics. Yet, it is conceivable that the presented defect can describe small-scale structures of space-time.

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[^0]:    ${ }^{1}$ Two space-times are taken to be equivalent if they are isometric.

[^1]:    ${ }^{2}$ Strictly speaking, equation (1.16) is still not well-behaved for $g=0$. The point of the modification is to replace equation (1.15) by its numerator, however.

[^2]:    ${ }^{1}$ In the following, we will refer to $\tilde{E}$ and $\tilde{L}$ simply as "energy" and "angular momentum".

[^3]:    ${ }^{2}$ It is conceivable that an observer will start, say, a rocket engine at some point. It then no longer moves on a geodesic.

[^4]:    ${ }^{1}$ It is, in principle, possible to set $d_{\Delta}$ to an arbitrary value. This corresponds to adding multiples of the first solution $R_{k l}^{(1)}$ to the second solution $R_{k l}^{(2)}$.

[^5]:    ${ }^{1}$ Notice that this relation is changed by an additional sign, when a past oriented Vierbein is used [15].

[^6]:    ${ }^{1}$ English translation of Latin original [60].

