Gauge dependence of Green functions and algebraic calculation of general two-loop self-energies

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Chapter 1

Introduction

The ultimate goal of elementary particle physics is to gain insight into the basic principles of nature. Accordingly, research aims at a description of a whole variety of seemingly different phenomena in terms of just a few fundamental laws. An important step in this direction was achieved with the Glashow–Salam–Weinberg (GSW) model [1], which provides a unified formulation of the electromagnetic and the weak interaction that is in agreement with all currently known experimental facts of electroweak origin. It is therefore also called the electroweak Standard Model (SM).

The GSW model is based on the principle of gauge invariance which relates the electroweak interaction to the internal symmetries classifying the elementary particles. Since also quantum chromodynamics (QCD), the theory of the strong interaction, and general relativity, the classical theory of gravity, are gauge theories there is some hope that the principle of gauge invariance may eventually open the way to a unification of all basic interactions.

Despite the impressive success of the SM in describing electroweak phenomena, the particle content predicted by the model has not yet been completely verified by experiment. Whereas the measurements at CDF now give some evidence for the existence of the top quark [2], the Higgs particle being crucial for the internal consistency of the SM is still missing. The direct search for these particles is limited by the energy range of the present accelerators and can only set lower limits on their masses. Moreover, in theories going beyond the SM one expects the existence of particle states at higher mass scales possibly lying outside the range of any accelerator that may go into operation in the near future.

An indirect tool for obtaining information about parts of the theory not directly accessible by experiment is provided by performing precision measurements at a lower energy scale and confronting them with the theoretical predictions. Experiments of this kind are currently carried out in particular at LEP1, where for example the uncertainty in measuring the width and mass of the Z-boson has been reduced to less than 1.5 per mille and 0.05 per mille, respectively [3]. This accuracy allows to test the theory on its quantum level. Since at this level all parameters of the theory enter the predictions for the experimentally measured quantities one can determine a mass range allowed for unknown particles of the model, i.e. both a lower and an upper limit.

In this way one has obtained a mass range for the top quark of $m_t = 169^{+16}_{-18}{}^{+17}_{-20}$ GeV [4], where for the central value a Higgs mass of 300 GeV is assumed and the second error corresponds to a variation of the Higgs mass from 60 to 1000 GeV. This result is in perfect agreement with the value favored by the direct search at CDF, 174 ± 16 GeV [2]. The perturbative evaluation of the quantum corrections used in this analysis was mainly restricted to one-loop calculations. In contrast to the top quark, the dependence of the SM predictions on the Higgs mass is relatively weak so that no reliable bound could be established so far. One hopes that a precise determination of the top mass and the improved measurement of the W-boson mass at LEP2 will open the way to constrain the Higgs mass. On the theoretical side, this will require more precise calculations, i.e. the evaluation of higher-order corrections. These are also needed in order to discriminate possible effects of new physics from the SM predictions.

At one-loop order, many calculations of electroweak virtual corrections have been carried out during the last years. Some leading second-order effects are usually taken into account by suitable resummations, but rather limited results have been obtained for irreducible two-loop corrections.

In the study of the latter the self-energies are a central building block. These so-called oblique corrections are universal, i.e. process-independent, in contrast to the process-specific contributions from vertices, boxes and bremsstrahlung. A number of authors (see e.g. Refs. [5–7]) recently stressed the importance of the oblique corrections for the analysis of precision experiments and in reformulations of electroweak radiative corrections using effective Lagrangians. Taking a practical point of view, it is often possible to reduce more complicated diagrams to graphs of self-energy type by neglecting the masses of some external particles. On the other hand, the evaluation of self-energies is usually needed for non-zero external momentum, for instance in order to carry out the renormalization.

In the SM so far no complete calculation of a two-loop self-energy has been carried out. This fact is due to the complicated structure and large number of the Feynman diagrams contributing at the two-loop level. Results were obtained treating the limiting cases of a heavy fermion doublet [8], a heavy top quark [9] and a large Higgs mass [10, 11]. In view of the fact that the mass expected for the top-quark (see above) is of the order of the heavy gauge-boson masses and that the Higgs mass may vary over a wide range, a calculation allowing for general values of the top and Higgs mass, the gauge-boson masses and the external momentum would be desirable.

In the first part of this thesis a systematic approach to the calculation of all two-loop self-energies in the general case is presented. The discussion focuses on the electroweak SM, but the method is valid for any renormalizable model. It generalizes the treatment of the gauge-boson self-energies in 't Hooft–Feynman gauge given in Ref. [12]. It is shown that a large part of the calculation can be carried out in a purely algebraic way. To this end a technique for the tensor decomposition of two-loop self-energy integrals is developed. This generalizes the well-known method worked out for one-loop integrals by Passarino and Veltman [13]. By making use of tensor decompositions and symmetry properties of the integrals it is shown that all Feynman amplitudes can be reduced to a set of two-loop standard scalar integrals. For a further analytical or numerical evaluation only three different types of two-loop integrals need to be considered.

Recently, a different method for the tensor decomposition of two-loop integrals based on numerical techniques was discussed in Ref. [14]. The algebraic approach pursued here allows a high degree of automatization which is of importance for calculations of two-loop self-energies in the SM possibly involving several thousands of Feynman diagrams. The results of the algebraic calculation are transparent and have the benefit that very stringent checks can easily be performed. Furthermore, important properties like the gauge dependence of the considered quantities can directly be studied at the algebraic level, i.e. without having to use any explicit analytical or numerical expression for the standard integrals. As an application, the algebraic calculation of the light-fermion contributions to the two-loop gauge-boson self-energies in the SM is presented.

Whereas the problems discussed above are of mainly technical nature, the second topic addressed in this thesis is a more conceptual one, namely the gauge dependence of Green functions. Although gauge invariance is used as the guiding principle in constructing the SM and QCD, one is forced to fix a gauge in order to quantize these theories. This leads to a number of complications. In particular, the Green functions — being the quantities one is primarily concerned with in field theory — are in general gauge-dependent. The gauge independence of the physical S matrix can only be restored at the very end by projecting onto physical degrees of freedom.

Before proceeding, one should briefly reconsider the notion of gauge invariance and gauge independence. The former means invariance of the Lagrangian under gauge transformations and gives rise to Ward identities between Green functions. The latter becomes relevant since quantization is done by fixing a gauge. It means independence of the method of gauge fixing.

The gauge dependence of Green functions is of particular relevance for the treatment of higher-order corrections. In many calculations a resummation of gauge-dependent higher-order contributions is performed in order to define running couplings or to take into account finite-width effects in resonance regions. Furthermore, as long as the two-loop evaluation of complete S-matrix elements is out of reach, calculations will often be restricted to certain Green functions. On the other hand, also many applications at one-loop order deal with off-shell Green functions. Frequently off-shell formfactors are discussed, e.g. for the neutrino or the top quark. As mentioned above, off-shell self-energies are often used to parametrize electroweak radiative corrections.

Owing to the gauge dependence, the actual form of a certain Green function is not uniquely determined but depends on the specific gauge used for its evaluation. Motivated by this fact, many attempts have been made in the literature to define new Green functions being distinguished from the conventional ones due to certain desirable theoretical properties (see e.g. Refs. [5, 6, 15–18]). Most previous proposals along this line aimed at eliminating the gauge-parameter dependence of the Green functions within a special class of gauges, usually the R_ξ gauges. The new "Green functions" are constructed by rearranging contributions between self-energies, vertex and box diagrams. As one can still shift arbitrary gauge-parameter-independent contributions between the different building blocks the resulting quantities are not unique. This freedom has been used in Refs. [5, 6] to demand several desirable properties of the new "self-energies", like a decent asymptotic behavior and the vanishing of the photon–Z-boson mixing at zero-momentum transfer. These proposals nevertheless resulted in different definitions of the new building blocks. Furthermore, these ad-hoc treatments only refer to four-fermion processes and do not give a general prescription which is applicable to other vertex functions.

A definite prescription for a rearrangement yielding gauge-parameter-independent quantities at one-loop order is provided by the pinch technique (PT) [15–18]. The "Green functions" of the PT were found to obey simple Ward identities and directly fulfill the desirable properties which had to be explicitly enforced in the former treatments mentioned above. Despite these successes, there are a number of problems related to the PT approach. It has been formulated only for the one-loop level and its extension to higher orders is not clear. Even at one-loop order the PT is not applicable in a straightforward manner to all possible Green functions. In addition to these technical difficulties, the PT approach is also plagued with several fundamental problems. Strictly speaking, the resulting building blocks of the S matrix should not be called Green functions since their field theoretical meaning has not been clarified. The process independence of the new "Green functions" constructed within the PT has not been proven. Moreover, the simple Ward identities and other desirable features have not been derived within the PT but only verified for specific one-loop examples.

In order to examine these problems in more detail, an extension of the PT to a two-loop example is studied. It is shown that the new "self-energy" obtained in this way is in fact gauge-parameter-independent. However, its construction turns out to be technically rather involved. Moreover, it requires an admixture of irreducible and reducible contributions which poses problems concerning its theoretical interpretation.

For a further exploration of the problems related to the gauge dependence of Green functions a completely different approach is chosen, which is based on the background-field method (BFM) [19–21]. While the treatments discussed above focus on the gauge-parameter dependence of Green functions, the BFM seeks to avoid the breaking of gauge invariance in the Lagrangian from the start. By decomposing the usual gauge field in the Lagrangian into a quantum field and a background field one can impose the gauge fixing necessary for quantization while keeping the gauge invariance of the effective action.

It is shown in this work that the BFM offers a well-founded framework in which the theoretical and practical problems inherent in the PT approach are resolved. The Green functions in the BFM are directly derived from a gauge-invariant effective action in all orders of perturbation theory. They are shown to fulfill simple QED-like Ward identities and, in comparison to their counterparts in the conventional R_ξ-gauge formalism, possess further desirable properties like an improved asymptotic, UV, and

IR behavior. These properties can directly be derived as a consequence of gauge invariance. The results obtained within the PT in QCD and the SM are shown to exactly coincide with a special case of the BFM results.

The BFM has previously been used mainly in gravity and supergravity [22] where it proved to facilitate computations both technically and conceptually. In QCD it was applied e.g. for the calculation of the β-function [21, 23]. The equivalence of the S matrix in the BFM to the conventional one has been proven in Refs. [24]. In the recent formulation of string motivated rules for more efficient computations in gauge theories, the BFM plays an important role [25]. Nevertheless, for spontaneously broken gauge theories the BFM has up to now only rarely been used. An application to the electroweak one-loop process $Z \to 3\gamma$ was presented in Ref. [26], while in Ref. [27] the one-loop β-function of the electromagnetic coupling in the SM was calculated. So far no complete formulation of the BFM for the SM has been given. In particular, the renormalization has not been worked out in detail.

In order to provide the tools necessary for an application of the BFM in the SM an explicit on-shell renormalization is carried out in accordance with the gauge invariance of the effective action. It is shown that the gauge invariance implies relations between the renormalization constants for parameters and fields and greatly simplifies the renormalization. Furthermore, a complete list of Feynman rules for the SM in the BFM is derived which includes all counterterms necessary for one-loop calculations.

The outline of this work is as follows. In Chapter 2 the electroweak SM is briefly sketched and the quantization is discussed both in the conventional formalism and in the BFM. Chapter 3 deals with the algebraic calculation of two-loop self-energies. A method for the tensor decomposition is worked out and shown to be applicable for all two-loop self-energies. The light-fermion contributions to the gauge-boson self-energies in the SM are calculated and their gauge-parameter dependence is discussed. In Chapter 4 the PT at the one-loop level is briefly reviewed. Its extension to the two-loop case is worked out for an example. Chapter 5 is concerned with the BFM. The Ward identities of the BFM are derived from the gauge-invariant effective action. The PT results are shown to coincide with a special case of the BFM. The desirable theoretical properties of the BFM Green functions are deduced from the Ward identities, and the renormalization of the SM in the BFM is worked out. In the appendix the Feynman rules for the SM in the BFM are listed.

Chapter 2

The electroweak Standard Model: conventional quantization and background-field method

The Glashow-Salam-Weinberg model [1], which is also known as the electroweak Standard Model (SM), gives a description of the electroweak interaction which is theoretically consistent, i.e. renormalizable [28] and free of anomalies, and in agreement with all experiments testing the electroweak sector.

The SM is a non-Abelian gauge theory based on the non-simple group $SU(2)_W \times U(1)_Y$. It is constructed following the principle of gauge invariance. In particular, the Higgs-Kibble mechanism [29] of spontaneous symmetry breaking is used to generate masses for the gauge bosons and fermions while retaining the gauge invariance of the Lagrangian. This is achieved by introducing a scalar field with non-vanishing vacuum expectation value which yields the spontaneous breaking of the $SU(2)_W \times U(1)_Y$ symmetry down to the electromagnetic subgroup $U(1)_{em}$. The gauge boson associated with $U(1)_{em}$ remains massless and can therefore be identified with the photon.

The classical Lagrangian constructed in this way is given explicitly in the first section of this chapter where the parameters and fields of the model and the conventions used in this work are specified. Whereas the classical Lagrangian is manifestly gauge-invariant, a gauge fixing is necessary in order to quantize the theory. In the conventional formalism, which is briefly sketched in Section 2.2, as a consequence of the gauge-fixing procedure the gauge symmetry is spoiled in intermediate steps of calculations and can only be restored at the very end in the S-matrix elements. This leads to a number of technical and conceptual complications. In particular, instead of the simple Ward identities related to the classical Lagrangian one has to deal with more complicated Slavnov–Taylor identities. In Section 2.3 the quantization in the framework of the background-field method (BFM) [19–21] is described. In the BFM, the explicit breaking of gauge symmetry is avoided by decomposing the usual gauge field into a quantum field and a background field. This allows to construct a gauge-invariant effective action.

2.1 The classical Lagrangian

The classical Lagrangian $\mathcal{L}_{\mathrm{C}}$ of the (minimal) electroweak SM consists of the Yang-Mills, the Higgs and the fermion part

$$\mathcal{L}_{\mathrm{C}} = \mathcal{L}_{\mathrm{YM}} + \mathcal{L}_{\mathrm{H}} + \mathcal{L}_{\mathrm{F}}. \tag{2.1}$$

The Yang-Mills part is given as

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{4}\left(\partial_\mu \hat{W}^a_\nu - \partial_\nu \hat{W}^a_\mu + g_2 \varepsilon^{abc} \hat{W}^b_\mu \hat{W}^c_\nu\right)^2 - \frac{1}{4}\left(\partial_\mu \hat{B}_\nu - \partial_\nu \hat{B}_\mu\right)^2, \tag{2.2}$$

where the isotriplet of gauge fields $\hat{W}^a_\mu$, $a = 1,2,3$, is associated with the generators I^a_{W} of the weak isospin group $\mathrm{SU}(2)_{\mathrm{W}}$ and the isosinglet $\hat{B}_\mu$ with the weak hypercharge Y_{W} of the group $\mathrm{U}(1)_{\mathrm{Y}}$. For later convenience we denote the classical fields with a caret. The gauge couplings of the $\mathrm{SU}(2)_{\mathrm{W}}$ and the $\mathrm{U}(1)_{\mathrm{Y}}$ group are denoted as g_2 and g_1, respectively. The Higgs part has the form

$$\mathcal{L}_{\mathrm{H}} = \left(\hat{D}_\mu \hat{\Phi}\right)^\dagger \left(\hat{D}^\mu \hat{\Phi}\right) - V(\hat{\Phi}) \tag{2.3}$$

with the covariant derivative

$$\hat{D}_\mu = \partial_\mu - i g_2 I^a_{\mathrm{W}} \hat{W}^a_\mu + i g_1 \frac{Y_{\mathrm{W}}}{2} \hat{B}_\mu. \tag{2.4}$$

In (2.3), $\hat{\Phi}(x)$ denotes the complex scalar $\mathrm{SU}(2)_{\mathrm{W}}$ doublet field of the minimal Higgs sector with hypercharge $Y^{\hat{\Phi}}_{\mathrm{W}} = 1$

$$\hat{\Phi}(x) = \begin{pmatrix} \hat{\phi}^+(x) \\ \hat{\phi}^0(x) \end{pmatrix}, \tag{2.5}$$

and the Higgs potential reads

$$V(\hat{\Phi}) = \frac{\lambda}{4}\left(\hat{\Phi}^\dagger \hat{\Phi}\right)^2 - \mu^2 \hat{\Phi}^\dagger \hat{\Phi}. \tag{2.6}$$

The left-handed fermions of each lepton ($\hat{L}$) and quark ($\hat{Q}$) generation are grouped into $\mathrm{SU}(2)_{\mathrm{W}}$ doublets (the color index is suppressed)

$$\hat{L}^{\mathrm{L}}_k = \omega_- \hat{L}_k = \begin{pmatrix} \hat{\nu}^{\mathrm{L}}_k \\ \hat{l}^{\mathrm{L}}_k \end{pmatrix}, \qquad \hat{Q}^{\mathrm{L}}_k = \omega_- \hat{Q}_k = \begin{pmatrix} \hat{u}^{\mathrm{L}}_k \\ \hat{d}^{\mathrm{L}}_k \end{pmatrix}, \tag{2.7}$$

the right-handed fermions into singlets

$$\hat{l}^{\mathrm{R}}_k = \omega_+ \hat{l}_k, \qquad \hat{u}^{\mathrm{R}}_k = \omega_+ \hat{u}_k, \qquad \hat{d}^{\mathrm{R}}_k = \omega_+ \hat{d}_k, \tag{2.8}$$

where $\omega_\pm = (1 \pm \gamma_5)/2$ are the projectors on right- and left-handed fields, respectively, k is the generation index, and $\hat{\nu}$, $\hat{l}$, $\hat{u}$ and $\hat{d}$ stand for neutrinos, charged leptons, up-type quarks and down-type quarks, respectively. The weak hypercharge Y_{W} is assigned according to the Gell-Mann Nishijima relation

$$Y_{\mathrm{W}} = 2(Q - I_{\mathrm{W}}^3), \tag{2.9}$$

where Q is the electric charge operator. We write the fermionic part of the Lagrangian as (neglecting quark mixing throughout this work)

$$\begin{aligned} \mathcal{L}_{\mathrm{F}} = & \sum_k \left(\bar{\hat{L}}_k^{\mathrm{L}} i \hat{\not{D}} \hat{L}_k^{\mathrm{L}} + \bar{\hat{Q}}_k^{\mathrm{L}} i \hat{\not{D}} \hat{Q}_k^{\mathrm{L}} \right) \\ & + \sum_k \left(\bar{\hat{l}}_k^{\mathrm{R}} i \hat{\not{D}} \hat{l}_k^{\mathrm{R}} + \bar{\hat{u}}_k^{\mathrm{R}} i \hat{\not{D}} \hat{u}_k^{\mathrm{R}} + \bar{\hat{d}}_k^{\mathrm{R}} i \hat{\not{D}} \hat{d}_k^{\mathrm{R}} \right) \\ & - \sum_k \left(\bar{\hat{L}}_k^{\mathrm{L}} G_k^l \hat{l}_k^{\mathrm{R}} \hat{\Phi} + \bar{\hat{Q}}_k^{\mathrm{L}} G_k^u \hat{u}_k^{\mathrm{R}} \hat{\tilde{\Phi}} + \bar{\hat{Q}}_k^{\mathrm{L}} G_k^d \hat{d}_k^{\mathrm{R}} \hat{\Phi} + \mathrm{h.c.} \right), \end{aligned} \tag{2.10}$$

where G_k^l, G_k^u and G_k^d denote the Yukawa couplings, $\hat{\tilde{\Phi}} = \left(\hat{\phi}^{0*}, -\hat{\phi}^-\right)^T$ is the charge-conjugated Higgs field, and $\hat{\phi}^- = \left(\hat{\phi}^+\right)^*$.

In order to generate masses for the gauge-boson and fermion fields a non-vanishing vacuum expectation value has been assigned to the scalar field, i.e. its classical ground state satisfies

$$|\langle 0|\hat{\Phi}|0\rangle|^2 = \frac{v^2}{2} \neq 0, \tag{2.11}$$

where v is the minimum of the classical Higgs potential, i.e.

$$v^2 = \frac{4\mu^2}{\lambda} \tag{2.12}$$

at lowest order. Since in perturbation theory one has to expand around the ground state the Higgs field is rewritten as

$$\hat{\Phi}(x) = \begin{pmatrix} \hat{\phi}^+(x) \\ \frac{1}{\sqrt{2}}(v + \hat{H}(x) + i\hat{\chi}(x)) \end{pmatrix}, \tag{2.13}$$

where the components $\hat{H}$, $\hat{\chi}$ and $\hat{\phi}^+$ possess zero vacuum expectation values. Whereas $\hat{H}$ corresponds to the physical Higgs field, $\hat{\chi}$, $\hat{\phi}^+$ and $\hat{\phi}^-$ are unphysical degrees of freedom which can be eliminated by a suitable gauge transformation. The gauge in which they are absent is called the unitary gauge. When inserting (2.13) into the Lagrangian, the vacuum expectation value v gives rise to mass terms for the fields in $\mathcal{L}_{\mathrm{C}}$. It also yields a term linear in the Higgs field $\hat{H}$ which we denote by $t\hat{H}$ with

$$t = v(\mu^2 - \frac{\lambda}{4}v^2). \tag{2.14}$$

This tadpole term vanishes at lowest order due to the choice (2.12) for v. At higher orders the minimum of the Higgs potential is shifted due to radiative corrections. This can be compensated for by a suitable renormalization (see Section 5.3).

The physical gauge-boson fields are obtained by diagonalizing the corresponding mass matrices

$$\hat{W}^{\pm}_{\mu} = \frac{1}{\sqrt{2}}\left(\hat{W}^1_{\mu} \mp i\hat{W}^2_{\mu}\right), \qquad \begin{pmatrix} \hat{Z}_{\mu} \\ \hat{A}_{\mu} \end{pmatrix} = \begin{pmatrix} c_{\mathrm{W}} & s_{\mathrm{W}} \\ -s_{\mathrm{W}} & c_{\mathrm{W}} \end{pmatrix} \begin{pmatrix} \hat{W}^3_{\mu} \\ \hat{B}_{\mu} \end{pmatrix}, \tag{2.15}$$

where

$$c_{\mathrm{W}} = \cos\theta_{\mathrm{W}} = \frac{g_2}{\sqrt{g_1^2+g_2^2}}, \qquad s_{\mathrm{W}} = \sin\theta_{\mathrm{W}} = \sqrt{1-c_{\mathrm{W}}^2}, \tag{2.16}$$

and θ_{W} is the weak mixing angle.

The masses of the physical fields are given as

$$M_{\mathrm{W}} = \frac{1}{2}g_2 v, \quad M_{\mathrm{Z}} = \frac{1}{2}\sqrt{g_1^2+g_2^2}\,v, \quad M_{\gamma} = 0, \quad M_{\mathrm{H}} = \sqrt{2}\mu, \quad m_f = G^f\frac{v}{\sqrt{2}}. \tag{2.17}$$

Due to the absence of right-handed neutrinos no mass-generating Yukawa coupling exists for the neutrinos which therefore remain massless. From (2.17) one obtains for the weak mixing angle

$$c_{\mathrm{W}} = \frac{M_{\mathrm{W}}}{M_{\mathrm{Z}}}. \tag{2.18}$$

The electromagnetic coupling is given by

$$e = \frac{g_1 g_2}{\sqrt{g_1^2+g_2^2}}. \tag{2.19}$$

The relations (2.17) and (2.19) allow to replace in $\mathcal{L}_{\mathrm{C}}$ the original set of parameters $g_1, g_2, \lambda, \mu^2, G^f$ by a set of parameters with direct physical meaning, i.e. the electric charge e and the particle masses $M_{\mathrm{W}}, M_{\mathrm{Z}}, M_{\mathrm{H}}, m_f$.

The classical Lagrangian specified above is invariant under local gauge transformations of the group $\mathrm{SU}(2)_{\mathrm{W}} \times \mathrm{U}(1)_{\mathrm{Y}}$. In terms of the physical fields the infinitesimal gauge transformations of the gauge-boson and scalar fields read

$$\begin{aligned}
\delta\hat{W}^{\pm}_{\mu} &= \partial_{\mu}\delta\hat{\theta}^{\pm} \mp ie\hat{W}^{\pm}_{\mu}(\delta\hat{\theta}^A - \frac{c_{\mathrm{W}}}{s_{\mathrm{W}}}\delta\hat{\theta}^Z) \pm ie(\hat{A}_{\mu} - \frac{c_{\mathrm{W}}}{s_{\mathrm{W}}}\hat{Z}_{\mu})\delta\hat{\theta}^{\pm}, \\
\delta\hat{Z}_{\mu} &= \partial_{\mu}\delta\hat{\theta}^Z - ie\frac{c_{\mathrm{W}}}{s_{\mathrm{W}}}(\hat{W}^+_{\mu}\delta\hat{\theta}^- - \hat{W}^-_{\mu}\delta\hat{\theta}^+), \\
\delta\hat{A}_{\mu} &= \partial_{\mu}\delta\hat{\theta}^A + ie(\hat{W}^+_{\mu}\delta\hat{\theta}^- - \hat{W}^-_{\mu}\delta\hat{\theta}^+), \\
\delta\hat{\phi}^{\pm} &= \pm\frac{ie}{2s_{\mathrm{W}}}(\hat{H} + v \pm i\hat{\chi})\delta\hat{\theta}^{\pm} \mp ie\hat{\phi}^{\pm}(\delta\hat{\theta}^A - \frac{c_{\mathrm{W}}^2 - s_{\mathrm{W}}^2}{2c_{\mathrm{W}}s_{\mathrm{W}}}\delta\hat{\theta}^Z), \\
\delta\hat{H} &= \frac{ie}{2s_{\mathrm{W}}}(\hat{\phi}^+\delta\hat{\theta}^- - \hat{\phi}^-\delta\hat{\theta}^+) + \frac{e}{2c_{\mathrm{W}}s_{\mathrm{W}}}\hat{\chi}\delta\hat{\theta}^Z, \\
\delta\hat{\chi} &= \frac{e}{2s_{\mathrm{W}}}(\hat{\phi}^+\delta\hat{\theta}^- + \hat{\phi}^-\delta\hat{\theta}^+) - \frac{e}{2c_{\mathrm{W}}s_{\mathrm{W}}}(\hat{H} + v)\delta\hat{\theta}^Z.
\end{aligned} \tag{2.20}$$

The corresponding transformations of the fermion fields are

$$
\begin{aligned}
\delta \hat{f}_u^{\mathrm{L}} &= \frac{ie}{\sqrt{2}s_{\mathrm{W}}}\hat{f}_d^{\mathrm{L}}\delta\hat{\theta}^+ - ie\left[Q_{f_u}\delta\hat{\theta}^A - \left(\frac{1}{2c_{\mathrm{W}}s_{\mathrm{W}}} - Q_{f_u}\frac{s_{\mathrm{W}}}{c_{\mathrm{W}}}\right)\delta\hat{\theta}^Z\right]\hat{f}_u^{\mathrm{L}}, \\
\delta \hat{f}_d^{\mathrm{L}} &= \frac{ie}{\sqrt{2}s_{\mathrm{W}}}\hat{f}_u^{\mathrm{L}}\delta\hat{\theta}^- - ie\left[Q_{f_d}\delta\hat{\theta}^A + \left(\frac{1}{2c_{\mathrm{W}}s_{\mathrm{W}}} + Q_{f_d}\frac{s_{\mathrm{W}}}{c_{\mathrm{W}}}\right)\delta\hat{\theta}^Z\right]\hat{f}_d^{\mathrm{L}}, \\
\delta \hat{f}^{\mathrm{R}} &= -ieQ_f(\delta\hat{\theta}^A + \frac{s_{\mathrm{W}}}{c_{\mathrm{W}}}\delta\hat{\theta}^Z)\hat{f}^{\mathrm{R}},
\end{aligned}
\tag{2.21}
$$

where $\hat{f}_u^{\mathrm{L}}$ stands for all left-handed up-type quarks and neutrinos of (2.7), $\hat{f}_d^{\mathrm{L}}$ denotes their isospin partners, and $\hat{f}^{\mathrm{R}}$ represents the right-handed singlets of (2.8).

2.2 Quantization in the conventional formalism

In the conventional formalism directly the fields $\hat{\varphi}$ appearing in the classical Lagrangian are quantized, i.e.

$$\mathcal{L}_{\mathrm{C}}(\hat{\varphi}) \to \mathcal{L}_{\mathrm{C}}(\varphi), \tag{2.22}$$

where φ denotes the quantum fields. A gauge-fixing term is added to $\mathcal{L}_{\mathrm{C}}$ which breaks the explicit gauge invariance. Most commonly, a renormalizable R_ξ gauge [30] of 't Hooft type is chosen with the linear gauge fixings

$$F^A = \partial^\mu A_\mu, \quad F^Z = \partial^\mu Z_\mu - \xi_Z M_{\mathrm{Z}}\chi, \quad F^\pm = \partial^\mu W_\mu^\pm \mp i\xi_W M_{\mathrm{W}}\phi^\pm, \tag{2.23}$$

yielding the gauge-fixing Lagrangian

$$\mathcal{L}_{\mathrm{GF}}^{\mathrm{conv}} = -\left[\frac{1}{2\xi_A}(F^A)^2 + \frac{1}{2\xi_Z}(F^Z)^2 + \frac{1}{\xi_W}F^+F^-\right]. \tag{2.24}$$

The R_ξ gauge is specified by one gauge parameter ξ_i $(i = A, Z, W)$ for each vector boson. It avoids lowest-order mixing between gauge fields and scalars.

In order to compensate contributions from the unphysical components of the gauge fields in $\mathcal{L}_{\mathrm{GF}}^{\mathrm{conv}}$, one furthermore has to introduce Faddeev–Popov ghost fields $u^\alpha(x)$ and $\bar{u}^\alpha(x)$, $\alpha = A, Z, \pm$, by adding the Lagrangian

$$\mathcal{L}_{\mathrm{FP}}^{\mathrm{conv}} = -\bar{u}^\alpha\frac{\delta F^\alpha}{\delta\hat{\theta}^\beta}u^\beta. \tag{2.25}$$

Here $\delta F^\alpha/\delta\hat{\theta}^\beta$ is the variation of the gauge-fixing terms F^α under the infinitesimal gauge transformations of the form (2.20), where the carets have to be omitted for the fields.

The complete Lagrangian for the SM in the conventional formalism therefore reads

$$\mathcal{L}_{\mathrm{GSW}}^{\mathrm{conv}} = \mathcal{L}_{\mathrm{C}}(\varphi) + \mathcal{L}_{\mathrm{GF}}^{\mathrm{conv}} + \mathcal{L}_{\mathrm{FP}}^{\mathrm{conv}}. \tag{2.26}$$

The predictions of the model are derived from the Lagrangian with the methods of quantum field theory. Usually the path integral formalism is applied to obtain the Green functions and from these via LSZ reduction (see e.g. [31]) the S-matrix elements. The generating functional $W_{\rm conv}$ of the Green functions

$$G^{(n),{\rm conv}}_{1\cdots n}(x_1,\dots,x_n) = \langle 0|T\varphi_1\cdots\varphi_n|0\rangle \tag{2.27}$$

is defined by

$$W_{\rm conv}[J,\omega,\bar\omega] = \int \mathcal{D}\varphi\,\mathcal{D}\bar u\,\mathcal{D}u\,\exp\left\{i\int d^4x\,[\mathcal{L}^{\rm conv}_{\rm GSW}+\varphi J+\bar u\omega+\bar\omega u]\right\}, \tag{2.28}$$

where J are the sources for the fields φ, and ω^α, $\bar\omega^\alpha$ denote the sources for the (anti)-ghost fields (the indices α have been suppressed in (2.28) for simplicity). The quantum fields φ, u and $\bar u$ appear as variables of integration in the path integral. The Green functions are obtained from $W_{\rm conv}[J,\omega,\bar\omega]$ by taking functional derivatives with respect to J:

$$G^{(n),{\rm conv}}_{1\cdots n}(x_1,\dots,x_n) = \frac{1}{i^n}\frac{\delta^n W_{\rm conv}}{\delta J_1\cdots\delta J_n}\bigg|_{J=\omega=\bar\omega=0}. \tag{2.29}$$

These are the full Green functions of the theory. The connected Green functions are generated by

$$Z_{\rm conv}[J,\omega,\bar\omega] = \ln W_{\rm conv}[J,\omega,\bar\omega]. \tag{2.30}$$

Finally, one defines an effective action by making the Legendre transformation

$$\Gamma_{\rm conv}[\tilde\varphi,\tilde u,\tilde{\bar u}] = Z_{\rm conv}[J,\omega,\bar\omega] - i\int d^4x\,[\tilde\varphi J+\tilde{\bar u}\omega+\bar\omega\tilde u], \tag{2.31}$$

where

$$\tilde\varphi = \frac{\delta Z_{\rm conv}}{i\delta J},\qquad \tilde u^\alpha = \frac{\delta Z_{\rm conv}}{i\delta\bar\omega^\alpha},\qquad \tilde{\bar u}^\alpha = -\frac{\delta Z_{\rm conv}}{i\delta\omega^\alpha}. \tag{2.32}$$

The derivatives of the effective action with respect to the fields $\tilde\varphi$ are the vertex functions of the theory, i.e. the one-particle irreducible (1PI) Green functions:

$$\Gamma^{(n),{\rm conv}}_{1\cdots n}(x_1,\dots,x_n) = \frac{\delta^n\Gamma_{\rm conv}}{\delta\tilde\varphi_1\cdots\delta\tilde\varphi_n}\bigg|_{\tilde\varphi=\tilde u=\tilde{\bar u}=0}. \tag{2.33}$$

The perturbative evaluation of the generating functionals given above leads to the Feynman rules of the theory from which the Green functions can be calculated order by order. The Feynman rules for the SM in the conventional formalism are given e.g. in Ref. [32], where the 't Hooft–Feynman gauge is used. In the R_ξ gauge (2.24) the lowest-order gauge-boson propagators have the form

$$\Delta^{j,(0)}_{\mu\nu}(k) = \frac{-ig_{\mu\nu}}{\left[k^2-m_j^2\right]} + \frac{i(1-\xi_j)k_\mu k_\nu}{\left[k^2-\xi_j m_j^2\right]\left[k^2-m_j^2\right]}, \tag{2.34}$$

where $j = A, Z, W$. The 't Hooft–Feynman gauge is realized by setting $\xi_j = 1$, the Landau gauge is defined by $\xi_j = 0$, while the limit $\xi_j \to \infty$ corresponds to the unitary gauge. The unphysical scalar and ghost propagators read in the R_ξ gauge

$$\Delta^{j,(0)}(k) = \frac{i}{\left[k^2 - \xi_j m_j^2\right]}. \tag{2.35}$$

As mentioned above, the gauge invariance present in $\mathcal{L}_{\mathrm{C}}$ is lost after the introduction of $\mathcal{L}_{\mathrm{GF}}^{\mathrm{conv}}$ and $\mathcal{L}_{\mathrm{FP}}^{\mathrm{conv}}$. By defining suitable transformations for the ghost fields it is possible to find a BRS transformation [33] under which $\mathcal{L}_{\mathrm{GSW}}^{\mathrm{conv}}$ is invariant. The resulting Slavnov–Taylor identities [34] for the Green functions have a rather complicated structure and in general involve ghost contributions (see e.g. Ref. [35]).

2.3 Quantization in the framework of the background-field method

Whereas in the conventional formalism directly the fields appearing in the classical Lagrangian are quantized, the quantization in the BFM [19–21] is carried out by splitting the fields $\hat{\varphi}$ of $\mathcal{L}_{\mathrm{C}}$ into classical background fields $\hat{\varphi}$ and quantum fields φ,

$$\mathcal{L}_{\mathrm{C}}(\hat{\varphi}) \to \mathcal{L}_{\mathrm{C}}(\hat{\varphi} + \varphi). \tag{2.36}$$

A gauge-fixing term is added which only breaks the gauge invariance of the quantum fields but retains the gauge invariance of the effective action with respect to the background fields.

In order to avoid tree-level mixing between the gauge bosons and the corresponding unphysical Higgs bosons, we use a generalization of the R_ξ gauge-fixing to the BFM [36].

$$\begin{aligned}\mathcal{L}_{\mathrm{GF}}^{\mathrm{BFM}} = & -\frac{1}{2\xi_Q^W}\left[(\delta^{ac}\partial_\mu + g_2\varepsilon^{abc}\hat{W}_\mu^b)W^{c,\mu} - ig_2\xi_Q^W\frac{1}{2}(\hat{\Phi}_m^\dagger\sigma_{mn}^a\Phi_n - \Phi_m^\dagger\sigma_{mn}^a\hat{\Phi}_n)\right]^2 \\ & -\frac{1}{2\xi_Q^B}\left[\partial_\mu B^\mu + ig_1\xi_Q^B\frac{1}{2}(\hat{\Phi}_m^\dagger\Phi_m - \Phi_m^\dagger\hat{\Phi}_m)\right]^2,\end{aligned} \tag{2.37}$$

where σ^a, $a = 1,2,3$, denote the Pauli matrices, and ξ_Q^W, ξ_Q^B are parameters associated with the gauge fixing of the quantum fields. Background-field gauge invariance restricts the number of quantum gauge parameters to two, one for $\mathrm{SU(2)_W}$ and one for $\mathrm{U(1)_Y}$. The background Higgs field $\hat{\Phi}$ has the usual non-vanishing vacuum expectation value v, while the one of the quantum Higgs field Φ is zero:

$$\hat{\Phi}(x) = \begin{pmatrix}\hat{\phi}^+(x) \\ \frac{1}{\sqrt{2}}(v + \hat{H}(x) + i\hat{\chi}(x))\end{pmatrix}, \qquad \Phi(x) = \begin{pmatrix}\phi^+(x) \\ \frac{1}{\sqrt{2}}(H(x) + i\chi(x))\end{pmatrix}. \tag{2.38}$$

In the spirit of the BFM, one should also split the fermion fields into background and quantum fields. However, for all fields that do not enter the gauge-fixing term, quantization in the BFM is equivalent to the conventional formalism. The Feynman rules for background and quantum fields are identical for these fields and there is no need to distinguish them. In the following, we therefore use a common symbol for the fermion fields, i.e. we drop the caret for the fermion background fields.

Next, we express the gauge-fixing term (2.37) by physical fields. In order to avoid tree-level mixing between the photon and the Z-boson one has to choose $\xi_Q = \xi_Q^W = \xi_Q^B$. This yields

$$\mathcal{L}_{\mathrm{GF}}^{\mathrm{BFM}} = -\frac{1}{2\xi_Q}\left[(G^A)^2 + (G^Z)^2 + 2G^+G^-\right], \tag{2.39}$$

where

$$\begin{aligned}
G^A &= \partial^\mu A_\mu + ie(\hat{W}_\mu^+ W^{-\mu} - W_\mu^+ \hat{W}^{-\mu}) + ie\xi_Q(\hat{\phi}^-\phi^+ - \hat{\phi}^+\phi^-), \\
G^Z &= \partial^\mu Z_\mu - ie\frac{c_{\mathrm{W}}}{s_{\mathrm{W}}}(\hat{W}_\mu^+ W^{-\mu} - W_\mu^+ \hat{W}^{-\mu}) - ie\xi_Q\frac{c_{\mathrm{W}}^2 - s_{\mathrm{W}}^2}{2c_{\mathrm{W}}s_{\mathrm{W}}}(\hat{\phi}^-\phi^+ - \hat{\phi}^+\phi^-) \\
&\quad + e\xi_Q\frac{1}{2c_{\mathrm{W}}s_{\mathrm{W}}}(\hat{\chi}H - \hat{H}\chi - v\chi), \\
G^\pm &= \partial^\mu W_\mu^\pm \pm ie(\hat{A}^\mu - \frac{c_{\mathrm{W}}}{s_{\mathrm{W}}}\hat{Z}^\mu)W_\mu^\pm \mp ie(A^\mu - \frac{c_{\mathrm{W}}}{s_{\mathrm{W}}}Z^\mu)\hat{W}_\mu^\pm \\
&\quad \mp ie\xi_Q\frac{1}{2s_{\mathrm{W}}}\left[(v + \hat{H} \mp i\hat{\chi})\phi^\pm - (H \mp i\chi)\hat{\phi}^\pm\right].
\end{aligned} \tag{2.40}$$

As can easily be checked by comparing (2.40) with (2.23), the gauge-fixing term (2.37) translates to the conventional one upon replacing the background Higgs field by its vacuum expectation value and omitting the background $\mathrm{SU(2)_W}$ triplet field $\hat{W}_\mu^a$.

Finally, we add a Faddeev–Popov part to the Lagrangian

$$\mathcal{L}_{\mathrm{FP}}^{\mathrm{BFM}} = -\bar{u}^\alpha\frac{\delta G^\alpha}{\delta\theta^\beta}u^\beta, \tag{2.41}$$

where $\alpha = A, Z, \pm$. Here $\delta G^\alpha/\delta\theta^\beta$ is the variation of the gauge-fixing terms G^α under infinitesimal quantum gauge transformations [20] which read

$$\begin{aligned}
\delta W_\mu^\pm &= \partial_\mu\delta\theta^\pm \mp ie(W_\mu^\pm + \hat{W}_\mu^\pm)(\delta\theta^A - \frac{c_{\mathrm{W}}}{s_{\mathrm{W}}}\delta\theta^Z) \pm ie\left[(A_\mu + \hat{A}_\mu) - \frac{c_{\mathrm{W}}}{s_{\mathrm{W}}}(Z_\mu + \hat{Z}_\mu)\right]\delta\theta^\pm, \\
\delta Z_\mu &= \partial_\mu\delta\theta^Z - ie\frac{c_{\mathrm{W}}}{s_{\mathrm{W}}}\left[(W_\mu^+ + \hat{W}_\mu^+)\delta\theta^- - (W_\mu^- + \hat{W}_\mu^-)\delta\theta^+\right], \\
\delta A_\mu &= \partial_\mu\delta\theta^A + ie\left[(W_\mu^+ + \hat{W}_\mu^+)\delta\theta^- - (W_\mu^- + \hat{W}_\mu^-)\delta\theta^+\right], \\
\delta\phi^\pm &= \pm\frac{ie}{2s_{\mathrm{W}}}\left[H + \hat{H} + v \pm i(\chi + \hat{\chi})\right]\delta\theta^\pm \mp ie(\phi^\pm + \hat{\phi}^\pm)(\delta\theta^A - \frac{c_{\mathrm{W}}^2 - s_{\mathrm{W}}^2}{2c_{\mathrm{W}}s_{\mathrm{W}}}\delta\theta^Z), \\
\delta H &= \frac{ie}{2s_{\mathrm{W}}}\left[(\phi^+ + \hat{\phi}^+)\delta\theta^- - (\phi^- + \hat{\phi}^-)\delta\theta^+\right] + \frac{e}{2c_{\mathrm{W}}s_{\mathrm{W}}}(\chi + \hat{\chi})\delta\theta^Z, \\
\delta\chi &= \frac{e}{2s_{\mathrm{W}}}\left[(\phi^+ + \hat{\phi}^+)\delta\theta^- + (\phi^- + \hat{\phi}^-)\delta\theta^+\right] - \frac{e}{2c_{\mathrm{W}}s_{\mathrm{W}}}(H + \hat{H} + v)\delta\theta^Z,
\end{aligned} \tag{2.42}$$

for the quantum fields and

$$\delta \hat{W}^{\pm}_{\mu} = \delta \hat{Z}_{\mu} = \delta \hat{A}_{\mu} = \delta \hat{\phi}^{\pm} = \delta \hat{H} = \delta \hat{\chi} = 0 \tag{2.43}$$

for the background fields.

The Lagrangian for the SM in the BFM is given as

$$\mathcal{L}^{\mathrm{BFM}}_{\mathrm{GSW}} = \mathcal{L}_{\mathrm{C}}(\varphi + \hat{\varphi}) + \mathcal{L}^{\mathrm{BFM}}_{\mathrm{GF}} + \mathcal{L}^{\mathrm{BFM}}_{\mathrm{FP}}. \tag{2.44}$$

Using this Lagrangian one can define quantities analogous to W_{conv}, Z_{conv} and Γ_{conv} in the BFM following Ref. [21]. This yields

$$W_{\mathrm{BFM}}[J, \hat{\varphi}] = \int \mathcal{D}\varphi \, \mathcal{D}\bar{u} \, \mathcal{D}u \exp\left\{ i \int d^4x \left[\mathcal{L}^{\mathrm{BFM}}_{\mathrm{GSW}} + \varphi J \right] \right\}. \tag{2.45}$$

Note that the variables of integration in the functional integral are the quantum fields. W_{BFM} depends both on the conventional sources J and the background fields $\hat{\varphi}$. The generating functional (2.28) of the conventional formalism contains also sources for the ghost fields which were introduced in order to cope with the appearance of ghost contributions in the Slavnov–Taylor identities. Since, as will be shown below, the Ward identities for the background-field vertex functions are free of ghost contributions we do not include a source term for the ghost fields in W_{BFM}. We furthermore define

$$Z_{\mathrm{BFM}}[J, \hat{\varphi}] = \ln W_{\mathrm{BFM}}[J, \hat{\varphi}] \tag{2.46}$$

and

$$\Gamma_{\mathrm{BFM}}[\bar{\varphi}, \hat{\varphi}] = Z_{\mathrm{BFM}}[J, \hat{\varphi}] - i \int d^4x \bar{\varphi} J, \tag{2.47}$$

where

$$\bar{\varphi} = \frac{\delta Z_{\mathrm{BFM}}}{i \delta J}. \tag{2.48}$$

We focus on the effective action obtained from (2.47) by putting $\bar{\varphi} = 0$. As can be shown in analogy to Ref. [21], the effective action $\Gamma_{\mathrm{BFM}}[0, \hat{\varphi}]$ is gauge-invariant, i.e. it is invariant under the gauge transformations (2.20) and (2.21) of the background fields. The effective action $\Gamma_{\mathrm{BFM}}[0, \hat{\varphi}]$ is the generating functional of the background-field vertex functions:

$$\Gamma^{(n),\mathrm{BFM}}_{1\cdots n}(x_1, \ldots, x_n) = \left. \frac{\delta^n \Gamma_{\mathrm{BFM}}[0, \hat{\varphi}]}{\delta \hat{\varphi}_1 \cdots \delta \hat{\varphi}_n} \right|_{\hat{\varphi}=0}. \tag{2.49}$$

The Ward identities for these vertex functions will be derived in Section 5.1. One can show [21] that the gauge-invariant effective action $\Gamma_{\mathrm{BFM}}[0, \hat{\varphi}]$ of the BFM coincides with the conventional effective action (2.31) calculated in an unusual gauge which depends on $\hat{\varphi}$.

The BFM vertex functions can be calculated from the Feynman rules associated with $\mathcal{L}^{\mathrm{BFM}}_{\mathrm{GSW}}$. These Feynman rules distinguish between quantum and background fields. Whereas the quantum fields appear only inside loops, the background fields are associated with the external lines. Apart from the doubling of the gauge and Higgs fields, the BFM Feynman rules differ from the conventional ones only owing to the gauge-fixing and ghost terms, which affect only vertices that involve both background and quantum fields. Since the gauge-fixing term (2.39) is non-linear in the fields, the gauge parameter enters also the gauge-boson vertices. As mentioned above, the lowest-order Feynman rules involving fermion fields are the same as in the conventional formalism. We have evaluated the complete set of BFM Feynman rules in the electroweak SM for arbitrary values of the quantum gauge parameter ξ_Q. They are listed in the appendix.

The S matrix is constructed in the usual way by forming trees with vertices from $\Gamma_{\mathrm{BFM}}[0,\hat{\varphi}]$ which are connected by lowest-order background-field propagators [24]. As a simple example, the one-loop process $\mathrm{Z} \to \mathrm{b}\bar{\mathrm{b}}$ has been calculated for arbitrary values of ξ_Q [37]. It has been verified that the resulting S-matrix element is in fact independent of ξ_Q and equal to the one obtained in the conventional formalism.

Despite the distinction between background and quantum fields, calculations in the BFM become in general simpler than in the conventional formalism. This is in particular the case in the 't Hooft–Feynman gauge ($\xi_Q = 1$) for the quantum fields where many vertices simplify considerably (see appendix). Moreover, the gauge fixing of the background fields, which is necessary to define background-field propagators, is totally unrelated to the gauge fixing of the quantum fields. This freedom can be used to choose a particularly suitable background gauge, e.g. the unitary gauge or a non-linear gauge [38]. In this way the number of Feynman diagrams contributing to an S-matrix element can be reduced drastically. The gauge fixing of the background fields does not affect $\Gamma_{\mathrm{BFM}}[0,\hat{\varphi}]$. It is only relevant for the construction of connected Green functions and S-matrix elements in which it only enters tree-level quantities. In particular, in linear background gauges only the tree-level propagators are concerned.

The cancellation of the background gauge parameters in complete S-matrix elements is a direct consequence of the BFM Ward identities. As an example, this can easily be checked for background R_ξ gauges in four-fermion processes. In this case, the BFM Ward identities imply the cancellation of the background gauge parameters separately for self-energy and vertex contributions.

In Ref. [26], the BFM was applied to the process $\mathrm{Z} \to 3\gamma$ at one-loop order. However, the gauge-fixing term used there breaks background-field gauge invariance since no background Higgs field has been introduced. This influences the vertex functions with external Higgs fields. Since for the specific process $\mathrm{Z} \to 3\gamma$ no such vertex function contributes, the results obtained there are nevertheless unaffected. In Ref. [26], the Feynman rules for vertices involving exactly two quantum fields and no background Higgs fields were given for the special case $\xi_Q = 1$. Putting $\xi_Q = 1$ in the corresponding rules given in appendix A we find agreement except for the ones in (A.49b) which differ by a factor 2.

Chapter 3

Algebraic calculation of general two-loop self-energies

In this chapter an algebraic method for the calculation of general two-loop self-energies is developed. It reduces all self-energies to a set of standard scalar integrals.

After studying the symmetry properties of the relevant integrals an algorithm for treating arbitrary two-loop tensor integrals of self-energy type is presented. Furthermore, certain integral relations are derived. The calculations are performed for arbitrary values of all particle masses, the invariant momentum p^2 and the space–time dimension D. The algebraic method is suited for working in a general class of renormalizable gauges where the gauge parameters are not fixed but kept as additional free parameters. The gauge dependence of the considered quantities can directly be studied at the algebraic level.

The algorithm presented here has been implemented in the computer-algebra program *TwoCalc* [39]. In analogy to the investigation of gauge dependence also Slavnov–Taylor identities and Ward identities can be checked directly at the algebraic result. The suitability of the algebraic treatment for calculating general two-loop self-energies is demonstrated by explicitly verifying Slavnov–Taylor identities for the self-energies of the neutral gauge bosons involving several thousand Feynman diagrams. Every graph is calculated separately and the results are summed up algebraically.

As an application of the algebraic method the light-fermion contributions to the two-loop gauge-boson self-energies in the SM are treated. This class of diagrams is of interest since it gives rise to an important contribution at one-loop order. The gauge dependence of the results is discussed.

3.1 Classification of two-loop self-energies

As the first step in the evaluation of two-loop self-energies the relevant Feynman diagrams have to be generated. These can be classified according to their topologies. For completeness we also include tadpole contributions and consider all propagator-type topologies which cannot be generated via Dyson-summation of one-loop diagrams. All topologies of this type possibly occurring in renormalizable gauge theories are shown in Fig. 3.1. In the following we will refer to the first eight topologies in Fig. 3.1 as "generic" two-loop topologies.

We focus on the unrenormalized self-energies, a proper renormalization can be done after the algebraic calculation has been carried out. In order to make the integrals mathematically meaningful we use dimensional regularization and work in an arbitrary space–time dimension D.

Inserting fields into the topologies and applying the Feynman rules for propagators and vertices leads to the Feynman amplitudes. The sum of the one-particle irreducible Feynman diagrams corresponds to the two-point vertex functions in momentum space, which are the Fourier transforms of (2.33) or (2.49). We define the self-energy as the one-particle irreducible diagrams multiplied by $(-i)$, i.e. for a scalar field we write

$$\Gamma^{SS}(p,-p) = i(p^2 - M^2) + i\Sigma^{SS}(p^2), \tag{3.1}$$

where p is the external momentum, M the mass of the particle, and as usual the lowest-order contribution has been taken out. The self-energies for the other types of fields are defined in precisely the same way. Throughout this work we will consider all momenta and fields in the vertex functions as incoming. The loop expansion of the self-energy yields

$$\Sigma^{SS}(p^2) = \Sigma^{SS,(1)}(p^2) + \Sigma^{SS,(2)}(p^2) + \mathcal{O}(e^6), \tag{3.2}$$

where the superscripts $^{(n)}$ indicate the one-loop and two-loop self-energies, respectively. The full propagators are obtained from the vertex functions by inversion, i.e.

$$\Delta^{SS}(p,-p) = -\left[\Gamma^{SS}(p,-p)\right]^{-1}. \tag{3.3}$$

For the neutral gauge bosons this amounts to a matrix inversion.

The self-energy defined in (3.1) contains no contribution from tadpole diagrams. For ease of notation it is often convenient to absorb the tadpole contributions in the self-energies. This will be indicated by a tilde, i.e. $\tilde{\Sigma}^{(2)}$ is understood to incorporate contributions from all topologies in Fig. 3.1.

In the following we will use the shorthand notation for the integrations

$$\langle\langle \ldots \rangle\rangle = \int \frac{d^D q_1}{i\pi^2 (2\pi\mu)^{D-4}} \int \frac{d^D q_2}{i\pi^2 (2\pi\mu)^{D-4}} (\ldots), \tag{3.4}$$

where q_1 and q_2 are the integration momenta of the loop integrals and μ is an arbitrary reference mass, and we write for a product of propagators

$$\Delta_{ij\ldots l} = \frac{1}{\left[k_i^2 - m_i^2\right]\left[k_j^2 - m_j^2\right]\cdots\left[k_l^2 - m_l^2\right]}, \tag{3.5}$$

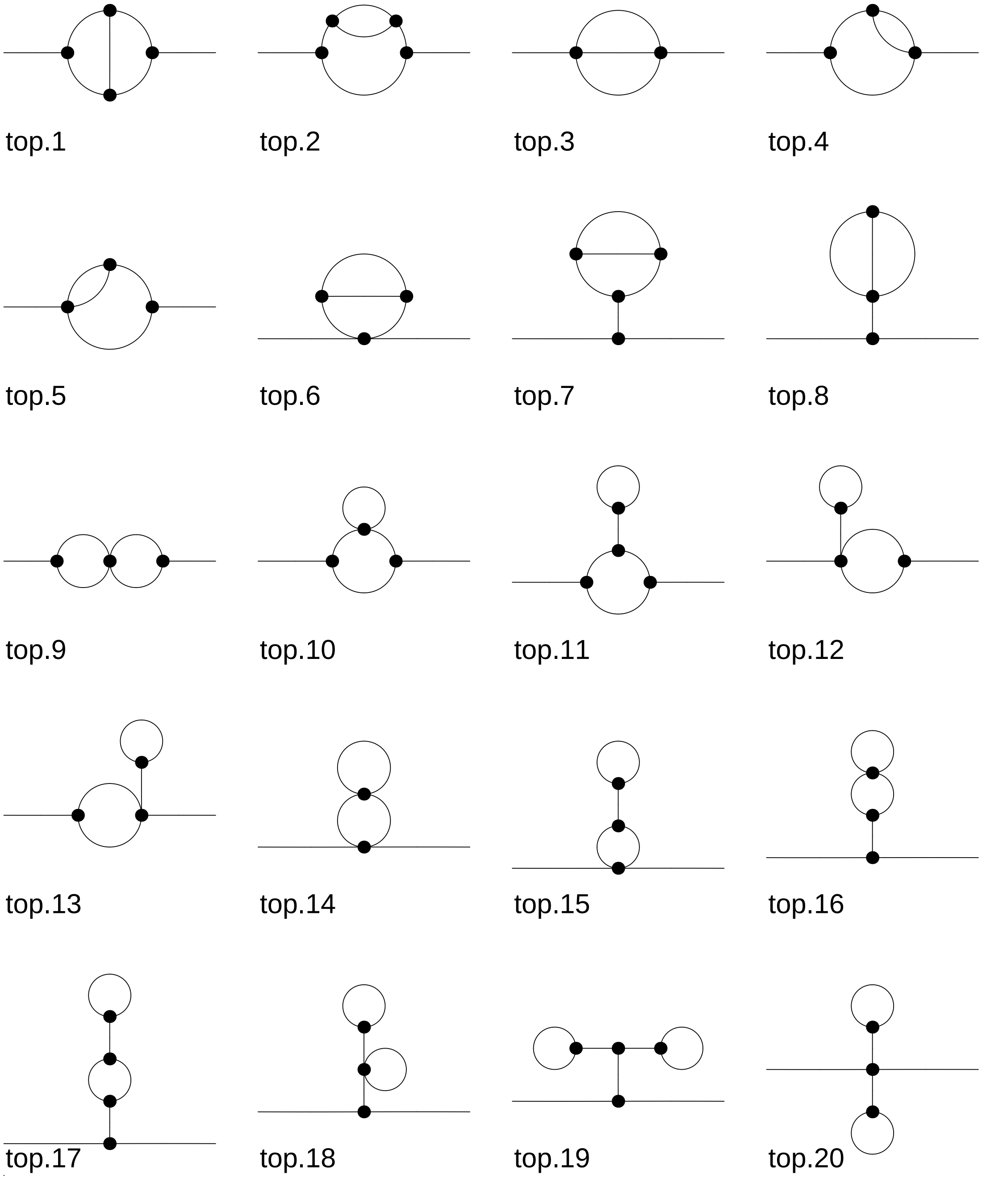


Figure 3.1: The 20 topologies possible for two-loop self-energies (including tadpoles) in renormalizable gauge theories.

where k_j is the momentum of the j-th propagator (respecting momentum conservation) and m_j its mass. It is understood that the masses in (3.5) carry small negative imaginary parts.

A general Feynman amplitude of a two-loop self-energy takes the form

$$\langle\langle \frac{f(k_i, m_i, \xi_i, D)}{\left[k_1^2 - m_1^2\right]\left[k_2^2 - m_2^2\right]\cdots\left[k_l^2 - m_l^2\right]}\rangle\rangle, \tag{3.6}$$

where in the denominator terms of the form $m_i\xi_i$ stemming from propagators of gauge bosons, unphysical scalars or ghosts (see e.g. (A.60) – (A.62)) are simply treated as mass parameters. The expression $f(k_i, m_i, \xi_i, D)$ in the numerator of (3.6) depends on the momenta, the particle masses, the gauge parameters and the space–time dimension D and contains the whole Lorentz and Dirac structure. It will be shown in this chapter that all Feynman amplitudes can be reduced to a form where the numerator consists only of quantities which are independent of the integration momenta and can therefore be pulled out of the integral. The denominator is still of the form (3.6). This means that the amplitudes can be expressed in terms of a class of scalar two-loop integrals

$$T_{i_1i_2\ldots i_l}(p^2; m_1^2, m_2^2, \ldots, m_l^2) = \langle\langle \frac{1}{\left[k_{i_1}^2 - m_1^2\right]\left[k_{i_2}^2 - m_2^2\right]\cdots\left[k_{i_l}^2 - m_l^2\right]}\rangle\rangle, \tag{3.7}$$

which we call T-integrals. This is reminiscent of the well-known result that every one-loop tensor integral can be reduced to the basic scalar integrals A_0, B_0, C_0 and D_0 [13, 40]. It should be noted, however, that it is by far not obvious that such a reduction is possible in the general two-loop case and we only claim it for self-energies.

We used the double index notation $T_{i_1i_2\ldots i_l}$ to indicate that the indices of the T-integrals refer to the corresponding momenta $k_{i_1}, k_{i_2}, \ldots, k_{i_l}$. If a propagator has mass zero, we indicate this with a prime at the corresponding index and drop the zero in the list of arguments, e.g.

$$T_{1'234}(p^2; m_a^2, m_b^2, m_c^2) = \langle\langle \frac{1}{k_1^2\left[k_2^2 - m_a^2\right]\left[k_3^2 - m_b^2\right]\left[k_4^2 - m_c^2\right]}\rangle\rangle. \tag{3.8}$$

The dependence on p^2 will be suppressed in the following. The masses will only be explicitly written as arguments if confusion is possible.

The T-integrals can be represented by the topologies listed in Fig. 3.1. A line in the topology carrying zero momentum contributes only a factor $(-1/m_i^2)$ which is irrelevant for the scalar integral. The topologies 6 and 7 in Fig. 3.1 therefore represent the same type of T-integral. The first two topologies in Fig. 3.1 are the most general ones since all scalar integrals corresponding to the other topologies can be obtained from these by omitting factors $\left[k_i^2 - m_i^2\right]$ in the denominator or equivalently shrinking the corresponding lines in the topology to a point. The integral corresponding to the first topology, being the only one where five propagators with different momenta

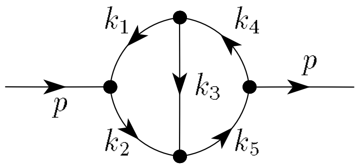


Figure 3.2: The topology of the "master integral".

occur, is sometimes called the "master integral"

$$T_{12345} = \langle\langle \frac{1}{\left[k_1^2 - m_1^2\right]\left[k_2^2 - m_2^2\right]\left[k_3^2 - m_3^2\right]\left[k_4^2 - m_4^2\right]\left[k_5^2 - m_5^2\right]} \rangle\rangle, \tag{3.9}$$

where the momenta are labeled as indicated in Fig. 3.2. In the following we do not manifestly impose momentum conservation, but keep the overcomplete set of momenta $k_1, \ldots, k_5$. This is convenient especially for treating the symmetry properties of the integrals and performing the tensor reductions. It is easy to reexpress the momenta $k_1, \ldots, k_5$ by the external momentum p and the integration momenta q_1 and q_2:

$$k_1 = q_1,\ k_2 = q_1 + p,\ k_3 = q_2 - q_1,\ k_4 = q_2,\ k_5 = q_2 + p. \tag{3.10}$$

This convention will be used throughout this work. Accordingly, the integral corresponding to the second topology in Fig. 3.1, for example, can be written as T_{11234}. We will see below that equivalent representations can be obtained by certain permutations of the indices.

3.2 Symmetries of the two-loop integrals

In order to work out the Feynman amplitudes it is convenient to begin with a decomposition into Lorentz scalars. For the gauge-boson self-energies it reads

$$\Sigma_{\mu\nu}^{\alpha,\beta}(p,-p) = \Big(-g_{\mu\nu} + \frac{p_\mu p_\nu}{p^2}\Big)\Sigma_{\mathrm{T}}^{\alpha,\beta}(p^2) - \frac{p_\mu p_\nu}{p^2}\Sigma_{\mathrm{L}}^{\alpha,\beta}(p^2), \tag{3.11}$$

from which the transverse part $\Sigma_{\mathrm{T}}^{\alpha,\beta}(p^2)$ and the longitudinal part $\Sigma_{\mathrm{L}}^{\alpha,\beta}(p^2)$ can easily be extracted,

$$\Sigma_{\mathrm{T}}^{\alpha,\beta}(p^2) = \frac{1}{D-1}\Big(-g^{\mu\nu} + \frac{p^\mu p^\nu}{p^2}\Big)\Sigma_{\mu\nu}^{\alpha,\beta}(p,-p)\,, \quad \Sigma_{\mathrm{L}}^{\alpha,\beta}(p^2) = -\frac{p^\mu p^\nu}{p^2}\Sigma_{\mu\nu}^{\alpha,\beta}(p,-p)\,, \tag{3.12}$$

where $\alpha, \beta = A, Z, W$.

We write for the mixing of gauge bosons and scalars

$$\Sigma_{\mu}^{\alpha,i}(p,-p) = p_\mu \Sigma^{\alpha,i}(p^2), \tag{3.13}$$

where $i = H, \chi, \phi$, and p is the momentum of the gauge boson. This yields

$$\Sigma^{\alpha,i}(p^2) = \frac{p^\mu}{p^2}\Sigma^{\alpha,i}_\mu(p,-p). \tag{3.14}$$

The fermion self-energies can be decomposed into a left-handed and a right-handed part, a scalar part and a pseudoscalar part according to

$$\Sigma^{f\bar{f}}(p,-p) = \not{p}\,\omega_- \Sigma^{f\bar{f}}_{\mathrm{L}}(p^2) + \not{p}\,\omega_+ \Sigma^{f\bar{f}}_{\mathrm{R}}(p^2) + m_f \Sigma^{f\bar{f}}_{\mathrm{S}}(p^2) + m_f \gamma_5 \Sigma^{f\bar{f}}_{\mathrm{P}}(p^2). \tag{3.15}$$

These parts can be isolated by taking suitable traces

$$\Sigma^{f\bar{f}}_{\mathrm{L}}(p^2) = \frac{1}{4p^2}\mathrm{Tr}\left(\omega_-\, \not{p}\,\Sigma^{f\bar{f}}(p,-p)\right), \quad \Sigma^{f\bar{f}}_{\mathrm{R}}(p^2) = \frac{1}{4p^2}\mathrm{Tr}\left(\omega_+\, \not{p}\,\Sigma^{f\bar{f}}(p,-p)\right), \tag{3.16}$$

and

$$\Sigma^{f\bar{f}}_{\mathrm{S}}(p^2) = \frac{1}{4m_f}\mathrm{Tr}\left(\Sigma^{f\bar{f}}(p,-p)\right), \quad \Sigma^{f\bar{f}}_{\mathrm{P}}(p^2) = \frac{1}{4m_f}\mathrm{Tr}\left(\gamma_5 \Sigma^{f\bar{f}}(p,-p)\right). \tag{3.17}$$

After extracting the scalar parts of the self-energies the contraction of Lorentz indices, reduction of the Dirac algebra and evaluation of Dirac traces can be worked out like in the one-loop case. This results in scalar products of momenta $(k_i{\cdot}k_j)$, $(k_i{\cdot}p)$, p^2 in the numerator of the Feynman amplitude. The denominator is unchanged. We now implicitly use momentum conservation and express all scalar products as sums of momentum squares, e.g. from $k_2 = k_1 + p$ one infers

$$(k_1 \cdot p) = \frac{1}{2}(k_2^2 - k_1^2 - p^2). \tag{3.18}$$

Subsequently, all k_i^2 appearing both in the numerator and the denominator are canceled via

$$k_i^2 = (k_i^2 - m_i^2) + m_i^2. \tag{3.19}$$

In cases where all k_i^2 appearing in the numerator can be canceled, this procedure directly leads to T-integrals. In general, however, another type of integral emerges which contains squares of momenta in the numerator not occurring in the denominator:

$$Y^{i\dots}_{jkl\dots} = \langle\langle k_i^2 \,\cdots\, \Delta_{jkl\dots}\rangle\rangle, \quad i,j,k,l = 1,\dots,5\ , \quad i \neq j,k,l. \tag{3.20}$$

Here the shorthand notation (3.5) has been used.

After these steps, the amplitudes are represented in terms of scalar T- and Y-integrals. The number of occurring integrals can considerably be reduced by taking into account their symmetries with respect to permutation of the k_i^2 or, equivalently, of the corresponding indices.

All T- and Y-integrals are invariant under the permutations

$$(12)(45), \quad (14)(25), \quad (15)(24), \tag{3.21}$$

where again $k_1, \ldots, k_5$ are labeled according to (3.10). Application of the first permutation yields for example

$$T_{1234}(p^2; m_a^2, m_b^2, m_c^2, m_d^2) = T_{1235}(p^2; m_b^2, m_a^2, m_c^2, m_d^2) = \langle\langle \frac{1}{\left[k_1^2 - m_b^2\right]\left[k_2^2 - m_a^2\right]\left[k_3^2 - m_c^2\right]\left[k_5^2 - m_d^2\right]} \rangle\rangle. \quad (3.22)$$

The validity of the symmetries listed in (3.21) can most easily be seen for the "master integral" (Fig. 3.2) or can be checked using (3.10) and the invariance properties of the integrals with respect to changes of the integration momenta.

Additional symmetry relations hold if an index does not occur in the integral, e.g.

$$(23) \quad \text{if 1 is absent.} \quad (3.23)$$

The symmetry relations (3.21) and (3.23) can be used to map every integral onto a standard representative, i.e. all integrals which are related by symmetry transformations are brought to the same form.

3.3 Tensor reduction for two-loop self-energy integrals

As claimed above, all self-energy amplitudes can be expressed by T-integrals. This means that all Y-integrals can further be reduced. To achieve this, we rewrite the Y-integrals in terms of tensor integrals and perform a tensor decomposition.

As a simple example we consider the integral

$$Y^1_{2345} = \langle\langle k_1^2 \Delta_{2345} \rangle\rangle. \quad (3.24)$$

Insertion of

$$k_1^2 = (k_2^2 - m_2^2) + (m_2^2 + p^2) - 2(p \cdot k_2) \quad (3.25)$$

yields

$$Y^1_{2345} = T_{345} + (m_2^2 + p^2) T_{2345} - 2 p_\mu \langle\langle k_2^\mu \Delta_{2345} \rangle\rangle. \quad (3.26)$$

Now one has to perform a tensor decomposition for the integral

$$S^{2,\mu}_{2345} = \langle\langle k_2^\mu \Delta_{2345} \rangle\rangle. \quad (3.27)$$

It is obvious that the Passarino–Veltman procedure [13] used for one-loop integrals, i.e. the ansatz

$$S^{2,\mu}_{2345} = p^\mu S(p^2), \quad (3.28)$$

does not lead to simpler integrals in this case. To determine the scalar quantity $S(p^2)$ one has to contract with p_μ, but it is not possible to cancel the resulting scalar product $(p \cdot k_2)$, since it is not expressible as a sum of squared momenta occurring in

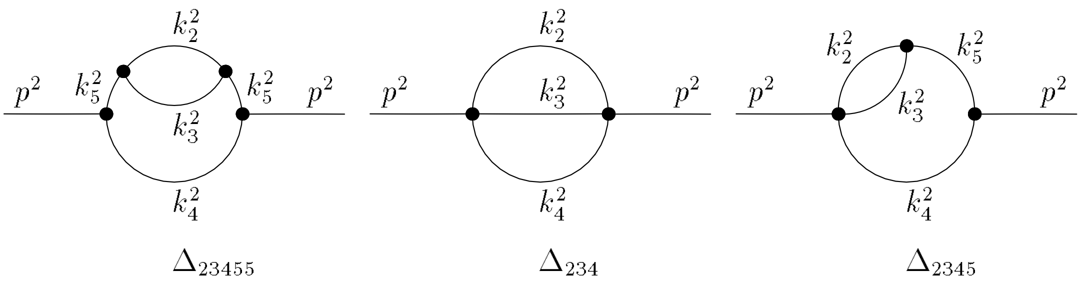


Figure 3.3: The topologies representing the denominators of tensor integrals for which the Passarino–Veltman technique is not directly applicable.

the denominator of the integral. This is due to the fact that the momenta p and k_2 belong to a four-vertex in the topology Δ_{2345} (see Fig. 3.3). This is a typical feature of two-loop topologies, whereas in one-loop order only three-vertices occur in the loop integral and the Passarino–Veltman technique always leads to scalar products that can be canceled.

In our approach we work not only with the tensor structure of the integral with respect to the external momentum p but also perform decompositions with respect to a subloop. In general, one can write for a subloop which has the structure of a first-rank tensor

$$s^\mu = p_1^\mu s_1 + p_2^\mu s_2 + \dots\,, \tag{3.29}$$

where the s_i ($i = 1, 2, \dots$) are Lorentz-scalars and the p_i are a set of independent external momenta of the subloop.

At first sight it is not obvious that the scalar products resulting from this decomposition can be canceled. Moreover, as is well known from one-loop calculations, contracting (3.29) with $p_{i\mu}$ and solving for the s_i leads to factors which are determinants of the momenta p_i. These, however, depend on the integration momentum of the second loop and cannot be pulled out of the tensor integral like in the one-loop case.

In order to show how this technique can successfully be applied to all two-loop self-energies, we first examine which types of denominators can occur in the tensor integrals after the steps performed in the previous section. The possible denominators can be read off from the topologies listed in Fig. 3.1. Topology 1 clearly is not possible, since at least one of its five different propagators can always be canceled. The topologies 9 through 20 are just products of one-loop contributions. For these the Passarino–Veltman technique can be applied in a straightforward way. The tensor decomposition for the topologies 6 through 8, having no dependence on the external momentum p, is also trivial.

We therefore have to consider the topologies 2 – 5. As shown in Fig. 3.3 they can be written as Δ_{23455}, Δ_{234} and Δ_{2345}. Note that the scalar integrals corresponding to topologies 4 and 5 are equivalent owing to the symmetry relations (3.21). The topologies we have to deal with all contain at least one subloop which is a self-energy insertion, in our terminology this is Δ_{23}. We do the decomposition for this subloop. The external momentum of the subloop Δ_{23} is k_5 (see Fig. 3.3). In the simplest case of a first-rank tensor we have

$$\langle k_2^\mu \Delta_{23}\rangle = k_5^\mu\, s(k_5^2), \tag{3.30}$$

which is just the example considered in (3.27). The Lorentz-scalar $s(k_5^2)$ is obtained from

$$s(k_5^2) = \frac{1}{k_5^2}\langle (k_5\cdot k_2)\Delta_{23}\rangle = \langle (k_5\cdot k_2)\Delta_{235'}\rangle. \tag{3.31}$$

Owing to the self-energy structure of Δ_{23}, the tensor decomposition does not yield a determinant but only a factor $1/k_5^2$ which can be written as a massless propagator. Insertion of (3.31) into (3.30) and (3.27) therefore leads to integrals which still belong to the class of T- and Y-integrals. We get

$$p_\mu S^{2,\mu}_{2345} = \langle\langle (p\cdot k_5)(k_5\cdot k_2)\Delta_{23455'}\rangle\rangle. \tag{3.32}$$

Since the momenta k_5 and k_2 belong to a three vertex in the topology Δ_{2345} their scalar product is expressible as a sum of squared momenta which can be canceled:

$$(k_5\cdot k_2) = \frac{1}{2}(k_2^2 - k_3^2 + k_5^2). \tag{3.33}$$

The same holds for $(p\cdot k_5)$

$$(p\cdot k_5) = \frac{1}{2}(k_5^2 - k_4^2 + p^2), \tag{3.34}$$

since one of the k_5^2-terms can be canceled against the massless propagator $\Delta_{5'}$.

We can therefore express Y^1_{2345} solely in terms of T-integrals. The result reads

$$\begin{aligned} Y^1_{2345} = {} & A_0(m_3^2)B_0(p^2;m_4^2,m_5^2) \\ & -\frac{1}{2}\Big\{\big[A_0(m_2^2)-A_0(m_3^2)\big]\Big[\frac{1}{m_5^2}A_0(m_5^2)-B_0(p^2;m_4^2,m_5^2)\Big] \\ & \quad + T_{234}-T_{235}-(m_2^2+m_3^2+m_4^2-m_5^2+p^2)T_{2345} \\ & \quad -(m_2^2-m_3^2)T_{2355'}+(m_4^2-p^2)\big[T_{2455'}-T_{3455'}\big] \\ & \quad -(m_2^2-m_3^2)(m_4^2-p^2)T_{23455'}\Big\}. \end{aligned} \tag{3.35}$$

The one-loop scalar integrals A_0 and B_0 appear since some T-integrals are expressible as products of one-loop integrals, e.g.

$$T_{345} = \langle\langle \Delta_{345}\rangle\rangle = \langle \Delta_3\rangle\langle\Delta_{45}\rangle = A_0(m_3^2)\,B_0(p^2;m_4^2,m_5^2), \tag{3.36}$$

where

$$A_0(m^2) = \langle \frac{1}{q^2 - m^2} \rangle, \tag{3.37}$$

$$B_0(p^2; m_1^2, m_2^2) = \langle \frac{1}{\left[q^2 - m_1^2\right]\left[(q+p)^2 - m_2^2\right]} \rangle. \tag{3.38}$$

Note that our definitions slightly differ from those used in Ref. [40].

The integrals containing a massive and a massless propagator with the same momentum can further be simplified by partial fractioning.

The topology Δ_{23455} is treated in analogy to Δ_{2345}. The corresponding integrals are easily obtained by taking the derivative with respect to m_5^2, e.g.

$$Y^1_{23455} = \frac{\partial}{\partial(m_5^2)} Y^1_{2345}\,. \tag{3.39}$$

The decomposition for the tensors of higher rank can be worked out in a similar manner as described above. In order to treat Y^{11}_{2345} we have to decompose the integral

$$\langle\langle (p \cdot k_2)(p \cdot k_2)\Delta_{2345}\rangle\rangle = p_\mu p_\nu \langle \Delta_{45} \langle k_2^\mu k_2^\nu \Delta_{23}\rangle\rangle. \tag{3.40}$$

We write

$$\langle k_2^\mu k_2^\nu \Delta_{23}\rangle = s_{00}(k_5^2) g^{\mu\nu} + s_{11}(k_5^2)\frac{k_5^\mu k_5^\nu}{k_5^2} \tag{3.41}$$

with the Lorentz-scalars $s_{00}(k_5^2)$ and $s_{11}(k_5^2)$. Contraction with $g_{\mu\nu}$ and $k_{5\mu}k_{5\nu}$, respectively, leads to two equations for the scalars s_{00} and s_{11}:

$$\begin{aligned} s_{00} D + s_{11} &= \langle k_2^2 \Delta_{23}\rangle, \\ s_{00} + s_{11} &= \frac{1}{k_5^2}\langle (k_5 \cdot k_2)(k_5 \cdot k_2)\Delta_{23}\rangle. \end{aligned} \tag{3.42}$$

Again, the factors $1/k_5^2$ appearing in (3.41) and (3.42) can be written as propagators with mass zero. Inspection of (3.42) and (3.40) shows that the decomposition for the integral (3.40) leads to T-integrals and one Y-integral Y^5_{234} which corresponds to a tensor of lower rank. The tensor reduction of this integral will be worked out below, where we discuss the integrals with the topology Δ_{234}. The complete reduction formula for Y^{11}_{2345} is rather lengthy. It is given in Ref. [41].

The generalization to tensors of higher than second rank should now be obvious. The decomposition of the tensor integrals always leads directly to T-integrals or to tensor integrals of lower rank.

So far we showed that the tensor decomposition works for all tensor integrals with the topologies Δ_{2345} and Δ_{23455}. The third topology we have to consider is Δ_{234}. This case is slightly more complicated owing to the presence of the two four-vertices in the topology. The Y-integrals which can occur are Y^1_{234}, Y^5_{234}, Y^{11}_{234}, Y^{15}_{234}, Y^{55}_{234}, etc. The integrals Y^1_{234} and Y^5_{234} are not independent of each other but are related through a symmetry relation which amounts to a permutation of the masses m_2 and m_4. The same holds, of course, for Y^{11}_{234} and Y^{55}_{234}.

We begin with the integral Y^1_{234} and apply exactly the same steps as for Y^1_{2345}. This yields

$$\begin{aligned} Y^1_{234} &= T_{34} + (m_2^2 + p^2)T_{234} - 2p_\mu \langle \Delta_4 \langle k_2^\mu \Delta_{23} \rangle\rangle \\ &= T_{34} + (m_2^2 + p^2)T_{234} - 2\langle\langle (p \cdot k_5)(k_5 \cdot k_2)\Delta_{2345'} \rangle\rangle, \end{aligned} \tag{3.43}$$

where (3.30) and (3.31) were used. The last integral looks very similar to (3.32) but here it is not possible to cancel the k_5^4 term resulting from the insertion of (3.33) and (3.34). Instead we obtain

$$Y^1_{234}(m_2^2, m_3^2, m_4^2) = -\frac{1}{2} Y^5_{234}(m_2^2, m_3^2, m_4^2) + f(T), \tag{3.44}$$

where $f(T)$ summarizes terms in which only T-integrals occur. In order to find an expression for Y^1_{234} in terms of T-integrals only, we use the last symmetry relation listed in (3.21) and obtain after appropriate relabeling of the masses

$$Y^5_{234}(m_2^2, m_3^2, m_4^2) = -\frac{1}{2} Y^1_{234}(m_2^2, m_3^2, m_4^2) + g(T)\ , \tag{3.45}$$

where $g(T)$ again depends on T-integrals only. Inserting this equation into (3.44) yields the result

$$\begin{aligned} Y^1_{234} = \frac{1}{3}\Big\{ & h(A_0, B_0) + (m_2^2 + m_3^2 + m_4^2 + p^2)T_{234} + 2(m_2^2 - m_3^2)T_{235'} \\ & + (m_3^2 - m_4^2)T_{1'34} + 2(m_2^2 - m_3^2)(m_4^2 - p^2)T_{2345'} \\ & + (m_2^2 - p^2)(m_3^2 - m_4^2)T_{1'234} \Big\}, \end{aligned} \tag{3.46}$$

where $h(A_0, B_0)$ represents a function containing only one-loop integrals.

Another complication can be seen by inspecting (3.46). Owing to the absence of the momentum k_5 the integral Y^1_{234} is symmetric with respect to the permutation of the momenta k_3 and k_4 or, equivalently, of the masses m_3 and m_4. However, the tensor decomposition artificially introduces the momentum k_5 into the terms on the right-hand side of (3.46) and the symmetry seems to be lost. It can be made manifest again by using the relation (3.50) stated below. The result for Y^1_{234} finally reads

$$\begin{aligned} Y^1_{234} = \frac{1}{3}\Big\{ & A_0(m_2^2)A_0(m_3^2) + A_0(m_2^2)A_0(m_4^2) + A_0(m_3^2)A_0(m_4^2) \\ & - (m_4^2 - p^2)\Big[A_0(m_2^2) - A_0(m_3^2)\Big] B_0(p^2; m_4^2, 0) \\ & - (m_3^2 - p^2)\Big[A_0(m_2^2) - A_0(m_4^2)\Big] B_0(p^2; m_3^2, 0) \\ & + (m_2^2 + m_3^2 + m_4^2 + p^2)T_{234} \end{aligned}$$

$$
\begin{aligned}
&+ (m_2^2 - m_3^2)T_{235'} + (m_2^2 - m_4^2)T_{235'}(m_2^2, m_4^2) \\
&+ (m_2^2 - m_3^2)(m_4^2 - p^2)T_{2345'} \\
&+ (m_3^2 - p^2)(m_2^2 - m_4^2)T_{2345'}(m_2^2, m_4^2, m_3^2)\Big\},
\end{aligned}
\tag{3.47}
$$

which displays the considered symmetry.

The integrals involving higher tensors are treated in exactly the same way. For the second-rank tensor we use (3.41). The reduction of Y_{234}^{11} leads to a formula involving Y_{234}^{15}. For this an expression analogous to (3.45) is obtained. The symmetry properties of the integral become manifest after using a relation involving T-integrals. The expressions for Y_{234}^{11} and Y_{234}^{15} are listed in Ref. [41].

With the technique for the tensor decomposition described above all Y-integrals can be reduced to scalar integrals of simpler structure where no integration momentum appears in the numerator, i.e. to T-integrals. The class of T-integrals therefore suffices to express all possible two-loop self-energies.

This result has been derived without referring to a specific gauge. For practical purposes we briefly examine the situation in the 't Hooft–Feynman gauge, i.e. $\xi_A = \xi_Z = \xi_W = 1$ in the conventional formalism and $\xi_Q = 1$ in the BFM. As explained above, the most complicated Y-integrals have denominators of the form shown in Fig. 3.3. The possible numerators can be obtained by power counting. In the 't Hooft–Feynman gauge the gauge-boson propagators do not contribute powers of momenta to the numerator of the Feynman integrals. Counting the powers arising from the other Feynman rules and from the decompositions (3.12), (3.14), (3.16) and (3.17) reveals that the relevant integrals are Y_{234}^{1}, Y_{2345}^{1}, Y_{2345}^{11}, Y_{23455}^{1} and Y_{23455}^{11}. Explicit results for these integrals are given in this section and in Ref. [41], where also some reduction formulae needed for general values of ξ_i are listed.

3.4 Relations between the scalar integrals

The results obtained for the two-loop self-energies via the steps described in the preceding sections consist of a Lorentz tensor specified in (3.11), (3.13) and (3.15), respectively, and a scalar part being a sum of T-integrals multiplied by rational functions in the invariant momentum p^2, the space–time dimension D, the particle masses m_j and the gauge parameters ξ_i. The algebraic structure of the result is convenient for studying the dependence on these variables. This can be done by inserting appropriate analytical or numerical expressions for the scalar integrals. However, as an important benefit of the algebraic approach pursued here, it is also possible to investigate many important properties of the considered quantity directly at the algebraic level.

The algebraic result contains in general several scalar integrals depending on different arguments. These integrals, which we denote as $T^1, T^2, \ldots, T^n$ for the moment, are not necessarily independent of each other at the algebraic level, i.e. in general there exist relations

$$c^1 T^1 + c^2 T^2 + \ldots + c^n T^n = 0. \tag{3.48}$$

The coefficients $c^1, \ldots, c^n$ are polynomials in p^2, D, m_j and the gauge parameters. The result can be made very transparent by using these relations to eliminate as many integrals as possible. For example, if a considered set of amplitudes is gauge-independent this property directly manifests itself in the algebraic result provided that the set of integrals used for expressing the result is sufficiently small. In this case all terms depending on the gauge parameters ξ_i disappear. This means that the prefactors of all gauge-dependent basic integrals algebraically add up to zero while all ξ_i-dependent terms multiplying a gauge-independent basic integral exactly cancel. In this way the gauge independence is revealed by purely algebraical means. No explicit analytical or numerical expressions of the standard integrals are needed for this consideration.

In the same way as the gauge independence, Slavnov–Taylor identities or any other relation involving two-loop self-energies can be checked algebraically. This is very useful for performing consistency checks on the result.

The question arises whether a sufficiently small basis of integrals can actually be found. We argue that after invoking the symmetry properties of the two-loop integrals and eliminating all Y-integrals by means of tensor decompositions only few additional relations between the scalar integrals are needed. In Section 3.6 we will verify Slavnov–Taylor identities valid for the self-energies of the photon and the Z-boson by adding up the results of several thousand Feynman amplitudes. For this application only the relations for the integrals $T_{2345'}(m_2^2, m_3^2, m_4^2)$ and $T_{2345'5'}(m_2^2, m_3^2, m_4^2)$ described below are used.

The first type of relations we want to consider involves integrals in which at least one propagator is massless. They are obtained indirectly by using symmetry arguments, momentum conservation and properties of the tensor decomposition. In this way we derive a relation between the integral $T_{2345'}(m_2^2, m_3^2, m_4^2)$ and the integrals of the same type where the masses are permuted. Momentum conservation at the four-vertices of the topology Δ_{234} obviously implies

$$\langle\langle \frac{p \cdot (k_2 + k_3 - k_4 - p)}{\left[k_2^2 - m_2^2\right]\left[k_3^2 - m_3^2\right]\left[k_4^2 - m_4^2\right]} \rangle\rangle = 0. \tag{3.49}$$

Writing (3.49) as a sum of four integrals, using their symmetries with respect to permutations of k_2, k_3 and k_4 and performing tensor decompositions for them leads to the relation

$$\begin{aligned} &\Big\{(m_2^2 - m_3^2)(m_4^2 - p^2)T_{2345'}(m_2^2, m_3^2, m_4^2) + \text{cycl.}\Big\} \\ &+ \Big\{(m_2^2 - m_3^2)T_{235'}(m_2^2, m_3^2) + \text{cycl.}\Big\} \\ &- \Big\{\Big[A_0(m_2^2) - A_0(m_3^2)\Big](m_4^2 - p^2)B_0(p^2; m_4^2, 0) + \text{cycl.}\Big\} = 0. \end{aligned} \tag{3.50}$$

Here "cycl." denotes cyclic permutation of the masses. Applying (3.50), one can eliminate one permutation of the $T_{2345'}$-integral. This relation was used in the last section to obtain (3.47). If two masses are equal, (3.50) becomes trivial.

Two similar relations hold for the permutations of the integral $T_{2345'5'}$. Using these, it is possible to obtain one standard permutation for every integral of this type. In contrast to (3.50), this relation remains nontrivial if two of the three masses are equal. It gives an expression for the integral $T_{2345'5'}(m^2, M^2, M^2)$ in terms of integrals with fewer propagators only [41]. Therefore it can completely be replaced by integrals of a simpler type. This relation takes a particularly simple form in the special case where $M^2 = 0$:

$$\begin{aligned} T_{23'4'5'5'}(m^2) = \frac{1}{Dm^2p^2}\Big\{ & A_0(m^2)\big[(D-4)B_0(p^2;0,0) + Dp^2 B_0'(p^2;0,0)\big] \\ & - (D-2)^2 T_{23'5'}(m^2) + (3D-8)T_{23'4'}(m^2) \\ & - (D-4)(m^2+p^2)T_{23'4'5'}(m^2)\Big\}. \end{aligned} \tag{3.51}$$

The one-loop integral B_0' is defined as

$$B_0'(p^2; m_1^2, m_2^2) = \frac{\partial}{\partial(m_1^2)} B_0(p^2; m_1^2, m_2^2). \tag{3.52}$$

Similar relations as for $T_{2345'}$ and $T_{2345'5'}$ can be obtained for integrals with more massless propagators, e.g. $T_{2345'5'5'}$ and $T_{2345'5'5'5'}$.

A second class of relations can be derived via the well-known method of integration by parts. This yields relations for integrals where at least one propagator appears with a power higher than one. A particularly interesting example is the formula for the integral corresponding to topology 6 in Fig. 3.1, i.e. $T_{1134}(m_1^2, m_1^2, m_3^2, m_4^2)$. Starting from

$$D\,T_{134} = \langle\langle \Delta_{134} \frac{\partial k_1^\mu}{\partial k_1^\mu} \rangle\rangle = -\langle\langle k_1^\mu \frac{\partial}{\partial k_1^\mu} \Delta_{134} \rangle\rangle \tag{3.53}$$

and using the symmetry of the integral T_{134} with respect to permutation of the masses m_1^2 and m_3^2 one obtains

$$\begin{aligned} T_{1134} = \frac{-1}{\lambda(m_1^2, m_3^2, m_4^2)}\Big\{ & (3-D)(m_1^2 - m_3^2 - m_4^2)T_{134} \\ & - (1-D/2)\Big[A_0(m_1^2)\Big(A_0(m_3^2) + A_0(m_4^2)\Big) - 2A_0(m_3^2)A_0(m_4^2)\Big] \\ & - (m_3^2 - m_4^2)\Big(A_0(m_3^2) - A_0(m_4^2)\Big) B_0(0; m_1^2, m_1^2)\Big\}, \end{aligned} \tag{3.54}$$

where

$$\lambda(m_1^2, m_3^2, m_4^2) = m_1^4 + m_3^4 + m_4^4 - 2(m_1^2 m_3^2 + m_1^2 m_4^2 + m_3^2 m_4^2). \tag{3.55}$$

The integral T_{1134} can therefore be expressed through T_{134} and products of one-loop integrals.

Finally, it should be noted that some integrals vanish trivially in the framework of dimensional regularization, e.g.

$$T_{11} = T_{12} = T_{112} = 0. \tag{3.56}$$

3.5 Computer-algebraic realization and further evaluation

The procedure for the algebraic calculation of two-loop self-energies is algorithmic and we implemented it into a computer-algebra program which we called *TwoCalc* [39]. It is written in *Mathematica* and linked to the packages *FeynArts* [42] and *FeynCalc* [43]. *FeynArts* creates the Feynman amplitudes and draws the Feynman graphs. *FeynCalc* is used here for the contraction of Lorentz indices and the evaluation of Dirac traces. In the calculations performed in this work an anticommuting γ_5 in D dimensions was used.

The three packages run fully automatically. This high degree of automation is very convenient for performing calculations which involve a large number of diagrams.

The algebraic method presented here has been shown to be valid for all two-loop self-energies in massive gauge theories. Accordingly, the task of evaluating any two-loop self-energy involving in general several thousand Feynman amplitudes is reduced to the problem of calculating a small number of two-loop scalar integrals. In fact, one only has to work out three different types of two-loop scalar integrals, namely T_{234}, T_{1234} and T_{12345}. Integrals containing propagators with higher powers, e.g. T_{11234}, can be obtained from those by partial fractioning or taking derivatives with respect to the masses. The vacuum integral T_{134} is given as the special case $p^2 = 0$ of T_{234} and need not be considered separately.

During the last years, these integrals have extensively been studied in the literature. Analytical results in terms of polylogarithmic functions, i.e. logarithms, di- and trilogarithms, have been derived for special values of the masses and the external momentum (see e.g. Refs. [44, 45]). In the general case it has been shown that this class of special functions is not sufficient for expressing the integrals [46]. Nevertheless, a number of results have been obtained for the general case using different methods. Small and large momentum expansions were worked out in Refs. [47]. For T_{234} and T_{1234} expressions in terms of generalized hypergeometric functions have been derived [48–50]. Furthermore, integral representations suitable for a numerical evaluation have been obtained for all three types of scalar integrals [11, 49–51]. The one-loop integrals arising from the calculation of irreducible two-loop contributions are in general needed up to $\mathcal{O}(D-4)$. Results for these integrals can be found in Refs. [45, 52].

3.6 Slavnov–Taylor identities for two-loop self-energies

In order to demonstrate the abilities of our algorithm for carrying out large calculations we explicitly verify some Slavnov–Taylor identities valid for two-loop self-energies in the conventional formalism of the SM. In this section we use the 't Hooft–Feynman gauge.

As mentioned above, the Slavnov–Taylor identities in general involve Green functions of ghost fields. An exception are the two-point functions of the gauge bosons, for which identities free of ghost contributions can be found. In Ref. [35] exact relations for the full gauge-boson propagators $\Delta^{\alpha\beta}_{\mu\nu}(p,-p)$, the mixing propagators $\Delta^{\alpha i}_{\mu}(p,-p)$ of gauge bosons and scalars and the unphysical Higgs propagators $\Delta^{ij}(p,-p)$ are listed

$$\begin{aligned}
&p^\mu p^\nu \Delta^{AA}_{\mu\nu}(p,-p) &&&&= -i,\\
&p^\mu p^\nu \Delta^{ZZ}_{\mu\nu}(p,-p) &&- 2iM_{\rm Z} p^\mu \Delta^{Z\chi}_{\mu}(p,-p) &&+ M_{\rm Z}^2 \Delta^{\chi\chi}(p,-p) &= -i,\\
&p^\mu p^\nu \Delta^{AZ}_{\mu\nu}(p,-p) &&- iM_{\rm Z} p^\mu \Delta^{A\chi}_{\mu}(p,-p) &&&= 0,\\
&p^\mu p^\nu \Delta^{WW}_{\mu\nu}(p,-p) &&+ 2M_{\rm W} p^\mu \Delta^{W\phi}_{\mu}(p,-p) &&+ M_{\rm W}^2 \Delta^{\phi\phi}(p,-p) &= -i.
\end{aligned} \tag{3.57}$$

Writing the propagators as the inverse of the two-point vertex functions as specified in Section 3.1, using (3.12) and (3.13) and expanding up to second order leads to the relations for the two-loop self-energies [12]

$$\begin{aligned}
&p^2\left[\tilde\Sigma_{\rm L}^{AA,(2)} - \left(\tilde\Sigma^{A\chi,(1)}\right)^2\right] &&= 0,\\
&p^2\left[\tilde\Sigma_{\rm L}^{ZZ,(2)} - 2iM_{\rm Z}\tilde\Sigma^{Z\chi,(2)}\right] - M_{\rm Z}^2\tilde\Sigma^{\chi\chi,(2)} - p^2\left(\tilde\Sigma^{Z\chi,(1)}\right)^2 + \tilde\Sigma_{\rm L}^{ZZ,(1)}\,\tilde\Sigma^{\chi\chi,(1)} &&= 0,\\
&p^2\left[\tilde\Sigma_{\rm L}^{AZ,(2)} - iM_{\rm Z}\tilde\Sigma^{A\chi,(2)}\right] - p^2\tilde\Sigma^{A\chi,(1)}\,\tilde\Sigma^{Z\chi,(1)} + iM_{\rm Z}\tilde\Sigma^{A\chi,(1)}\,\tilde\Sigma^{\chi\chi,(1)} &&= 0,\\
&p^2\left[\tilde\Sigma_{\rm L}^{WW,(2)} + 2M_{\rm W}\tilde\Sigma^{W\phi,(2)}\right] - M_{\rm W}^2\tilde\Sigma^{\phi\phi,(2)} - p^2\left(\tilde\Sigma^{W\phi,(1)}\right)^2 + \tilde\Sigma_{\rm L}^{WW,(1)}\,\tilde\Sigma^{\phi\phi,(1)} &&= 0,
\end{aligned} \tag{3.58}$$

where the tadpole contributions are included in the self-energies as indicated by writing a tilde. The corresponding relations for the one-loop self-energies are given in Ref. [35].

The first relation in (3.58) indicates that unlike in pure QED the longitudinal part of the two-loop photon self-energy does not vanish in the conventional formalism of the SM. We used *TwoCalc* to calculate explicitly every Feynman amplitude contributing to the left-hand side of this relation. We did this in the full electroweak SM with the only restriction that the fermions are limited to one doublet, which we chose to be the u- and d-quark. The calculations for all other fermions can be performed analogously.

We had to consider 1024 graphs for the two-loop photon self-energy (including tadpole contributions), 162 of which contain the u- and d-quarks, the remaining ones are purely bosonic. The most involved diagrams are of course those stemming from the generic two-loop topologies. These are 418 diagrams belonging to the first 6 topologies in Fig. 3.1. The remaining ones lead to integrals expressible solely as products of one-

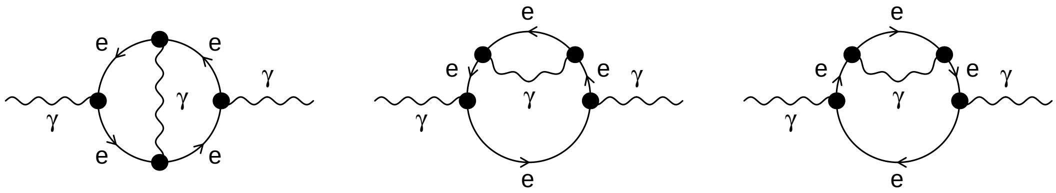


Figure 3.4: The diagrams contributing to the two-loop photon self-energy in QED.

loop integrals. Altogether, 13 of the 20 possible topologies are realized for the photon self-energy and the corresponding tadpole contributions.

The one-loop $(A\chi)$ mixing energy vanishes for fermionic graphs, while 4 purely bosonic graphs contribute. Summing up the results of all amplitudes gives exactly zero, i.e. the prefactor of every standard integral vanishes separately.

We also explicitly verified the second relation stated in (3.58). Here, of course, a lot more Feynman diagrams have to be calculated. Again we restricted the fermions to u- and d-quark. This gives 1552 diagrams for $\tilde{\Sigma}_{\mathrm{L}}^{ZZ,(2)}$, 1155 for $\tilde{\Sigma}^{Z\chi,(2)}$ and 1142 for $\tilde{\Sigma}^{\chi\chi,(2)}$. From these, 1787 diagrams arise from the 8 generic two-loop topologies.

For the Z-boson and the unphysical Higgs field χ all 20 topologies possible for two-loop self-energies and tadpoles contribute. The self-energies in one-loop order give rise to 65 diagrams. Adding those to the 3849 two-loop graphs by using the coefficients specified in (3.58) gives exactly zero.

3.7 Results for two-loop gauge-boson self-energies

3.7.1 Photon self-energy in QED

As an application of the techniques described above we treat the gauge-boson self-energies in the electroweak SM. We begin with the special case of the photon self-energy in pure QED. The three contributing diagrams are shown in Fig. 3.4, where as in the following we denote the photon in the Feynman diagrams by γ.

We decompose the photon self-energy according to (3.11). The longitudinal part vanishes owing to the well-known Ward identity. We checked this by explicit calculation. The result for the transverse part reads

$$\begin{aligned}
\Sigma_{\mathrm{T,QED}}^{AA,(2)}(p^2) = \frac{e^4}{128\pi^4(D-1)} \Big\{ & \Big[(2-D)^3/m_{\mathrm{e}}^2\Big] A_0^2(m_{\mathrm{e}}^2) \\
& - \Big[(2-D)^2(m_{\mathrm{e}}^2 - p^2)/m_{\mathrm{e}}^2\Big] A_0(m_{\mathrm{e}}^2) B_0(p^2; 0, m_{\mathrm{e}}^2) \\
& - \Big[(2-D)(4m_{\mathrm{e}}^2 + 2p^2 - Dp^2)/m_{\mathrm{e}}^2\Big] A_0(m_{\mathrm{e}}^2) B_0(p^2; m_{\mathrm{e}}^2, m_{\mathrm{e}}^2) \\
& - (8m_{\mathrm{e}}^2 - 14p^2 + 9Dp^2 - D^2p^2) B_0^2(p^2; m_{\mathrm{e}}^2, m_{\mathrm{e}}^2) \\
& - (2-D)\Big[2(4m_{\mathrm{e}}^2 - 2p^2 + Dp^2) A_0(m_{\mathrm{e}}^2) B_0'(p^2; m_{\mathrm{e}}^2, m_{\mathrm{e}}^2)
\end{aligned}$$

$$
\begin{aligned}
&\qquad + (2-D)T_{13'4'} - 2(1-D)T_{134'} - (6-D)T_{234'} \\
&\qquad + (2-D)(m_e^2 - p^2)T_{1'234'}\Big] \\
&+ 8(4m_e^2 - 2p^2 + Dp^2)\Big[T_{1234'} + m_e^2 T_{11234'}\Big] \\
&+ 2(2m_e^2 - p^2)(4m_e^2 - 2p^2 + Dp^2)T_{123'45}\Bigg\}.
\end{aligned} \tag{3.59}
$$

As above, a prime at the indices of the T-integrals denotes that the corresponding propagator has mass zero. All other propagators carry the electron mass m_e.

The calculation was carried out in an arbitrary R_ξ gauge. The absence of the gauge parameter ξ_A indicates that the result is gauge-independent as it has to be. The two-loop integrals appearing in (3.59) are expressible in terms of polylogarithmic functions. They were studied for example in Refs. [44].

Calculations of the two-loop photon self-energy in QED were performed in Refs. [53]. Similar calculations focusing on the $\mathcal{O}(\alpha\alpha_s)$ corrections due to the gluon exchange in a quark loop of gauge-boson self-energies were done in Refs. [54].

In passing we note that in contrast to the photon self-energy the two-loop electron self-energy in QED is not gauge-independent. The explicit algebraic result will be presented elsewhere.

3.7.2 Results in the conventional formalism of the Standard Model

In the electroweak SM results for two-loop gauge-boson self-energies restricted to the region $p^2 = 0$ and asymptotic values of the top and Higgs mass, respectively, were worked out in Refs. [8–11]. A calculation allowing for general values of the invariant momentum p^2 and the masses of the gauge bosons, the top quark and the Higgs field has to deal with an enormous number of Feynman diagrams having in general a very complicated structure. With the methods outlined above we can express these in terms of a small number of standard integrals, the result being valid for general values of all parameters involved.

This was explicitly carried out in Section 3.6 for all graphs contributing to the longitudinal parts of the photon and Z-boson self-energies. The results for the complete transverse parts are of course rather lengthy and we do not present them here. Instead we focus on the light-fermion contributions, a class of diagrams particularly well suited for separate treatment which is expected to yield a substantial contribution to the complete result.

With light-fermion contributions we mean all diagrams containing any fermion other than the top-quark. The masses of these fermions are negligible in comparison to the boson masses (M_Z, M_W, M_H) and also to the invariant momentum p^2 if the latter is sufficiently large. All T-integrals resulting from these graphs therefore contain one massless fermion loop. They can be calculated analytically leading to polylogarithmic functions [45].

In this section we work in the conventional formalism of the SM and use the R_ξ gauge specified in (2.23) and (2.24)*.

In order to study the structure of the results it is sufficient to restrict the fermions to one doublet in the same way as in Section 3.6. This yields 27 diagrams contributing to the two-loop photon self-energy, 32 graphs for the Z self-energy and 45 diagrams for the self-energy of the $W^\pm$.

As shown in Fig. 3.5 it is convenient to subdivide the photon self-energy into several sets of graphs. The graphs depicted in Fig. 3.5a are of QED-type having a photon exchanged in a quark loop. As indicated in the picture the same set of graphs also exists for the u-quark. The results are in exact analogy to (3.59) and we do not write them out explicitly. As a consequence of (3.59), the QED-type graphs form gauge-independent subsets of the photon self-energy.

In the graphs shown in Fig. 3.5b a Z-boson is exchanged instead of the photon. We give the result neglecting the mass of the d-quark:

$$\begin{aligned}
\Sigma^{AA,(2)}_{\mathrm{T,Z}}(p^2) = {} & \frac{e^4(2-D)(9-12s_{\mathrm{W}}^2+8s_{\mathrm{W}}^4)}{(4\pi)^4(1-D)D108c_{\mathrm{W}}^2 s_{\mathrm{W}}^2} \\
& \times \Big\{ 2(4-6D+D^2)A_0(M_{\mathrm{Z}}^2)B_0(p^2;0,0) \\
& - D(2M_{\mathrm{Z}}^2+7p^2-Dp^2)\Big(B_0(p^2;0,0)\Big)^2 \\
& + 2(4-6D+D^2)T_{13'4'}(M_{\mathrm{Z}}^2) + 2(8-4D+D^2)T_{23'4'}(M_{\mathrm{Z}}^2) \\
& - 2(4-8D+D^2)(M_{\mathrm{Z}}^2+p^2)T_{1'2'34'}(M_{\mathrm{Z}}^2) \\
& - D(2M_{\mathrm{Z}}^4+8M_{\mathrm{Z}}^2p^2-DM_{\mathrm{Z}}^2p^2+2p^4)T_{1'2'34'5'}(M_{\mathrm{Z}}^2)\Big\}. \qquad (3.60)
\end{aligned}$$

The subscript Z refers to the Z-boson exchange. It can be seen from (3.60) that for non-exceptional values of p^2 no mass singularities are induced by setting the fermion mass to zero (see Ref. [55]).

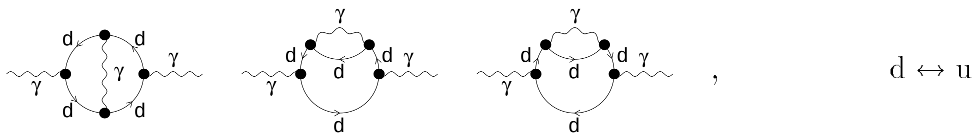
There is no gauge-parameter dependence left in (3.60). Therefore all contributions from "neutral currents", i.e. photon and Z exchange, are gauge-independent (within the class of R_ξ gauges).

The "charged currents" mediated by $W^\pm$ and $\phi^\pm$ give rise to the 15 diagrams shown in Fig. 3.5c. The result for the transverse part reads

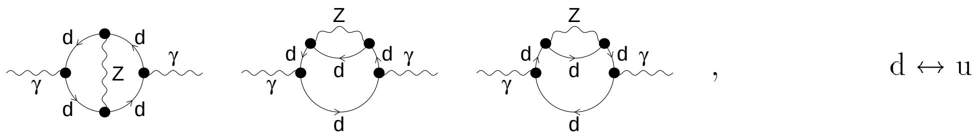
$$\begin{aligned}
\Sigma^{AA,(2)}_{\mathrm{T,W}}(p^2) = {} & \frac{e^4(2-D)}{(4\pi)^4(1-D)^2 12M_{\mathrm{W}}^4 s_{\mathrm{W}}^2}\Big\{ f(p^2,M_{\mathrm{W}}^2,D) + 9\,g(\xi_W,p^2,M_{\mathrm{W}}^2,D) \\
& + F(p^2,M_{\mathrm{W}}^2,D) - 9M_{\mathrm{W}}^2(p^2-2M_{\mathrm{W}}^2)\,G(\xi_W,p^2,M_{\mathrm{W}}^2,D)\Big\}, \qquad (3.61)
\end{aligned}$$

where in $f(p^2,M_{\mathrm{W}}^2,D)$ and $g(\xi_W,p^2,M_{\mathrm{W}}^2,D)$ we collected the contributions involving

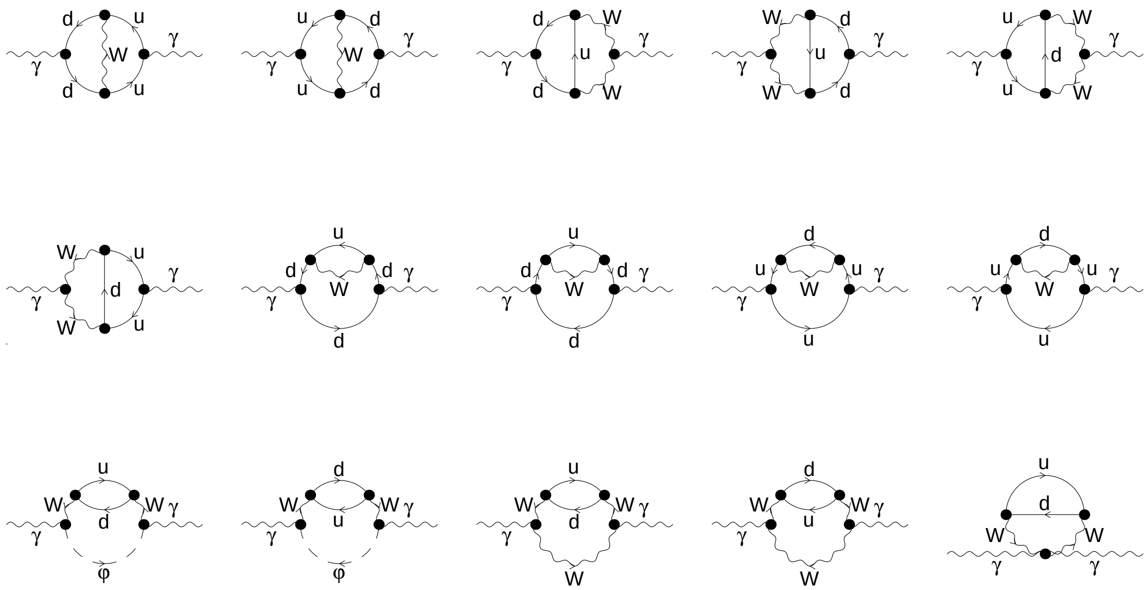
*Note that the gauge parameters defined in Ref. [41] are the inverse of the ones used in this work.



(a)



(b)



(c)

Figure 3.5: The light-fermion contributions to the two-loop photon self-energy.

only one-loop integrals

$$
\begin{aligned}
f(p^2, M_{\mathrm{W}}^2, D) = {} & 2M_{\mathrm{W}}^2 \Big[\Big(\frac{40}{D} - 91 + 70D - 10D^2\Big)M_{\mathrm{W}}^2 - 18(2-D)p^2\Big] \\
& \quad \times A_0(M_{\mathrm{W}}^2)B_0(p^2;0,0) \\
& + 8(1-D)M_{\mathrm{W}}^4 \Big[2M_{\mathrm{W}}^2 + (7-D)p^2\Big] \Big(B_0(p^2;0,0)\Big)^2 \\
& - 9(2M_{\mathrm{W}}^2 - p^2)\Big[4(1-D)M_{\mathrm{W}}^4 + 4(3-2D)M_{\mathrm{W}}^2 p^2 - p^4\Big] \\
& \quad \times B_0(p^2;0,0)B_0(p^2;M_{\mathrm{W}}^2,M_{\mathrm{W}}^2), \qquad (3.62) \\
g(\xi_W, p^2, M_{\mathrm{W}}^2, D) = {} & -2\xi_W M_{\mathrm{W}}^2(M_{\mathrm{W}}^2 - p^2)A_0(M_{\mathrm{W}}^2)B_0(p^2;0,0) \\
& + 2M_{\mathrm{W}}^2\Big[\xi_W(M_{\mathrm{W}}^2 - p^2) - M_{\mathrm{W}}^2 + 2(2-D)p^2\Big] \\
& \quad \times A_0(\xi_W M_{\mathrm{W}}^2)B_0(p^2;0,0) \\
& - 2(M_{\mathrm{W}}^2 - p^2)\Big[M_{\mathrm{W}}^4(\xi_W - 1)^2 - 2(\xi_W + 3 - 2D)M_{\mathrm{W}}^2 p^2 + p^4\Big] \\
& \quad \times B_0(p^2;0,0)B_0(p^2;M_{\mathrm{W}}^2,\xi_W M_{\mathrm{W}}^2) \\
& + p^4(4\xi_W M_{\mathrm{W}}^2 - p^2)B_0(p^2;0,0)B_0(p^2;\xi_W M_{\mathrm{W}}^2,\xi_W M_{\mathrm{W}}^2), \qquad (3.63)
\end{aligned}
$$

and $F(p^2, M_{\mathrm{W}}^2, D)$ and $G(\xi_W, p^2, M_{\mathrm{W}}^2, D)$ contain the generic two-loop contributions

$$
\begin{aligned}
F(p^2, M_{\mathrm{W}}^2, D) = {} & M_{\mathrm{W}}^2\Big\{\Big[2\Big(\frac{40}{D} - 118 + 187D - 100D^2 + 18D^3\Big)M_{\mathrm{W}}^2 \\
& \quad - 9(7-4D)(3-D)p^2\Big]T_{13'4'}(M_{\mathrm{W}}^2) \\
& + \Big[4\Big(\frac{40}{D} - 96 + 61D - 14D^2\Big)M_{\mathrm{W}}^2 + 9p^2\Big]T_{23'4'}(M_{\mathrm{W}}^2) \\
& - 20(1/D - 1)(4 - 8D + D^2)M_{\mathrm{W}}^2(M_{\mathrm{W}}^2 + p^2)T_{1'2'34'}(M_{\mathrm{W}}^2) \\
& + 18(4M_{\mathrm{W}}^2 - p^2)\Big[2(1-D)M_{\mathrm{W}}^2 + (3-2D)p^2\Big] \\
& \quad \times T_{123'4'}(M_{\mathrm{W}}^2, M_{\mathrm{W}}^2) \\
& + 9(4M_{\mathrm{W}}^2 - p^2)\Big[4(1-D)M_{\mathrm{W}}^4 + 4(3-2D)M_{\mathrm{W}}^2 p^2 - p^4\Big] \\
& \quad \times T_{1123'4'}(M_{\mathrm{W}}^2, M_{\mathrm{W}}^2, M_{\mathrm{W}}^2) \\
& + 72(1-D)M_{\mathrm{W}}^4(M_{\mathrm{W}}^2 + 2p^2)T_{123'4'5'}(M_{\mathrm{W}}^2, M_{\mathrm{W}}^2) \\
& + 8(1-D)M_{\mathrm{W}}^2\Big[2(M_{\mathrm{W}}^4 + p^4) + (8-D)M_{\mathrm{W}}^2 p^2\Big] \\
& \quad \times T_{1'2'34'5'}(M_{\mathrm{W}}^2)\Big\}, \qquad (3.64) \\
G(\xi_W, p^2, M_{\mathrm{W}}^2, D) = {} & \xi_W(D-3)T_{13'4'}(M_{\mathrm{W}}^2) + T_{23'4'}(\xi_W M_{\mathrm{W}}^2) \\
& - 2\Big[(\xi_W - 1)M_{\mathrm{W}}^2 + (3-2D)p^2\Big]T_{123'4'}(M_{\mathrm{W}}^2, \xi_W M_{\mathrm{W}}^2) \\
& + \Big[M_{\mathrm{W}}^4(\xi_W - 1)^2 - 2(\xi_W + 3 - 2D)M_{\mathrm{W}}^2 p^2 + p^4\Big] \\
& \times T_{1123'4'}(M_{\mathrm{W}}^2, M_{\mathrm{W}}^2, \xi_W M_{\mathrm{W}}^2). \qquad (3.65)
\end{aligned}
$$

The functions $f(p^2, M_{\mathrm{W}}^2, D)$ and $F(p^2, M_{\mathrm{W}}^2, D)$ are independent of the gauge parameter ξ_W. The occurrence of ξ_W in $g(\xi_W, p^2, M_{\mathrm{W}}^2, D)$ and $G(\xi_W, p^2, M_{\mathrm{W}}^2, D)$ indicates

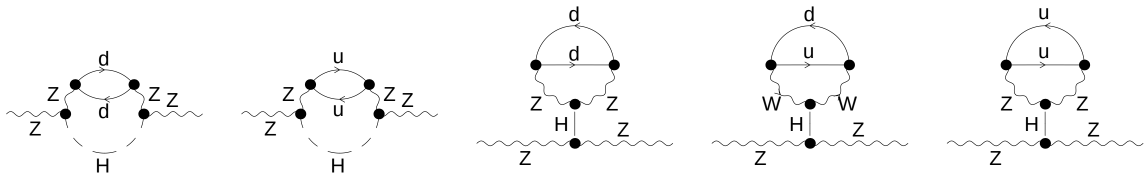


Figure 3.6: Higgs-dependent contributions to the Z self-energy including tadpoles.

that the contribution from the "charged current" graphs is gauge-dependent. This feature is expected since also in one-loop order the graphs containing W-bosons yield gauge-dependent contributions to the photon self-energy (see for example Ref. [56]).

As mentioned above, the integrals occurring in (3.61) can be solved analytically for general values of ξ_W. Using the expressions given in [45] the dependence on the variables $p^2, M_{\rm W}^2, D$ and ξ_W can be studied.

For all graphs contributing to the photon self-energy we also calculated the longitudinal part. It vanishes separately for each set of diagrams specified above and without restriction on the values of the gauge parameters. We also studied the limit $p^2 \to 0$ for all results given above and checked that they give zero for all values of the gauge parameters and the dimension D.

Next, we focus on the self-energy of the Z-boson. We first note that there are contributions from 27 graphs corresponding to the ones considered for the photon self-energy, i.e. from the diagrams shown in Fig. 3.5 where the in- and outgoing photon is substituted by a Z. In addition, five light-fermion graphs occur containing the Higgs-field, three of which are tadpole graphs. These diagrams are depicted in Fig. 3.6.

We begin with the diagrams which are of the same form as in the case of the photon. The transverse parts of the "neutral current" graphs are treated in precisely the same way as described above. The results have the same form and contain exactly the same integrals as the corresponding ones for the photon self-energy. Since we are only interested in the structure of the results we do not list them here explicitly. As above, these graphs yield gauge-independent contributions.

For the 15 "charged current" graphs we obtain:

$$\begin{aligned}\Sigma_{\rm T,W}^{ZZ,(2)}(p^2) = \frac{e^4(2-D)}{(4\pi)^4(1-D)^2 12 M_{\rm W}^2 M_{\rm Z}^2 s_{\rm W}^4}\Big\{ & f_{\rm Z}(p^2, M_{\rm W}^2, M_{\rm Z}^2, D) \\ & + 9\, g(\xi_W, p^2, M_{\rm W}^2, D) + F_{\rm Z}(p^2, M_{\rm W}^2, M_{\rm Z}^2, D) \\ & - 9(p^2 - M_{\rm Z}^2)(p^2 - 2M_{\rm W}^2 + M_{\rm Z}^2)\frac{M_{\rm W}^2}{p^2}\, G(\xi_W, p^2, M_{\rm W}^2, D)\Big\}. \end{aligned} \quad (3.66)$$

The gauge-dependent functions $g(\xi_W, p^2, M_{\rm W}^2, D)$ and $G(\xi_W, p^2, M_{\rm W}^2, D)$ are the same as for the photon self-energy. They have been given in (3.63) and (3.65). The expressions for the gauge-parameter-independent functions $f_{\rm Z}(p^2, M_{\rm W}^2, M_{\rm Z}^2, D)$ and $F_{\rm Z}(p^2, M_{\rm W}^2, M_{\rm Z}^2, D)$ read

$$f_{\rm Z}(p^2, M_{\rm W}^2, M_{\rm Z}^2, D) = 2\Big[(1/D-1)(4-6D+D^2)\Big(9M_{\rm W}^4 + (M_{\rm W}^2 - M_{\rm Z}^2)^2\Big)$$

$$
\begin{aligned}
&\quad + 9M_\mathrm{W}^2\Big(M_\mathrm{W}^2 - 2(2-D)p^2\Big)\Big] A_0(M_\mathrm{W}^2) B_0(p^2;0,0) \\
&+ (1-D)(4M_\mathrm{W}^2 - M_\mathrm{Z}^2)(2M_\mathrm{W}^2 + M_\mathrm{Z}^2)\Big[2M_\mathrm{W}^2 + (7-D)p^2\Big] \\
&\quad \times \Big(B_0(p^2;0,0)\Big)^2 \\
&- 9(2M_\mathrm{W}^2 - p^2)\Big[4(1-D)M_\mathrm{W}^4 + 4(3-2D)M_\mathrm{W}^2 p^2 - p^4\Big] \\
&\quad \times B_0(p^2;0,0) B_0(p^2;M_\mathrm{W}^2,M_\mathrm{W}^2),
\end{aligned} \tag{3.67}
$$

$$
\begin{aligned}
F_\mathrm{Z}(p^2, M_\mathrm{W}^2, M_\mathrm{Z}^2, D) = {} & \Big[2\Big(\frac{40}{D} - 118 + 187D - 100D^2 + 18D^3\Big) M_\mathrm{W}^4 \\
&\quad - \Big(\frac{8}{D} + 7 + 5D - 2D^2 + 9(2-D)\frac{M_\mathrm{W}^2}{p^2}\Big) M_\mathrm{Z}^2 (2M_\mathrm{W}^2 - M_\mathrm{Z}^2) \\
&\quad - 9(7-4D)(3-D) M_\mathrm{W}^2 p^2\Big] T_{13'4'}(M_\mathrm{W}^2) \\
&+ \Big[4(\frac{40}{D} - 96 + 61D - 14D^2) M_\mathrm{W}^4 \\
&\quad - 2(1/D - 1)(8 - 4D + D^2) M_\mathrm{Z}^2 (2M_\mathrm{W}^2 - M_\mathrm{Z}^2) \\
&\quad + 9M_\mathrm{W}^2 p^2\Big] T_{23'4'}(M_\mathrm{W}^2) \\
&- 2(1/D - 1)(4 - 8D + D^2)(M_\mathrm{W}^2 + p^2)\Big(9M_\mathrm{W}^4 + (M_\mathrm{W}^2 - M_\mathrm{Z}^2)^2\Big) \\
&\quad \times T_{1'2'34'}(M_\mathrm{W}^2) \\
&+ 18M_\mathrm{W}^2(4M_\mathrm{W}^2 - p^2)\Big[2(1-D)M_\mathrm{W}^2 + (3-2D)p^2\Big] \\
&\quad \times T_{123'4'}(M_\mathrm{W}^2, M_\mathrm{W}^2) \\
&+ 9M_\mathrm{W}^2(4M_\mathrm{W}^2 - p^2)\Big[4(1-D)M_\mathrm{W}^4 + 4(3-2D)M_\mathrm{W}^2 p^2 - p^4\Big] \\
&\quad \times T_{1123'4'}(M_\mathrm{W}^2, M_\mathrm{W}^2, M_\mathrm{W}^2) \\
&+ 72(1-D) M_\mathrm{W}^6 (M_\mathrm{W}^2 + 2p^2) T_{123'4'5'}(M_\mathrm{W}^2, M_\mathrm{W}^2) \\
&+ (1-D)(4M_\mathrm{W}^2 - M_\mathrm{Z}^2)(2M_\mathrm{W}^2 + M_\mathrm{Z}^2) \\
&\quad \times \Big[2(M_\mathrm{W}^4 + p^4) + (8-D) M_\mathrm{W}^2 p^2\Big] T_{1'2'34'5'}(M_\mathrm{W}^2).
\end{aligned} \tag{3.68}
$$

For $M_\mathrm{Z}^2 = 0$ the functions $f_\mathrm{Z}(p^2, M_\mathrm{W}^2, M_\mathrm{Z}^2, D)$ and $F_\mathrm{Z}(p^2, M_\mathrm{W}^2, M_\mathrm{Z}^2, D)$ coincide with the corresponding ones appearing in the result of the photon self-energy, i.e. $f(p^2, M_\mathrm{W}^2, D)$ and $F(p^2, M_\mathrm{W}^2, D)$.

The characteristic feature of the result (3.66) is that all gauge-dependent contributions involving two-loop integrals are proportional to $(p^2 - M_\mathrm{Z}^2)$. This can be understood either from the fact that $p^2 = M_\mathrm{Z}^2$ is the mass shell condition and from the gauge independence of the mass renormalization (see also the discussion in Section 5.3) or, in the framework of the pinch technique (see Chapter 4), by observing that this factor is needed to cancel a lowest-order Z propagator. Note that even for $p^2 = M_\mathrm{Z}^2$ a residual gauge-parameter dependence remains in the term $g(\xi_W, p^2, M_\mathrm{W}^2, D)$ containing only scalar one-loop integrals. In the mass renormalization its gauge dependence cancels with contributions from counterterm insertions in one-loop diagrams. The implication of this residual gauge-parameter dependence for the pinch technique will be discussed below.

In contrast to the photon case the light-fermion contributions to the Z self-energy depend on the Higgs mass. Whereas the contributions from the three tadpole graphs are compensated by the usual choice of the renormalization constant, the first two graphs in Fig. 3.6 give a p^2-dependent contribution to the Z self-energy. We obtain for the first graph

$$\begin{aligned}\Sigma_{\mathrm{T,H}}^{ZZ,(2)}(p^2) = {} & \frac{e^4(2-D)(9-12s_{\mathrm{W}}^2+8s_{\mathrm{W}}^4)}{(4\pi)^4(1-D)^2 48p^2 c_{\mathrm{W}}^4 s_{\mathrm{W}}^4} \\ & \times\Big\{\Big[(3-D)(p^2-M_{\mathrm{H}}^2)+(2-D)M_{\mathrm{Z}}^2\Big]T_{13'4'}(M_{\mathrm{Z}}^2) \\ & + M_{\mathrm{Z}}^2 T_{23'4'}(M_{\mathrm{H}}^2) - 2M_{\mathrm{Z}}^2\Big[M_{\mathrm{H}}^2-M_{\mathrm{Z}}^2+(3-2D)p^2\Big]T_{123'4'}(M_{\mathrm{Z}}^2,M_{\mathrm{H}}^2) \\ & + M_{\mathrm{Z}}^2\Big[(M_{\mathrm{H}}^2-M_{\mathrm{Z}}^2)^2-2p^2\Big(M_{\mathrm{H}}^2+(3-2D)M_{\mathrm{Z}}^2\Big)+p^4\Big] \\ & \quad\times T_{1123'4'}(M_{\mathrm{Z}}^2,M_{\mathrm{Z}}^2,M_{\mathrm{H}}^2)\Big\},\end{aligned} \tag{3.69}$$

which is obviously gauge-parameter-independent. The subscript H indicates the dependence on the Higgs mass which appears in the scalar integrals and as coefficients M_{H}^2 and M_{H}^4. The result for the second graph in Fig. 3.6 is derived from (3.69) by using the appropriate coupling of the u-quark.

The result for the three tadpole graphs reads

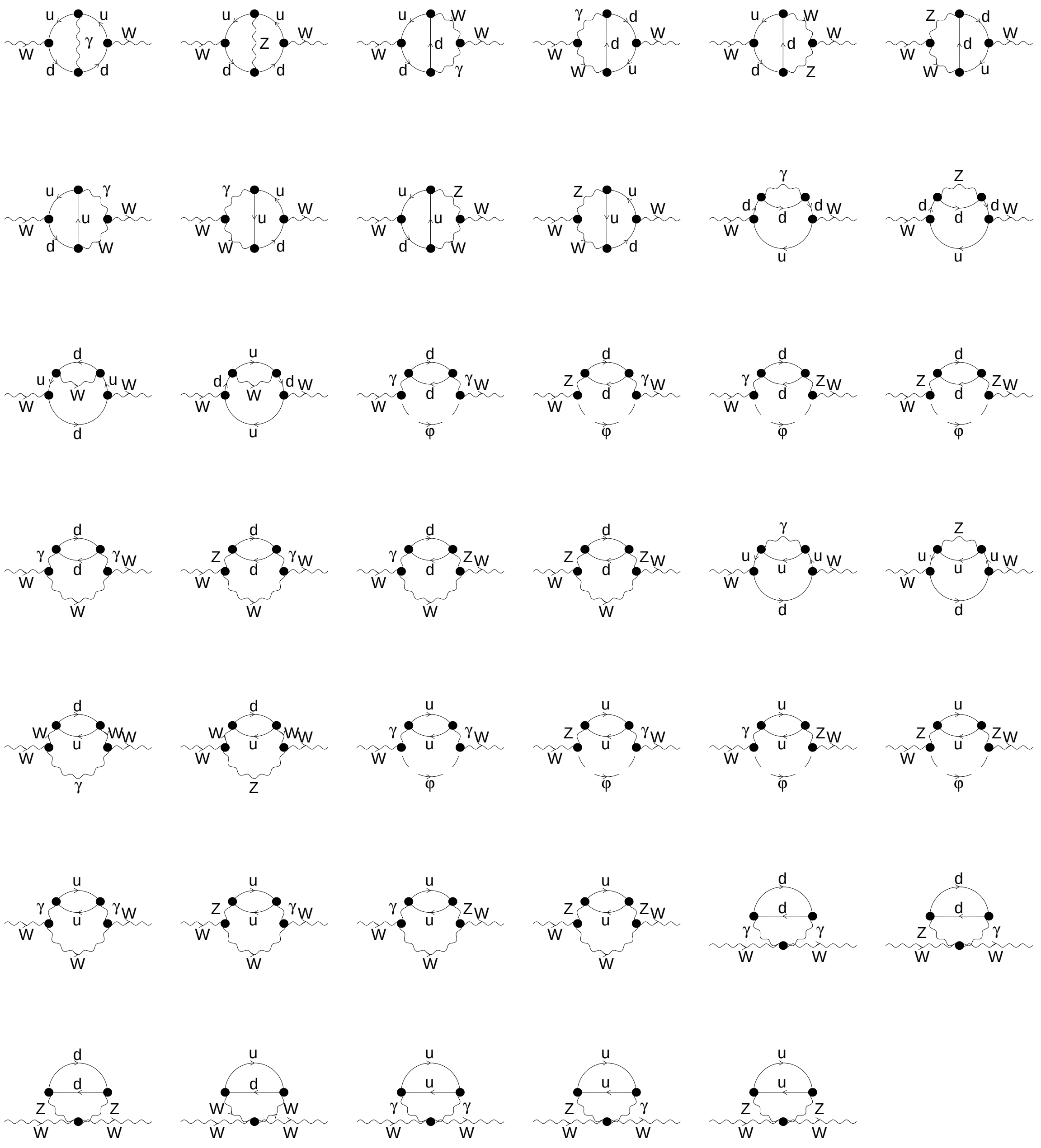
$$\tilde{\Sigma}_{\mathrm{T,tad}}^{ZZ,(2)} = \frac{e^4(2-D)^2M_{\mathrm{Z}}^2}{(4\pi)^4 12M_{\mathrm{H}}^2 s_{\mathrm{W}}^4}\Big\{18T_{13'4'}(M_{\mathrm{W}}^2)+\Big[11\frac{(M_{\mathrm{W}}^2-M_{\mathrm{Z}}^2)^2}{M_{\mathrm{W}}^4}+9\Big]T_{13'4'}(M_{\mathrm{Z}}^2)\Big\}, \tag{3.70}$$

showing that they are gauge-parameter-independent. This is also true for each of the graphs separately.

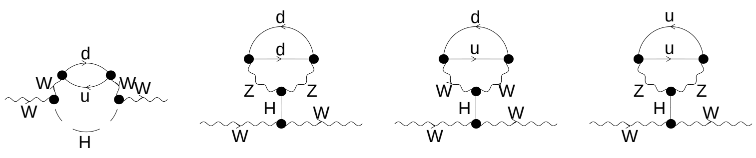
In the study of the mixing energy $\Sigma_{\mathrm{T}}^{AZ,(2)}(p^2)$ no new features appear. The diagrams are of the same type as for the photon, i.e. we have contributions from the 27 graphs shown in Fig. 3.5 with one of the external photons substituted by a Z-boson. Since the results have precisely the same structure as in the case of the photon we do not list them in detail.

The graphs relevant for the $\mathrm{W}^\pm$ self-energy are shown in Fig. 3.7. As can be seen in Fig. 3.7a there is no natural way to make a subdivision into "neutral" and "charged current" graphs. Like in the case of the Z-boson we get contributions from diagrams depending on the Higgs mass. These are shown in Fig. 3.7b.

We first consider the 41 graphs listed in Fig. 3.7a. The result in 't Hooft–Feynman gauge, i.e. $\xi_A=\xi_Z=\xi_W=1$, is given in Ref. [41]. Inspection of the integrals appearing in formula (A.1) of Ref. [41] shows that there is only one type of integral which has not yet been encountered in the results of the neutral gauge-bosons, namely $T_{123'4'5'}(M_{\mathrm{Z}}^2,M_{\mathrm{W}}^2)$. Like the integrals considered above it can be solved analytically [45].



(*a*)



(*b*)

Figure 3.7: The light-fermion contributions to the W self-energy including tadpoles.

The result for general values of the gauge parameters is very lengthy and will not be presented here. It depends on all three gauge parameters ξ_A, ξ_Z and ξ_W. Therefore in this case also the neutral gauge-bosons yield a gauge-dependent contribution. In analogy to the Z self-energy all gauge-dependent terms associated with the generic two-loop integrals are proportional to $(p^2 - M_{\mathrm{W}}^2)$. The integral types appearing in this result are the same as in the case of the 't Hooft–Feynman gauge.

The results for the Higgs-dependent graphs in Fig. 3.7b are of the same form as those obtained for the Z self-energy in (3.69) and (3.70), where in (3.69) M_{Z} has to be substituted by M_{W}.

3.7.3 Results in the background-field method

In the last section we have listed results for the light-fermion contributions to the gauge-boson self-energies in the conventional formalism of the SM. Using the Feynman rules of the BFM given in the appendix the corresponding results in the BFM can be obtained.

The "neutral current" contributions in the BFM, i.e. the graphs following from Fig. 3.5a and Fig. 3.5b by replacing the in- and outgoing photons by background fields $\hat{\gamma}$ or $\hat{\mathrm{Z}}$, are the same as in the conventional formalism. This is due to the fact that the lowest-order couplings of the fermions are identical in the two formulations. Similarly, it can be inferred that the Higgs-mass dependent contributions to the self-energies of $\hat{\mathrm{Z}}$ and Z are the same.

For brevity we give here only one example from the other sets of graphs, namely the "charged current" contribution to the $\hat{\mathrm{Z}}$ self-energy. The relevant diagrams follow from Fig. 3.5c by replacing the in- and outgoing photons by background $\hat{\mathrm{Z}}$-bosons. The result for these graphs has the form

$$\begin{aligned}\Sigma_{\mathrm{T,W}}^{\hat{Z}\hat{Z},(2)}(p^2) = &\frac{e^4(2-D)}{(4\pi)^4(1-D)^2 12 M_{\mathrm{W}}^2 M_{\mathrm{Z}}^2 s_{\mathrm{W}}^4}\Bigg\{ f_{\hat{\mathrm{Z}}}(p^2, M_{\mathrm{W}}^2, M_{\mathrm{Z}}^2, D) \\ &+ 9\, g_{\hat{\mathrm{Z}}}(\xi_Q, p^2, M_{\mathrm{W}}^2, D) + F_{\hat{\mathrm{Z}}}(p^2, M_{\mathrm{W}}^2, M_{\mathrm{Z}}^2, D) \\ &- 9(p^2 - M_{\mathrm{Z}}^2)(p^2 + M_{\mathrm{Z}}^2)\frac{M_{\mathrm{W}}^2}{p^2}\, G_{\hat{\mathrm{Z}}}(\xi_Q, p^2, M_{\mathrm{W}}^2, D)\Bigg\}. \end{aligned} \qquad (3.71)$$

The expressions $f_{\hat{\mathrm{Z}}}(p^2, M_{\mathrm{W}}^2, M_{\mathrm{Z}}^2, D)$ and $F_{\hat{\mathrm{Z}}}(p^2, M_{\mathrm{W}}^2, M_{\mathrm{Z}}^2, D)$ are gauge-parameter-independent. They read

$$\begin{aligned} f_{\hat{\mathrm{Z}}}(p^2, M_{\mathrm{W}}^2, M_{\mathrm{Z}}^2, D) = &\; f_{\mathrm{Z}}(p^2, M_{\mathrm{W}}^2, M_{\mathrm{Z}}^2, D) \\ &- 18 M_{\mathrm{W}}^2 (M_{\mathrm{W}}^2 + p^2) A_0(M_{\mathrm{W}}^2) B_0(p^2; 0, 0), \end{aligned} \qquad (3.72)$$

$$\begin{aligned} F_{\hat{\mathrm{Z}}}(p^2, M_{\mathrm{W}}^2, M_{\mathrm{Z}}^2, D) = &\; F_{\mathrm{Z}}(p^2, M_{\mathrm{W}}^2, M_{\mathrm{Z}}^2, D) \\ &- 18(p^2 - M_{\mathrm{Z}}^2)\frac{M_{\mathrm{W}}^2}{p^2}\Big[(2-D)M_{\mathrm{W}}^2 + (3-D)p^2\Big] T_{13'4'}(M_{\mathrm{W}}^2), \end{aligned} \qquad (3.73)$$

where the quantities $f_{\mathrm{Z}}(p^2, M_{\mathrm{W}}^2, M_{\mathrm{Z}}^2, D)$ and $F_{\mathrm{Z}}(p^2, M_{\mathrm{W}}^2, M_{\mathrm{Z}}^2, D)$ have been defined in (3.67) and (3.68). We obtain for the gauge-dependent functions $g_{\hat{\mathrm{Z}}}(\xi_Q, p^2, M_{\mathrm{W}}^2, D)$ and $G_{\hat{\mathrm{Z}}}(\xi_Q, p^2, M_{\mathrm{W}}^2, D)$

$$\begin{aligned}
g_{\hat{\mathrm{Z}}}(\xi_Q, p^2, M_{\mathrm{W}}^2, D) = {} & 2\xi_Q M_{\mathrm{W}}^2 p^2 A_0(M_{\mathrm{W}}^2) B_0(p^2; 0, 0) \\
& - 2M_{\mathrm{W}}^2 p^2 (\xi_Q - 5 + 2D) A_0(\xi_Q M_{\mathrm{W}}^2) B_0(p^2; 0, 0) \\
& + 2p^2 \Big[M_{\mathrm{W}}^4 (\xi_Q - 1)^2 - 2(\xi_Q + 3 - 2D) M_{\mathrm{W}}^2 p^2 + p^4 \Big] \\
& \quad \times B_0(p^2; 0, 0) B_0(p^2; M_{\mathrm{W}}^2, \xi_Q M_{\mathrm{W}}^2) \\
& + p^2 (2M_{\mathrm{W}}^2 + p^2)(4\xi_Q M_{\mathrm{W}}^2 - p^2) \\
& \quad \times B_0(p^2; 0, 0) B_0(p^2; \xi_Q M_{\mathrm{W}}^2, \xi_Q M_{\mathrm{W}}^2), \qquad (3.74) \\
G_{\hat{\mathrm{Z}}}(\xi_Q, p^2, M_{\mathrm{W}}^2, D) = {} & \xi_Q (D - 3) T_{13'4'}(M_{\mathrm{W}}^2) + T_{23'4'}(\xi_Q M_{\mathrm{W}}^2) \\
& - 2\Big[(\xi_Q - 1) M_{\mathrm{W}}^2 + (3 - 2D) p^2 \Big] T_{123'4'}(M_{\mathrm{W}}^2, \xi_Q M_{\mathrm{W}}^2) \\
& + \Big[M_{\mathrm{W}}^4 (\xi_Q - 1)^2 - 2(\xi_Q + 3 - 2D) M_{\mathrm{W}}^2 p^2 + p^4 \Big] \\
& \quad \times T_{1123'4'}(M_{\mathrm{W}}^2, M_{\mathrm{W}}^2, \xi_Q M_{\mathrm{W}}^2). \qquad (3.75)
\end{aligned}$$

As above, the gauge-dependent contribution involving the scalar two-loop integrals is proportional to $(p^2 - M_{\mathrm{Z}}^2)$. Note that $G_{\hat{\mathrm{Z}}}(\xi_Q, p^2, M_{\mathrm{W}}^2, D)$ has precisely the same form as the function $G(\xi_W, p^2, M_{\mathrm{W}}^2, D)$ obtained in the conventional formalism (see (3.65)). From (3.73) it furthermore follows that the gauge-parameter-independent expression $F_{\hat{\mathrm{Z}}}(p^2, M_{\mathrm{W}}^2, M_{\mathrm{Z}}^2, D)$ coincides for $p^2 = M_{\mathrm{Z}}^2$ with the corresponding quantity in the conventional formalism. Therefore at the pole all generic two-loop contributions are the same in the conventional formalism and the BFM. This reflects the fact that the mass renormalization must be the same in both formulations.

Chapter 4

Gauge-parameter dependence of Green functions: the pinch technique

In the last chapter the issue of gauge-parameter dependence has been discussed for two-loop self-energies. As mentioned above, many attempts have been made at one-loop order to construct distinguished new "Green functions" with improved theoretical properties by eliminating their gauge-parameter dependence within a certain class of gauges. To this end the conventional definition of a Green function is altered by shifting contributions between different Green functions. For instance, the gauge-parameter dependence of a self-energy is compensated by combining it with suitable contributions extracted from vertex- and box-type diagrams. Whereas in the treatments pursued in Refs. [5, 6] the rearrangement was performed in such a way that certain desirable theoretical properties were explicitly enforced, the pinch technique (PT) [15–18] provides a definite and more general prescription for identifying so-called "pinch parts" which have to be subtracted from the original Green functions and combined with others. The new "Green functions" obtained by using this prescription were found to possess desirable properties and to fulfill simple QED-like Ward identities without a further adjustment.

While the PT can successfully be applied to obtain well-behaved "Green functions" at one-loop order, it is not clear whether this approach can be generalized to higher orders. In this chapter an extension of the PT to a two-loop example is studied. It is shown that a suitable generalization of the one-loop prescription leads to a gauge-parameter-independent result also in this case. On the other hand, this example illustrates the difficulties and limitations inherent in the PT approach.

4.1 The pinch technique at one-loop order

In order to explain the concept of the PT we briefly consider its application at one-loop order in the SM. The original formulation of the PT was given for QCD [15, 16]. The extension to a spontaneously broken gauge theory was considered in Ref. [17], and the application of the PT to the SM was accomplished in Refs. [18, 57].

A general S-matrix element for the elastic scattering of two fermions of masses m_1 and m_2 can be written as

$$S(s,t,m_1,m_2) = S_1(s) + S_2(s,m_1,m_2) + S_3(s,t,m_1,m_2), \tag{4.1}$$

where $s = (p_1+p_2)^2 = p^2$ and $t = (p_1-p_3)^2$ are the Mandelstam variables and $S_1(s)$ denotes self-energy graphs (including tadpoles), $S_2(s,m_1,m_2)$ are the vertex graphs (including wave-function corrections) and $S_3(s,t,m_1,m_2)$ summarizes the box graphs. These contributions taken individually are in general gauge-parameter-dependent, whereas the complete S-matrix element $S(s,t,m_1,m_2)$ is gauge-independent.

The vertex and box graphs contain parts in which the dependence on the external fermions has been canceled and which are therefore propagator-like. By combining contributions of this kind with S_1 one can define a new gauge-parameter-independent "self-energy". In an analogous way one can also construct new gauge-parameter-independent "vertex" and "box" contributions. Since one can always shift gauge-parameter-independent contributions between these parts, such a rearrangement is not unique. From the infinitely many possible choices the PT singles out a specific one by giving a definite prescription for extracting the pinch parts.

The pinch parts are identified as terms emerging when a longitudinal momentum k_α is contracted with a Dirac matrix γ^α of the external fermion current. This gives rise to an elementary Ward identity, i.e. it can be written in terms of inverse fermion propagators and mass terms

$$k_\alpha\gamma^\alpha = (\not{p}_f + \not{k} - m_1) - (\not{p}_f - m_2) + m_1 - m_2, \tag{4.2}$$

where p_f denotes the momentum of the external fermion. The first term on the right-hand side cancels the propagator of the external fermion and gives rise to the pinch part, while on-shell the other terms yield contributions proportional to the fermion masses.

The relevant k_α-terms are taken from the longitudinal part of the gauge-boson propagators in the R_ξ gauge (see (2.34)) and from the three-gauge-boson vertices. Up to group theoretical factors these vertices can be decomposed into a pinch part and a free part

$$\Gamma_{\alpha\mu\lambda}(k,q,-q-k) = \Gamma^{\mathrm{P}}_{\alpha\mu\lambda}(k,q,-q-k) + \Gamma^{\mathrm{F}}_{\alpha\mu\lambda}(k,q,-q-k), \tag{4.3}$$

where

$$\begin{aligned}\Gamma^{\mathrm{P}}_{\alpha\mu\lambda}(k,q,-q-k) &= (q+k)_\lambda g_{\alpha\mu} + k_\alpha g_{\lambda\mu},\\ \Gamma^{\mathrm{F}}_{\alpha\mu\lambda}(k,q,-q-k) &= 2q_\alpha g_{\lambda\mu} - 2q_\lambda g_{\alpha\mu} - (2k+q)_\mu g_{\alpha\lambda}.\end{aligned} \tag{4.4}$$

The momenta $(q+k)_\lambda$ and k_α in $\Gamma^{\mathrm{P}}_{\alpha\mu\lambda}$ have the same form as those arising from the longitudinal part of the relevant gauge-boson propagators. Therefore these contributions are also regarded as pinch parts.

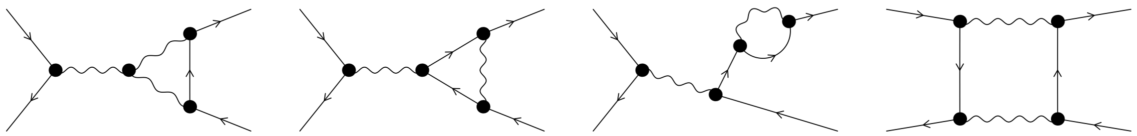


Figure 4.1: The types of vertex and box diagrams from which pinch parts are extracted.

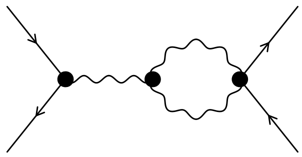


Figure 4.2: The pinch part corresponding to the first diagram in Fig. 4.1.

As an example we treat the construction of the one-loop gauge-boson "self-energies" in the PT. For simplicity, we consider a four-fermion process with massless external fermions. The relevant types of vertex and box diagrams are drawn in Fig. 4.1, where the wavy lines genericly denote gauge-boson propagators. The pinch parts are extracted from these diagrams by canceling the propagators of the external fermions according to the prescription given above. The pinch part corresponding to the first diagram in Fig. 4.1 is illustrated in Fig. 4.2. In order to combine a pinch part with the original self-energy the spinor chains have to be stripped off resulting in a propagator-like diagram which then has to be truncated by multiplying with the inverse propagators of the relevant gauge bosons. Since these propagator-like diagrams (see Fig. 4.2) possess at least one lowest-order propagator less than the diagrams associated with the usual one-loop propagator, the truncated pinch parts are by construction proportional to $(p^2 - M^2)$, where M denotes the mass of the gauge boson. For the mixing energy of photon and Z the truncation yields a factor $(2p^2 - M_Z^2)$.

We calculated the pinch parts in a general R_ξ gauge and checked that after adding these contributions to the original self-energies the gauge parameters actually drop out. The results for the one-loop "self-energies" in the PT read

$$\Sigma_{\mathrm{T}}^{AA,(1),\mathrm{PT}}(p^2) = \Sigma_{\mathrm{T}}^{AA,(1)}(p^2)\Big|_{\xi_i=1} - \frac{e^2}{4\pi^2} p^2 B_0(p^2; M_{\mathrm{W}}^2, M_{\mathrm{W}}^2), \tag{4.5}$$

$$\Sigma_{\mathrm{T}}^{AZ,(1),\mathrm{PT}}(p^2) = \Sigma_{\mathrm{T}}^{AZ,(1)}(p^2)\Big|_{\xi_i=1} + \frac{e^2 c_{\mathrm{W}}}{8\pi^2 s_{\mathrm{W}}}(2p^2 - M_{\mathrm{Z}}^2) B_0(p^2; M_{\mathrm{W}}^2, M_{\mathrm{W}}^2), \tag{4.6}$$

$$\tilde{\Sigma}_{\mathrm{T}}^{ZZ,(1),\mathrm{PT}}(p^2) = \tilde{\Sigma}_{\mathrm{T}}^{ZZ,(1)}(p^2)\Big|_{\xi_i=1} - \frac{e^2 c_{\mathrm{W}}^2}{4\pi^2 s_{\mathrm{W}}^2}(p^2 - M_{\mathrm{Z}}^2) B_0(p^2; M_{\mathrm{W}}^2, M_{\mathrm{W}}^2), \tag{4.7}$$

$$\begin{aligned}\tilde{\Sigma}_{\mathrm{T}}^{WW,(1),\mathrm{PT}}(p^2) &= \tilde{\Sigma}_{\mathrm{T}}^{WW,(1)}(p^2)\Big|_{\xi_i=1} \\ &\quad - \frac{e^2}{4\pi^2}(p^2 - M_{\mathrm{W}}^2)\Big[B_0(p^2; 0, M_{\mathrm{W}}^2) + \frac{c_{\mathrm{W}}^2}{s_{\mathrm{W}}^2} B_0(p^2; M_{\mathrm{Z}}^2, M_{\mathrm{W}}^2)\Big],\end{aligned} \tag{4.8}$$

where as before the tilde indicates that we have included the tadpole contributions in the self-energies of Z and W. The first entry on the right-hand side of each equation denotes the usual self-energy in the 't Hooft–Feynman gauge (see e.g. Ref. [32]). The form of the extra terms is as discussed above. These results coincide with the ones given in Ref. [18].

At the pole, the PT "self-energies" are identical to the usual self-energies. This fact is of course crucial for the notion of a "self-energy", since the usual one-loop self-energies (including tadpoles) derived in field theory are gauge-independent and unique at the propagator pole where they can directly be related to a physical observable, namely the mass of the particle. Note, however, that the same is not obvious for the PT "self-energies" which include contributions from vertex and box diagrams.

In passing we note that we have obtained the results (4.5) – (4.8) also in an alternative way by applying the so-called intrinsic pinch technique which had previously been used only in QCD [16].

Inserting in (4.6) the explicit expression for $\Sigma_{\rm T}^{AZ,(1)}(p^2)\Big|_{\xi_i=1}$ (see e.g. Ref. [32]) it has been found [18] that for zero momentum

$$\Sigma_{\rm T}^{AZ,(1),{\rm PT}}(0) = 0. \tag{4.9}$$

This is in contrast to the usual formalism where the vanishing of the photon–Z mixing at zero momentum is explicitly enforced through a renormalization condition in the on-shell scheme (see e.g. Ref. [32]).

In Ref. [18] it has been noted that the asymptotic behavior of running couplings defined via a Dyson summation of the PT "self-energies" listed above is automatically governed by the renormalization group. In Refs. [17, 18, 58, 59] further desirable properties of the PT "Green functions" have been found by an explicit computation of the one-loop quantities. In particular, it could be verified for some examples that the PT "Green functions" fulfill simple QED-like Ward identities.

4.2 Pinch technique at the two-loop level

In order to discuss an extension of the PT to the two-loop level we treat the example of the "charged current" light-fermion contributions to the two-loop Z self-energy which was considered in Section 3.7.2. The result in the R_ξ gauge of the conventional formalism has been given in (3.66).

For extracting the pinch parts we consider a four-fermion process at two-loop order. The different types of one-particle irreducible (1PI) two-loop vertex (containing also the wave-function corrections) and box diagrams relevant for this application are shown in Fig. 4.3. They can be obtained from the one-loop diagrams depicted in Fig. 4.1 by inserting a fermion loop in all possible ways.

The precise definition of the pinch parts at two-loop order is a priori not clear. In analogy to the one-loop case we identify the pinch parts as terms generated by

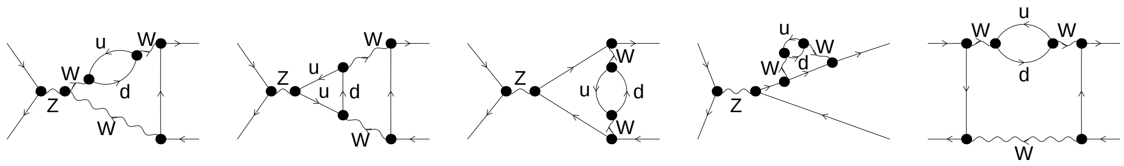


Figure 4.3: Two-loop vertex and box graphs containing propagator-like pinch parts.

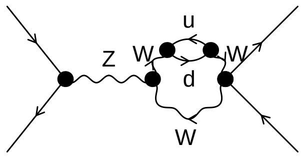


Figure 4.4: The pinch part of the first diagram in Fig. 4.3.

longitudinal momenta k_α stemming from lowest-order propagators and elementary three-gauge-boson vertices. The pinch part corresponding to the first diagram in Fig. 4.3 is drawn in Fig. 4.4.

We denote the sum of the pinch parts extracted from all diagrams of the form shown in Fig. 4.3 as $\Sigma^{ZZ,(2)}_{\mathrm{T,W,pin}}(p^2)$. Combining this quantity with the usual self-energy $\Sigma^{ZZ,(2)}_{\mathrm{T,W}}(p^2)$ given in (3.66) we find a result of the form

$$\Sigma^{ZZ,(2)}_{\mathrm{T,W}}(p^2) + \Sigma^{ZZ,(2)}_{\mathrm{T,W,pin}}(p^2) = e^4 \Big\{ f^{\mathrm{PT}}_{\mathrm{Z}}(p^2, M^2_{\mathrm{W}}, M^2_{\mathrm{Z}}, D) + g^{\mathrm{PT}}_{\mathrm{Z}}(\xi_W, p^2, M^2_{\mathrm{W}}, D) \\ + F^{\mathrm{PT}}_{\mathrm{Z}}(p^2, M^2_{\mathrm{W}}, M^2_{\mathrm{Z}}, D) \Big\}. \tag{4.10}$$

Here $f^{\mathrm{PT}}_{\mathrm{Z}}(p^2, M^2_{\mathrm{W}}, M^2_{\mathrm{Z}}, D)$ and $g^{\mathrm{PT}}_{\mathrm{Z}}(\xi_W, p^2, M^2_{\mathrm{W}}, D)$ denote contributions involving only scalar one-loop integrals, and $F^{\mathrm{PT}}_{\mathrm{Z}}(p^2, M^2_{\mathrm{W}}, M^2_{\mathrm{Z}}, D)$ contains the generic two-loop integrals. While $f^{\mathrm{PT}}_{\mathrm{Z}}$ and $F^{\mathrm{PT}}_{\mathrm{Z}}$ are gauge-parameter-independent, the quantity $g^{\mathrm{PT}}_{\mathrm{Z}}$ explicitly depends on the gauge parameter ξ_W.

This means that application of the PT to the 1PI two-loop vertex and box graphs only removes the gauge-parameter dependence associated with the generic two-loop integrals, while the contributions involving only one-loop integrals remain gauge-parameter-dependent. This feature was in fact anticipated from the structure of the result (3.66). As discussed in Section 3.7.2, its one-loop terms remain gauge-dependent even for $p^2 = M^2_{\mathrm{Z}}$. Since the pinch parts extracted from two-loop triangle and box graphs are by construction proportional to $(p^2 - M^2_{\mathrm{Z}})$, they cannot cancel this residual gauge-parameter dependence.

In the framework of the PT it is therefore not possible to construct a gauge-parameter-independent two-loop "self-energy" on the basis of 1PI contributions. It is easy to infer that also the one-particle reducible contributions in the PT are gauge-parameter-dependent. To this end we consider the complete two-loop contributions associated with the gauge-boson "propagators" in the PT. For a gauge boson of mass M the PT "propagator" is given as

$$\Delta_{\mathrm{T}}^{(2),\mathrm{PT}}(p^2) = -i\left\{\frac{\Sigma_{\mathrm{T}}^{(2)}(p^2) + \Sigma_{\mathrm{T,pin}}^{(2)}(p^2)}{\left[p^2 - M^2\right]^2} - \frac{\left(\Sigma_{\mathrm{T}}^{(1)}(p^2) + \Sigma_{\mathrm{T,pin}}^{(1)}(p^2)\right)^2}{\left[p^2 - M^2\right]^3}\right\}, \quad (4.11)$$

where the first term denotes the 1PI two-loop contributions as given in (4.10), while the second term contains all propagator-like contributions from reducible two-loop diagrams. For the neutral gauge-bosons this also incorporates mixing terms between photon and Z. Since $\Sigma_{\mathrm{T,pin}}^{(1)}(p^2)$ originates only from reducible graphs, it is not the full one-loop pinch part. In particular, it contains no contributions from one-loop box diagrams. However, these are necessary to cancel the gauge-parameter dependence of the one-loop gauge boson self-energy $\Sigma_{\mathrm{T}}^{(1)}(p^2)$. The reducible contribution in (4.11) is therefore gauge-parameter-dependent and does not correspond to the one-loop "self-energy" in the PT, $\Sigma_{\mathrm{T}}^{(1),\mathrm{PT}}(p^2)$, which was given in (4.5) – (4.8).

To identify the quantity which has to be regarded as the two-loop "self-energy" in the PT framework, we write the "propagator" (4.11) as

$$\Delta_{\mathrm{T}}^{(2),\mathrm{PT}}(p^2) = -i\left\{\frac{\Sigma_{\mathrm{T}}^{(2),\mathrm{PT}}(p^2)}{\left[p^2 - M^2\right]^2} - \frac{\left(\Sigma_{\mathrm{T}}^{(1),\mathrm{PT}}(p^2)\right)^2}{\left[p^2 - M^2\right]^3}\right\}. \quad (4.12)$$

This yields

$$\begin{aligned}\Sigma_{\mathrm{T}}^{(2),\mathrm{PT}}(p^2) &= \Sigma_{\mathrm{T}}^{(2)}(p^2) + \Sigma_{\mathrm{T,pin}}^{(2)}(p^2) \\ &\quad - \frac{1}{p^2 - M^2}\left[\left(\Sigma_{\mathrm{T}}^{(1)}(p^2) + \Sigma_{\mathrm{T,pin}}^{(1)}(p^2)\right)^2 - \left(\Sigma_{\mathrm{T}}^{(1),\mathrm{PT}}(p^2)\right)^2\right]. \quad (4.13)\end{aligned}$$

This means that a two-loop "self-energy" constructed via the PT necessarily involves both one-particle irreducible and reducible contributions.

Using the definition (4.13) we have evaluated the "charged current" light-fermion contribution to the Z "self-energy" in the PT and found that it is in fact independent of the gauge parameters. The result reads

$$\begin{aligned}\Sigma_{\mathrm{T,W}}^{ZZ,(2),\mathrm{PT}}(p^2) = \frac{e^4(2-D)}{(4\pi)^4(1-D)^2 12 M_{\mathrm{W}}^2 M_{\mathrm{Z}}^2 s_{\mathrm{W}}^4}\Big\{ & \\ & 2(1/D - 1)(4 - 6D + D^2)\left(9M_{\mathrm{W}}^4 + (M_{\mathrm{W}}^2 - M_{\mathrm{Z}}^2)^2\right) \\ & \quad \times A_0(M_{\mathrm{W}}^2) B_0(p^2; 0, 0) \\ & + (1-D)(4M_{\mathrm{W}}^2 - M_{\mathrm{Z}}^2)(2M_{\mathrm{W}}^2 + M_{\mathrm{Z}}^2)\left[2M_{\mathrm{W}}^2 + (7-D)p^2\right] \\ & \quad \times \left(B_0(p^2; 0, 0)\right)^2 \\ & - 36(1-D) M_{\mathrm{W}}^4 (2M_{\mathrm{W}}^2 + 3p^2) B_0(p^2; 0, 0) B_0(p^2; M_{\mathrm{W}}^2, M_{\mathrm{W}}^2)\end{aligned}$$

$$
\begin{aligned}
&+ \Big[4\Big(\frac{20}{D} - 68 + 98D - 50D^2 + 9D^3\Big) M_{\mathrm{W}}^4 \\
&\qquad - 4(1/D - 1)(4 - 6D + D^2) M_{\mathrm{W}}^2 M_{\mathrm{Z}}^2 \\
&\qquad + \Big(\frac{8}{D} + 7 + 5D - 2D^2 - 9\frac{M_{\mathrm{W}}^2}{p^2}\Big) M_{\mathrm{Z}}^4 \\
&\qquad - 36(2 - D)(3 - D) M_{\mathrm{W}}^2 p^2\Big] T_{13'4'}(M_{\mathrm{W}}^2) \\
&+ \Big[4\Big(\frac{40}{D} - 96 + 61D - 14D^2\Big) M_{\mathrm{W}}^4 \\
&\qquad - 2(1/D - 1)(8 - 4D + D^2) M_{\mathrm{Z}}^2 (2M_{\mathrm{W}}^2 - M_{\mathrm{Z}}^2) \\
&\qquad + 9\frac{M_{\mathrm{W}}^2}{p^2} M_{\mathrm{Z}}^4\Big] T_{23'4'}(M_{\mathrm{W}}^2) \\
&- 2(1/D - 1)(4 - 8D + D^2)(M_{\mathrm{W}}^2 + p^2)\Big(9M_{\mathrm{W}}^4 + (M_{\mathrm{W}}^2 - M_{\mathrm{Z}}^2)^2\Big) \\
&\quad \times T_{1'2'34'}(M_{\mathrm{W}}^2) \\
&+ 18 M_{\mathrm{W}}^2 \Big[8(1 - D) M_{\mathrm{W}}^4 - (3 - 2D) M_{\mathrm{Z}}^4 + 2(5 - 3D) M_{\mathrm{W}}^2 p^2\Big] \\
&\quad \times T_{123'4'}(M_{\mathrm{W}}^2, M_{\mathrm{W}}^2) \\
&+ 9 M_{\mathrm{W}}^2 \Big[4 M_{\mathrm{W}}^2 \Big(4(1 - D) M_{\mathrm{W}}^4 - (2 - D) M_{\mathrm{Z}}^4\Big) \\
&\qquad\quad + \Big(4(11 - 7D) M_{\mathrm{W}}^4 + M_{\mathrm{Z}}^4\Big) p^2 \\
&\qquad\quad - 4(2 - D) M_{\mathrm{W}}^2 p^4\Big] T_{1123'4'}(M_{\mathrm{W}}^2, M_{\mathrm{W}}^2, M_{\mathrm{W}}^2) \\
&+ 72(1 - D) M_{\mathrm{W}}^6 (M_{\mathrm{W}}^2 + 2p^2) T_{123'4'5'}(M_{\mathrm{W}}^2, M_{\mathrm{W}}^2) \\
&+ (1 - D)(4M_{\mathrm{W}}^2 - M_{\mathrm{Z}}^2)(2M_{\mathrm{W}}^2 + M_{\mathrm{Z}}^2) \\
&\quad \times \Big[2(M_{\mathrm{W}}^4 + p^4) + (8 - D) M_{\mathrm{W}}^2 p^2\Big] T_{1'2'34'5'}(M_{\mathrm{W}}^2)\Big\}.
\end{aligned}
\tag{4.14}
$$

The difference of this quantity to the usual two-loop self-energy (3.66) in the 't Hooft–Feynman gauge has explicitly been given in Ref. [50].

Inspection of (4.14) reveals that it precisely coincides with the special case $\xi_Q = 1$ of the result (3.71) evaluated in the BFM. It will be shown in the next chapter that also the results obtained within the PT at one-loop order in QCD and the SM correspond to the special case $\xi_Q = 1$ of the relevant BFM results.

4.3 Discussion

In this chapter an extension of the PT to a two-loop example has been worked out and a gauge-parameter-independent two-loop "self-energy" has been constructed. To achieve this, however, an involved rearrangement was necessary not only between Green functions of self-energy, vertex and box type, but also between 1PI and reducible contributions. This is very unsatisfactory from a field theoretical point of view since in field theory the 1PI contributions are usually treated completely separately from the reducible ones. The full S-matrix elements can be generated solely from 1PI graphs.

Even for the relatively simple example considered here the construction of the PT "self-energy" turned out to be technically rather cumbersome. Moreover, for more complicated two-loop applications the proper identification of suitable pinch parts is not clear.

A problem already present at one-loop order is posed by the use of a specific process for constructing the PT "Green functions". The process independence of these "Green functions" has not been proven but only checked for some examples. In addition, the fact that the one-loop "self-energies" in the PT coincide with the usual self-energies at the pole does not follow from a general argument but merely as a matter of construction. Also the desirable theoretical properties noted for the PT "Green functions", e.g. the validity of simple Ward identities, have not been derived but only established by explicit calculation at one-loop order. The PT therefore does not provide a theoretical understanding of these properties.

The two-loop result obtained within the PT in this chapter has been shown to coincide with a special case of the BFM result. In contrast to the PT, the Green functions in the BFM are derived from Feynman rules in a straightforward manner. They are well-defined objects in field theory and their process independence is guaranteed. In the next chapter the theoretical properties of the BFM Green functions will be discussed on the basis of gauge invariance.

Chapter 5

The background-field method: Green functions from a gauge-invariant effective action

It will be shown in this chapter that the BFM provides a general framework for deriving Green functions with very desirable theoretical properties. The BFM is applicable to all orders of perturbation theory and for all vertex functions. The gauge-invariant effective action of the BFM gives rise to simple QED-like Ward identities which are valid to all orders.

After deriving the Ward identities for the SM, a comparison between the BFM and the PT is performed. The BFM is used to calculate those quantities in QCD and the SM which were obtained in the literature via the PT. The PT results are shown to be given as the special case of the BFM results where the quantum gauge parameter is equal to one. The desirable properties of the BFM vertex functions hold for arbitrary values of the quantum gauge parameter. They can be shown to be a direct consequence of gauge invariance.

Besides the study of off-shell Green functions, the BFM is also very useful for a technically and conceptually simplified evaluation of S-matrix elements. In order to carry this out in the SM one has to establish a consistent renormalization which does not violate the explicit gauge invariance, i.e. which does not alter the form of the Ward identities. This is done by working out an explicit on-shell renormalization of the SM at one-loop order. The gauge invariance of the BFM is shown to considerably simplify the renormalization.

5.1 Ward identities for the Standard Model

In Section 2.3 the gauge-invariant effective action $\Gamma_{\mathrm{BFM}}[0,\hat{\varphi}]$ of the BFM has been derived. Since in the remainder of this work we will only be concerned with this effective action, we from now on drop its subscript "BFM". Furthermore, we make the dependence on the different types of background fields explicit and write the gauge-invariant effective action of the BFM as $\Gamma[\hat{V},\hat{S},F,\bar{F}]$, where $\hat{V}$ collectively denotes the background gauge fields, $\hat{S}$ the background Higgs fields and F the fermion fields.

The invariance of $\Gamma[\hat{V},\hat{S},F,\bar{F}]$ under the background gauge transformations specified in (2.20) and (2.21),

$$\begin{aligned}\delta\Gamma = {}& \sum_i \frac{\delta\Gamma[\hat{V},\hat{S},F,\bar{F}]}{\delta\hat{V}_i}\delta\hat{V}_i + \sum_j \frac{\delta\Gamma[\hat{V},\hat{S},F,\bar{F}]}{\delta\hat{S}_j}\delta\hat{S}_j \\ &+ \sum_k \left(\delta\bar{F}_k \frac{\delta\Gamma[\hat{V},\hat{S},F,\bar{F}]}{\delta\bar{F}_k} - \frac{\delta\Gamma[\hat{V},\hat{S},F,\bar{F}]}{\delta F_k}\delta F_k\right) = 0, \end{aligned} \tag{5.1}$$

where i,j,k run over all background gauge fields, background Higgs fields and fermion fields, respectively, gives rise to simple Ward identities. Since the gauge invariance has been retained in the background-field formulation, these are precisely the Ward identities related to the classical Lagrangian. This is in contrast to the conventional formalism where, owing to the gauge-fixing procedure, the explicit gauge invariance is lost and one is left with the more complicated Slavnov–Taylor identities [34] derived from the invariance under BRS transformations [33].

The BFM Ward identities follow from differentiating (5.1) with respect to the fields and are valid in all orders of perturbation theory. Note that the identities hold for arbitrary values of the quantum gauge parameter.

We first consider the self-energies which are obtained from the two-point vertex functions as follows

$$\begin{aligned}
\Gamma^{\hat{V}\hat{V}'}_{\mu\nu}(k,-k) &= i(-g_{\mu\nu}k^2 + k_\mu k_\nu + g_{\mu\nu}M^2_{\hat{V}})\delta_{\hat{V}\hat{V}'} \\
&\quad + i\left(-g_{\mu\nu} + \frac{k_\mu k_\nu}{k^2}\right)\Sigma^{\hat{V}\hat{V}'}_{\mathrm{T}}(k^2) - i\frac{k_\mu k_\nu}{k^2}\Sigma^{\hat{V}\hat{V}'}_{\mathrm{L}}(k^2), \\
\Gamma^{\hat{W}^\pm\hat{\phi}^\mp}_{\mu}(k,-k) &= ik_\mu\left[\pm M_{\mathrm{W}} + \Sigma^{\hat{W}^\pm\hat{\phi}^\mp}(k^2)\right], \\
\Gamma^{\hat{Z}\hat{\chi}}_{\mu}(k,-k) &= ik_\mu\left[iM_{\mathrm{Z}} + \Sigma^{\hat{Z}\hat{\chi}}(k^2)\right], \\
\Gamma^{\hat{A}\hat{\chi}}_{\mu}(k,-k) &= ik_\mu\Sigma^{\hat{A}\hat{\chi}}(k^2), \\
\Gamma^{\hat{H}\hat{H}}(k,-k) &= i(k^2 - M^2_{\mathrm{H}}) + i\Sigma^{\hat{H}\hat{H}}(k^2), \\
\Gamma^{\hat{\chi}\hat{\chi}}(k,-k) &= ik^2 + i\Sigma^{\hat{\chi}\hat{\chi}}(k^2), \\
\Gamma^{\hat{\phi}^+\hat{\phi}^-}(k,-k) &= ik^2 + i\Sigma^{\hat{\phi}^+\hat{\phi}^-}(k^2), \\
\Gamma^{\hat{H}} &= iT^{\hat{H}}, \\
\Gamma^{\bar{f}f}(p,-p) &= -i(\not{p} + m_f) - i\not{p}\omega_-\Sigma^{\bar{f}f}_{\mathrm{L}}(p^2) - i\not{p}\omega_+\Sigma^{\bar{f}f}_{\mathrm{R}}(p^2) + im_f\Sigma^{\bar{f}f}_{\mathrm{S}}(p^2),
\end{aligned} \tag{5.2}$$

where $\hat{V},\hat{V}'$ indicate vector fields, and the definition (3.1) and the decompositions (3.11), (3.13) and (3.15) have been used. In the following we omit the second argument of the two-point vertex functions which is fixed by momentum conservation. Note that no gauge-fixing terms for the background fields are included in the vertex functions, i.e. the lowest-order contributions to the vertex functions follow directly from $\mathcal{L}_{\mathrm{C}}$. The self-energies contain no tadpole contributions; these appear explicitly as $T^{\hat{H}}$. We obtain the following Ward identities for the self-energies

$$\Sigma_{\mathrm{L}}^{\hat{A}\hat{A}}(k^2) = 0, \tag{5.3}$$
$$\Sigma_{\mathrm{L}}^{\hat{A}\hat{Z}}(k^2) = 0, \tag{5.4}$$
$$\Sigma^{\hat{A}\hat{\chi}}(k^2) = 0, \tag{5.5}$$
$$\Sigma_{\mathrm{L}}^{\hat{Z}\hat{Z}}(k^2) - iM_{\mathrm{Z}}\Sigma^{\hat{Z}\hat{\chi}}(k^2) = 0, \tag{5.6}$$
$$k^2\Sigma^{\hat{Z}\hat{\chi}}(k^2) - iM_{\mathrm{Z}}\Sigma^{\hat{\chi}\hat{\chi}}(k^2) + i\frac{e}{2c_{\mathrm{W}}s_{\mathrm{W}}}T^{\hat{H}} = 0, \tag{5.7}$$
$$\Sigma_{\mathrm{L}}^{\hat{W}^\pm\hat{W}^\mp}(k^2) \mp M_{\mathrm{W}}\Sigma^{\hat{W}^\mp\hat{\phi}^\pm}(k^2) = 0, \tag{5.8}$$
$$k^2\Sigma^{\hat{W}^\pm\hat{\phi}^\mp} \mp M_{\mathrm{W}}\Sigma^{\hat{\phi}^\pm\hat{\phi}^\mp} \pm \frac{e}{2s_{\mathrm{W}}}T^{\hat{H}} = 0. \tag{5.9}$$

From the Ward identities (5.3) and (5.4) and the analyticity of $\Gamma_{\mu\nu}^{\hat{A}\hat{A}}(k)$ and $\Gamma_{\mu\nu}^{\hat{A}\hat{Z}}(k)$ at $k^2 = 0$ one can infer

$$\Sigma_{\mathrm{T}}^{\hat{A}\hat{A}}(0) = 0, \tag{5.10}$$

and

$$\Sigma_{\mathrm{T}}^{\hat{A}\hat{Z}}(0) = 0. \tag{5.11}$$

Whereas the QED relation (5.3) is valid in the BFM to all orders, it only holds at one-loop order in the conventional formalism (see (3.58)). The identities (5.4), (5.5) and (5.11) have no conventional counterpart. As mentioned above, the vanishing of the photon–Z-boson mixing at zero momentum can in the conventional formalism only be accomplished by using an explicit renormalization condition. In the BFM it is automatically fulfilled already for the unrenormalized self-energy. Equation (5.5) shows that in contrast to the R_ξ gauges of the conventional formalism the photon does not mix with the unphysical scalar χ in the BFM.

For the Z-boson self-energy one has in the usual formalism at one-loop order

$$k^2\left[\Sigma_{\mathrm{L}}^{ZZ,(1)}(k^2) - 2iM_{\mathrm{Z}}\Sigma^{Z\chi,(1)}(k^2)\right] - M_{\mathrm{Z}}^2\Sigma^{\chi\chi,(1)}(k^2) + \frac{eM_{\mathrm{Z}}}{2c_{\mathrm{W}}s_{\mathrm{W}}}T^{H,(1)} = 0, \tag{5.12}$$

and the two-loop relation has been given in (3.58). In the BFM, this relation decouples into two simpler Ward identities, (5.6) and (5.7), which are valid to all orders.

The three-point function $\Gamma_\mu^{\hat{A}\bar{f}f}$ obeys

$$k^\mu\Gamma_\mu^{\hat{A}\bar{f}f}(k,\bar{p},p) = -eQ_f\left[\Gamma^{\bar{f}f}(\bar{p}) - \Gamma^{\bar{f}f}(-p)\right], \tag{5.13}$$

i.e. just the QED Ward identity. Note that despite the $\mathrm{U}(1)_{\mathrm{em}}$ gauge invariance of the classical Lagrangian the conventional formalism of the SM does not yield the QED-type Ward identity. The BFM Ward identity for the $\hat{Z}\bar{f}f$ vertex reads

$$k^\mu\Gamma_\mu^{\hat{Z}\bar{f}f}(k,\bar{p},p) - iM_{\mathrm{Z}}\Gamma^{\hat{\chi}\bar{f}f}(k,\bar{p},p) = e\left[\Gamma^{\bar{f}f}(\bar{p})(v_f - a_f\gamma_5) - (v_f + a_f\gamma_5)\Gamma^{\bar{f}f}(-p)\right], \tag{5.14}$$

where $v_f = (I^3_{\mathrm{W},f} - 2s^2_{\mathrm{W}}Q_f)/(2c_{\mathrm{W}}s_{\mathrm{W}})$ and $a_f = I^3_{\mathrm{W},f}/(2c_{\mathrm{W}}s_{\mathrm{W}})$. For the $\hat{W}\bar{f}f$ vertex we obtain

$$
\begin{aligned}
k^\mu\Gamma^{\hat{W}^+\bar{f}_u f_d}_\mu(k,\bar{p},p) - M_{\mathrm{W}}\Gamma^{\hat{\phi}^+\bar{f}_u f_d}(k,\bar{p},p) &= \frac{e}{\sqrt{2}s_{\mathrm{W}}}\left[\Gamma^{\bar{f}_u f_u}(\bar{p})\omega_- - \omega_+\Gamma^{\bar{f}_d f_d}(-p)\right],\\
k^\mu\Gamma^{\hat{W}^-\bar{f}_d f_u}_\mu(k,\bar{p},p) + M_{\mathrm{W}}\Gamma^{\hat{\phi}^-\bar{f}_d f_u}(k,\bar{p},p) &= \frac{e}{\sqrt{2}s_{\mathrm{W}}}\left[\Gamma^{\bar{f}_d f_d}(\bar{p})\omega_- - \omega_+\Gamma^{\bar{f}_u f_u}(-p)\right]. \qquad (5.15)
\end{aligned}
$$

Also the triple gauge-boson vertex $\hat{A}\hat{W}^+\hat{W}^-$ fulfills a QED-like Ward identity,

$$
k^\mu\Gamma^{\hat{A}\hat{W}^+\hat{W}^-}_{\mu\rho\sigma}(k,k_+,k_-) = e\left[\Gamma^{\hat{W}^+\hat{W}^-}_{\rho\sigma}(k_+) - \Gamma^{\hat{W}^+\hat{W}^-}_{\rho\sigma}(-k_-)\right]. \qquad (5.16)
$$

Furthermore it obeys two identities relating it to the $\hat{A}\hat{\phi}\hat{W}$ vertex:

$$
\begin{aligned}
k_+^\rho\Gamma^{\hat{A}\hat{W}^+\hat{W}^-}_{\mu\rho\sigma}(k,k_+,k_-) - M_{\mathrm{W}}\Gamma^{\hat{A}\hat{\phi}^+\hat{W}^-}_{\mu\sigma}(k,k_+,k_-) &= \\
e\left[\Gamma^{\hat{W}^+\hat{W}^-}_{\mu\sigma}(-k_-) - \Gamma^{\hat{A}\hat{A}}_{\mu\sigma}(k) + \frac{c_{\mathrm{W}}}{s_{\mathrm{W}}}\Gamma^{\hat{A}\hat{Z}}_{\mu\sigma}(k)\right], &\qquad (5.17)\\
k_-^\sigma\Gamma^{\hat{A}\hat{W}^+\hat{W}^-}_{\mu\rho\sigma}(k,k_+,k_-) + M_{\mathrm{W}}\Gamma^{\hat{A}\hat{W}^+\hat{\phi}^-}_{\mu\rho}(k,k_+,k_-) &= \\
-e\left[\Gamma^{\hat{W}^-\hat{W}^+}_{\mu\rho}(-k_+) - \Gamma^{\hat{A}\hat{A}}_{\mu\rho}(k) + \frac{c_{\mathrm{W}}}{s_{\mathrm{W}}}\Gamma^{\hat{A}\hat{Z}}_{\mu\rho}(k)\right]. &\qquad (5.18)
\end{aligned}
$$

Analogous Ward identities hold for the $\hat{Z}\hat{W}^+\hat{W}^-$ vertex, where in the relation corresponding to (5.16) the $\hat{\chi}\hat{W}^+\hat{W}^-$ vertex enters as an additional term. An example involving vertex functions with Higgs bosons is:

$$
k_Z^\mu\Gamma^{\hat{Z}\hat{\chi}\hat{H}}_\mu(k_Z,k_\chi,k_H) - iM_{\mathrm{Z}}\Gamma^{\hat{\chi}\hat{\chi}\hat{H}}(k_Z,k_\chi,k_H) = -i\frac{e}{2s_{\mathrm{W}}c_{\mathrm{W}}}\left[\Gamma^{\hat{H}\hat{H}}(k_H) - \Gamma^{\hat{\chi}\hat{\chi}}(k_\chi)\right]. \qquad (5.19)
$$

The validity of all these Ward identities has been checked at tree level and one-loop order by explicit computation for arbitrary ξ_Q [60, 61]. As a two-loop example, we verified that the light-fermion contributions (see Section 3.7.3) fulfill the Ward identity (5.6) for arbitrary values of ξ_Q.

5.2 Comparison between background-field method and pinch technique

5.2.1 Quantum chromodynamics

The relation between the BFM and the PT can be most easily seen in QCD. The Feynman rules for QCD within the BFM have been given in Ref. [21]. They are particularly simple in the 't Hooft–Feynman gauge, i.e. $\xi_Q = 1$. For example, the coupling of a background gluon with momentum q^μ to two quantum gluons reads

$$
\begin{aligned}
\left.\Gamma^{abc,\mathrm{BFM}}_{\alpha\mu\lambda}(k,q,-q-k)\right|_{\xi_Q=1} &= gf^{abc}\left[2q_\alpha g_{\lambda\mu} - 2q_\lambda g_{\alpha\mu} - (2k+q)_\mu g_{\alpha\lambda}\right]\\
&= gf^{abc}\Gamma^{\mathrm{F}}_{\alpha\mu\lambda}(k,q,-q-k), \qquad (5.20)
\end{aligned}
$$

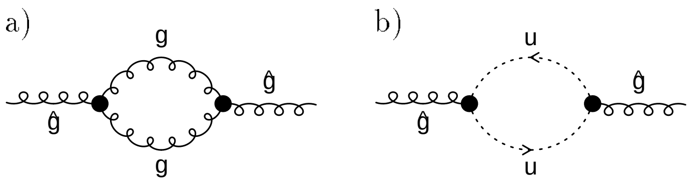


Figure 5.1: Feynman diagrams contributing to the one-loop gluon self-energy.

where f^{abc} are the structure constants of the gauge group SU(N), and $\Gamma^{\mathrm{F}}_{\alpha\mu\lambda}(k,q,-q-k)$ has been defined in (4.4). Straightforward application of the BFM Feynman rules for $\xi_Q = 1$ yields for the one-loop gluon self-energy within dimensional regularization (we omit the fermion contribution)

$$\begin{aligned}\Sigma^{ab,(1)}_{\mu\nu}(q) = -\delta^{ab}\frac{ig^2N}{(2\pi)^D}\mu^{4-D}\int\frac{d^Dk}{k^2(q+k)^2}\Big[\frac{1}{2}\Gamma^{\mathrm{F}}_{\alpha\mu\lambda}(k,q,-q-k)\Gamma^{\mathrm{F}}_{\alpha\nu\lambda}(k,q,-q-k)\\ -(2k+q)_\mu(2k+q)_\nu\Big],\end{aligned} \tag{5.21}$$

where the first term originates from the gluon loop (Fig. 5.1a) and the second term from the ghost loop (Fig. 5.1b). The graphs involving quartic couplings vanish within dimensional regularization. The result (5.21) precisely coincides with the one obtained for the gluon "self-energy" within the PT as given in formula (3.9) of Ref. [16]. Calculating $\Sigma^{ab,(1)}_{\mu\nu}(q)$ in the BFM for arbitrary ξ_Q yields

$$\Sigma^{ab,(1)}_{\mu\nu}(q) = \delta^{ab}g^2(-g_{\mu\nu}q^2+q_\mu q_\nu)\left[-c\frac{2}{\epsilon}+c\log\frac{|q^2|}{\mu^2}+\text{UV-finite constant}\right], \tag{5.22}$$

where

$$c = \frac{11N}{48\pi^2}. \tag{5.23}$$

The coefficient c of the renormalization-scale-dependent logarithm does not depend on ξ_Q and coincides with the coefficient of $-g^3$ in the β function $\beta(g)$ of the minimal-subtraction (MS) scheme [21]. As will be discussed in more detail for the SM in Section 5.4, this fact is a direct consequence of gauge invariance which in the BFM relates the β function to the anomalous dimension and thus to the field renormalization constant of the gauge boson.

For the one-loop contribution to the three-gluon vertex one obtains in the 't Hooft–Feynman gauge from the diagrams in Fig. 5.2 ($k_1 = k+q_2$, $k_2 = k-q_1$, $k_3 = k$)

$$\begin{aligned}\Gamma^{abc,(1)}_{\mu\nu\rho}(q_1,q_2,q_3) = -igf^{abc}\frac{g^2N}{2(2\pi)^D}\mu^{4-D}\int d^Dk\\ \times\Bigg\{\frac{1}{k_1^2k_2^2k_3^2}\Big[\Gamma^{\mathrm{F}}_{\alpha\mu\beta}(k_2,q_1,-k_3)\Gamma^{\mathrm{F}}_{\beta\nu\lambda}(k_3,q_2,-k_1)\Gamma^{\mathrm{F}}_{\lambda\rho\alpha}(k_1,q_3,-k_2)\\ +2(k_2+k_3)_\mu(k_1+k_3)_\nu(k_1+k_2)_\rho\Big]\end{aligned}$$

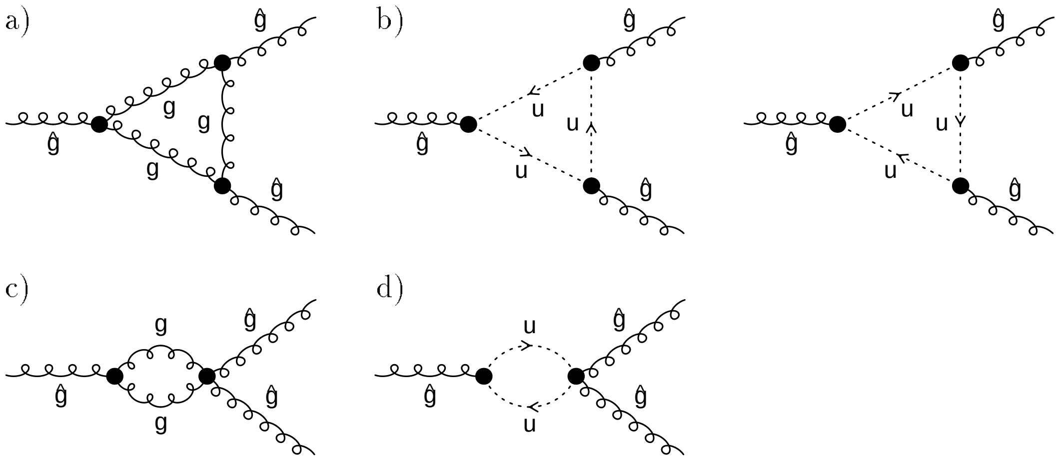


Figure 5.2: Feynman diagrams contributing to the three-gluon vertex at the one-loop level. Each graph in the second line represents three different permutations.

$$
\begin{aligned}
& - \frac{1}{k_2^2 k_3^2} 8(q_{1\rho} g_{\mu\nu} - q_{1\nu} g_{\mu\rho}) - \frac{1}{k_1^2 k_3^2} 8(q_{2\mu} g_{\rho\nu} - q_{2\rho} g_{\mu\nu}) \\
& - \frac{1}{k_1^2 k_2^2} 8(q_{3\nu} g_{\mu\rho} - q_{3\mu} g_{\nu\rho}) \Big\}. \qquad (5.24)
\end{aligned}
$$

Here again the first term results from the gauge-boson loop (Fig. 5.2a) and the second term from the two oppositely directed ghost loops (Fig. 5.2b). The two-point contributions originate from the diagrams represented in Fig. 5.2c; the diagrams of Fig. 5.2d vanish. We have applied the PT and checked that it yields exactly the same result.*

In the PT, the expressions (5.21) and (5.24) were obtained after an involved rearrangement of terms between different Green functions of the conventional formalism. In the BFM, however, (5.21) and (5.24) just correspond to the ordinary Feynman rules for $\xi_Q = 1$. They have the usual field-theoretical interpretation as a two-point and three-point vertex function, respectively, and are manifestly process-independent. Furthermore, the Ward identities which have been noted for the three-gluon "vertex function" [16] and four-gluon "vertex function" [58] within the PT by explicit computation at one-loop order are just the BFM Ward identities for QCD. They can be derived in the BFM from gauge invariance in the same way as shown for the SM in Section 5.1 and are valid to all orders and for arbitrary values of ξ_Q.

Shortly after publication of the results presented in this section [60], the agreement between the BFM results for $\xi_Q = 1$ in QCD and the corresponding PT results has also been noted in Ref. [62]. An application of the PT to the quark self-energy has recently been considered in Ref. [63]. The result coincides with the BFM result for $\xi_Q = 1$, which in this case is the same as the conventional Feynman-gauge result.

*There is a sign error in equation (3.18) of Ref. [16].

5.2.2 The electroweak Standard Model

Using the BFM Feynman rules listed in the appendix, we have evaluated in the BFM for arbitrary ξ_Q the quantities that have been treated in the literature via the PT, i.e. the one-loop gauge-boson self-energies, the one-loop fermion–gauge-boson vertices and the one-loop triple-gauge-boson vertices. We find that all of them coincide for $\xi_Q = 1$ with those constructed within the PT [18, 59, 64].

As an example, we give here the result for the one-loop Z self-energy in the BFM

$$
\begin{aligned}
\hat{\Sigma}_{\mathrm{T}}^{\hat{Z}\hat{Z},(1)}(p^2) = {} & \hat{\Sigma}_{\mathrm{T}}^{ZZ,(1)}(p^2)\Big|_{\xi_i=1} \\
& + \frac{e^2}{(4\pi)^2(1-D)4M_{\mathrm{W}}^2 M_{\mathrm{Z}}^2 s_{\mathrm{W}}^2}(p^2 - M_{\mathrm{Z}}^2)\Big\{ \\
& \quad \frac{2M_{\mathrm{W}}^2}{p^2}\Big(M_{\mathrm{Z}}^2 + (9-4D)p^2 - (M_{\mathrm{Z}}^2 + p^2)\xi_Q\Big)\Big[A_0(\xi_Q M_{\mathrm{W}}^2) - A_0(M_{\mathrm{W}}^2)\Big] \\
& \quad - \Big[4M_{\mathrm{W}}^4(8-4D) - 4(3-2D)M_{\mathrm{W}}^2 M_{\mathrm{Z}}^2 - 8(2-D)M_{\mathrm{W}}^2 p^2 \\
& \qquad + M_{\mathrm{Z}}^2 p^2 + p^4\Big] B_0(p^2; M_{\mathrm{W}}^2, M_{\mathrm{W}}^2) \\
& \quad + 2\frac{p^2 + M_{\mathrm{Z}}^2}{p^2}\Big[(1-\xi_Q)^2 M_{\mathrm{W}}^4 - 2(3-2D+\xi_Q)M_{\mathrm{W}}^2 p^2 + p^4\Big] \\
& \qquad \times B_0(p^2; M_{\mathrm{W}}^2, \xi_Q M_{\mathrm{W}}^2) \\
& \quad - (4M_{\mathrm{W}}^2 + M_{\mathrm{Z}}^2 + p^2)(p^2 - 4\xi_Q M_{\mathrm{W}}^2) B_0(p^2; \xi_Q M_{\mathrm{W}}^2, \xi_Q M_{\mathrm{W}}^2)\Big\}, \qquad (5.25)
\end{aligned}
$$

where the first term on the right-hand side is the Z self-energy evaluated in the 't Hooft–Feynman gauge of the conventional formalism. The tadpole contributions are included in the self-energies. Putting $\xi_Q = 1$ in (5.25) we obtain

$$
\hat{\Sigma}_{\mathrm{T}}^{\hat{Z}\hat{Z},(1)}(p^2)\Big|_{\xi_Q=1} = \hat{\Sigma}_{\mathrm{T}}^{ZZ,(1)}(p^2)\Big|_{\xi_i=1} - \frac{e^2 c_{\mathrm{W}}^2}{4\pi^2 s_{\mathrm{W}}^2}(p^2 - M_{\mathrm{Z}}^2) B_0(p^2; M_{\mathrm{W}}^2, M_{\mathrm{W}}^2), \qquad (5.26)
$$

which is identical to the PT result given in (4.7). It can been seen from (5.25) that at the pole the BFM self-energy coincides for arbitrary values of ξ_Q with the self-energy of the conventional formalism. In contrast to the PT, this fact follows in the BFM directly from the gauge independence of the propagator pole.

The desirable properties noted for the PT "vertex functions" by explicit calculation have a natural explanation in the BFM and hold for arbitrary values of ξ_Q. In particular, the Ward identities obtained within the PT for the W-boson "self-energy" [57], the $At\bar{t}$ "vertex function" [64], the $Wf\bar{f}$ "vertex function" [57] and the AW^+W^- "vertex function" [59] are in fact just the BFM Ward identities (5.8), (5.9), (5.13), (5.15) and (5.16), respectively†. Whereas the origin of these identities remains unclear in the PT, they have been derived from gauge invariance in the BFM in Section 5.1.

†The relations given for the W "self-energy" in Ref. [57] agree with the BFM Ward identities (5.8) and (5.9) up to tadpole contributions which are not treated explicitly in Ref. [57].

As discussed above, the BFM Ward identities imply the vanishing of the photon–Z-boson mixing at zero momentum. It will be shown in Section 5.4 that the Ward identities also give rise to a decent asymptotic behavior of the BFM self-energies. In contrast to the PT, the IR finiteness of the self-energies is obvious within the BFM. It was noted in Ref. [18] that in the PT the fermion–gauge-boson "vertex functions" including fermion wave-function renormalization are UV-finite. In the BFM, this feature will be explained in Section 5.4.

In the PT formalism the "vertex functions" are by construction independent of the gauge parameters within a certain class of gauges. The gauge parameters still appear in the tree propagators joining the vertex functions. In the BFM, these gauge parameters correspond to the background gauge parameters ξ_B. In Refs. [17, 57] the PT has been applied to four-fermion processes in a gauge theory with spontaneous symmetry breaking. It turned out that the "self-energy" and "vertex" contributions to these processes could be formulated in different ways. Within the BFM these different possibilities correspond just to different background gauges. The representations (4.4) and (4.11) of Ref. [17] correspond to the unitary gauge ($\xi_B \to \infty$), whereas the representations (4.16) and (4.21) correspond to the Landau gauge ($\xi_B = 0$). Thus, these results follow naturally within the BFM.

The properties of the $A\bar{t}t$ and $Z\bar{t}t$ vertices have been discussed within the PT in Ref. [64]. From our results for arbitrary ξ_Q we have explicitly confirmed the validity of the Ward identities (5.13) and (5.14) for the top-quark and recovered the PT results for $\xi_Q = 1$. The BFM result for the magnetic-dipole-moment form factor (MDM) of the top quark coincides with the one obtained within the conventional R_ξ-gauge formalism if in the latter result ξ_i is substituted by ξ_Q ($i = A, Z, W$). In contrast to the statements made in the preprint version of Ref. [64] but in agreement with Ref. [30], we find that the MDM vanishes in the limit $|k^2| \to \infty$ for all ξ_Q. This has been checked both numerically and analytically. Moreover, it can be inferred from a simple power-counting argument for all renormalizable gauges.

In Ref. [59] the AW^+W^- and ZW^+W^- vertices were constructed within the PT for W bosons coupled to conserved currents and the Ward identity (5.16) was verified for this case. These results cannot be used as suitable building blocks in processes where this restriction does not apply as e.g. in $\gamma\gamma \to W^+W^-$. Projecting our general BFM results onto the case of conserved currents, we find agreement with formula (3.16) of Ref. [59].[‡] For our general off-shell result we have explicitly checked the Ward identity (5.16) and the other two Ward identities (5.17) and (5.18) which have not been mentioned in the PT context. In the BFM the anomalous-magnetic-moment form factor and the electric-quadrupole-moment form factor as defined in Ref. [65] are IR-finite and vanish in the limit $|k^2| \to \infty$. These features can be derived from the BFM Ward identities and power-counting arguments.

The analysis performed in this section has shown that the desirable theoretical properties noted within the PT hold for arbitrary ξ_Q in the BFM. As a consequence,

[‡]This equation contains an incorrect relative sign. This has been confirmed by checking the Ward identities and redoing the PT calculation.

these features are not related to the gauge-parameter independence of the PT "vertex functions" as could be supposed from the PT point of view. From our results it follows in particular that the choice $\xi_Q = 1$ corresponding to the PT is not distinguished on physical grounds but is only one of arbitrarily many equivalent possibilities. Of course, the background-field 't Hooft–Feynman gauge technically facilitates the calculations. The ambiguity of the vertex functions — quantified in the BFM by ξ_Q — is also inherent in the PT and all similar constructions, since the definite prescription to eliminate the gauge-parameter dependence is just a matter of convention. On the other hand, we showed that the desirable properties of the background-field vertex functions are a direct consequence of the BFM Ward identities which reflect the underlying gauge invariance.

5.3 Renormalization of the electroweak Standard Model

In this section we consider the renormalization of the SM in the framework of the BFM. It will be shown that the gauge invariance present in the BFM has important consequences for the structure of the renormalization constants necessary to render Green functions and S-matrix elements finite. The arguments which we give in the following are made explicit for the one-loop level. It is easy, however, to extend them by induction to arbitrary orders in perturbation theory.

Following the QCD treatment of Ref. [21], we introduce field renormalization only for the background fields. We start with the following set of renormalization constants for the parameters

$$
\begin{aligned}
e_0 &= Z_e e = (1 + \delta Z_e) e, \\
M_{\mathrm{W},0}^2 &= M_{\mathrm{W}}^2 + \delta M_{\mathrm{W}}^2, \\
M_{\mathrm{Z},0}^2 &= M_{\mathrm{Z}}^2 + \delta M_{\mathrm{Z}}^2, \\
M_{\mathrm{H},0}^2 &= M_{\mathrm{H}}^2 + \delta M_{\mathrm{H}}^2, \\
m_{f,0} &= m_f + \delta m_f, \\
t_0 &= t + \delta t,
\end{aligned}
\tag{5.27}
$$

and fields

$$
\begin{aligned}
\hat{W}_0^{\pm} &= Z_{\hat{W}}^{1/2} \hat{W}^{\pm} = (1 + \frac{1}{2}\delta Z_{\hat{W}}) \hat{W}^{\pm}, \\
\begin{pmatrix} \hat{Z}_0 \\ \hat{A}_0 \end{pmatrix} &= \begin{pmatrix} Z_{\hat{Z}\hat{Z}}^{1/2} & Z_{\hat{Z}\hat{A}}^{1/2} \\ Z_{\hat{A}\hat{Z}}^{1/2} & Z_{\hat{A}\hat{A}}^{1/2} \end{pmatrix} \begin{pmatrix} \hat{Z} \\ \hat{A} \end{pmatrix} = \begin{pmatrix} 1 + \frac{1}{2}\delta Z_{\hat{Z}\hat{Z}} & \frac{1}{2}\delta Z_{\hat{Z}\hat{A}} \\ \frac{1}{2}\delta Z_{\hat{A}\hat{Z}} & 1 + \frac{1}{2}\delta Z_{\hat{A}\hat{A}} \end{pmatrix} \begin{pmatrix} \hat{Z} \\ \hat{A} \end{pmatrix}, \\
\hat{H}_0 &= Z_{\hat{H}}^{1/2} \hat{H} = (1 + \frac{1}{2}\delta Z_{\hat{H}}) \hat{H}, \\
\hat{\chi}_0 &= Z_{\hat{\chi}}^{1/2} \hat{\chi} = (1 + \frac{1}{2}\delta Z_{\hat{\chi}}) \hat{\chi},
\end{aligned}
$$

$$
\begin{aligned}
\hat{\phi}_0^{\pm} &= Z_{\hat{\phi}}^{1/2}\hat{\phi}^{\pm} = (1+\frac{1}{2}\delta Z_{\hat{\phi}})\hat{\phi}^{\pm}, \\
f_0^{\mathrm{L}} &= \left(Z_f^{\mathrm{L}}\right)^{1/2} f^{\mathrm{L}} = (1+\frac{1}{2}\delta Z_f^{\mathrm{L}})f^{\mathrm{L}}, \\
f_0^{\mathrm{R}} &= \left(Z_f^{\mathrm{R}}\right)^{1/2} f^{\mathrm{R}} = (1+\frac{1}{2}\delta Z_f^{\mathrm{R}})f^{\mathrm{R}}.
\end{aligned} \tag{5.28}
$$

The tadpole counterterm δt corrects the shift in the minimum of the Higgs potential caused by radiative corrections.

In order to preserve the background-field gauge invariance when renormalizing the theory it is necessary to require that the renormalized vertex functions fulfill Ward identities of the same form as the unrenormalized ones. As a consequence, also the counterterms have to fulfill these Ward identities. This yields relations between the counterterms.

For example, from (5.4) one obtains immediately

$$
0 = \Sigma_{\mathrm{L}}^{\hat{A}\hat{Z},\mathrm{ren}}(k^2) = \Sigma_{\mathrm{L}}^{\hat{A}\hat{Z}}(k^2) - M_{\mathrm{Z}}^2\frac{1}{2}\delta Z_{\hat{Z}\hat{A}} = -M_{\mathrm{Z}}^2\frac{1}{2}\delta Z_{\hat{Z}\hat{A}}, \tag{5.29}
$$

i.e.

$$
\delta Z_{\hat{Z}\hat{A}} = 0. \tag{5.30}
$$

Expressing bare quantities in the QED Ward identity (5.13) through renormalized ones and counterterms yields

$$
k^{\mu}\Gamma_{\mu}^{\hat{A}\bar{f}f}(k,\bar{p},p) = k^{\mu}\Gamma_{\mu}^{\hat{A}\bar{f}f,\mathrm{ren}}(k,\bar{p},p) + ieQ_f \not{k}\left(\delta Z_e + \frac{1}{2}\delta Z_{\hat{A}\hat{A}} + \delta Z_e^{\mathrm{R}}\omega_+ + \delta Z_e^{\mathrm{L}}\omega_-\right),
$$

where (5.30) was used, and

$$
-eQ_f\left[\Gamma^{\bar{f}f}(\bar{p}) - \Gamma^{\bar{f}f}(-p)\right] = -eQ_f\left[\Gamma^{\bar{f}f,\mathrm{ren}}(\bar{p}) - \Gamma^{\bar{f}f,\mathrm{ren}}(-p)\right] + ieQ_f\not{k}\left(\delta Z_e^{\mathrm{R}}\omega_+ + \delta Z_e^{\mathrm{L}}\omega_-\right).
$$

Using the Ward identity both for bare and renormalized quantities implies

$$
\delta Z_{\hat{A}\hat{A}} = -2\delta Z_e. \tag{5.31}
$$

This is just the famous relation between the renormalizations of field and coupling known from QED. In contrast to the conventional formalism, the BFM yields this relation also for the electroweak SM. Note that after fixing the charge renormalization there is no more freedom to impose an extra condition for the field renormalization. Just as in QED, the on-shell definition of the electric charge together with gauge invariance automatically fixes the residue of the photon propagator to unity. This can be derived using the Ward identities (5.11) and (5.13). Instead of considering (5.13), the relation (5.31) can equivalently be obtained from the Ward identity (5.16) for the non-Abelian coupling.

From the Ward identities (5.14), (5.15) and (5.19) one derives in a similar way the following relations between the renormalization constants

$$\begin{aligned}
\delta Z_{\hat{A}\hat{Z}} &= 2\frac{c_{\mathrm{W}}}{s_{\mathrm{W}}}\frac{\delta c_{\mathrm{W}}^2}{c_{\mathrm{W}}^2},\\
\delta Z_{\hat{Z}\hat{Z}} &= -2\delta Z_e - \frac{c_{\mathrm{W}}^2 - s_{\mathrm{W}}^2}{s_{\mathrm{W}}^2}\frac{\delta c_{\mathrm{W}}^2}{c_{\mathrm{W}}^2},\\
\delta Z_{\hat{W}} &= -2\delta Z_e - \frac{c_{\mathrm{W}}^2}{s_{\mathrm{W}}^2}\frac{\delta c_{\mathrm{W}}^2}{c_{\mathrm{W}}^2},\\
\delta Z_{\hat{H}} &= \delta Z_{\hat{\chi}} = \delta Z_{\hat{\phi}} = -2\delta Z_e - \frac{c_{\mathrm{W}}^2}{s_{\mathrm{W}}^2}\frac{\delta c_{\mathrm{W}}^2}{c_{\mathrm{W}}^2} + \frac{\delta M_{\mathrm{W}}^2}{M_{\mathrm{W}}^2},
\end{aligned} \tag{5.32}$$

where

$$\frac{\delta c_{\mathrm{W}}^2}{c_{\mathrm{W}}^2} = \frac{\delta M_{\mathrm{W}}^2}{M_{\mathrm{W}}^2} - \frac{\delta M_{\mathrm{Z}}^2}{M_{\mathrm{Z}}^2}.$$

Finally, we get for the field renormalizations of the fermions

$$\delta Z_f^{\mathrm{L}} = \delta Z_{f_u}^{\mathrm{L}} = \delta Z_{f_d}^{\mathrm{L}}, \tag{5.33}$$

i.e. the field renormalization constants for the two left-handed fermions in a doublet must be equal.

The relations (5.30) – (5.32) express the field renormalization constants of all gauge bosons and scalars completely in terms of the renormalization constants of the electric charge and the particle masses. If the renormalized parameters are identified with the physical electron charge and the physical particle masses, they are manifestly gauge-independent. Moreover, the bare quantities $g_{1,0}$, $g_{2,0}$, λ_0, μ_0 and $G_{i,0}^f$ in the Lagrangian (see Section 2.1) obviously are also gauge-independent, as they represent free parameters of the theory. According to (2.16) and (2.19), the same is true for the bare charge and the bare weak mixing angle. Consequently, the counterterms δZ_e and δc_{W}^2 for the gauge couplings are gauge-independent. The relations (5.31) and (5.32) therefore imply that the field renormalizations of all gauge-boson fields are gauge-independent. This is in contrast to the conventional formalism where the field renormalizations in the on-shell scheme are gauge-dependent.

It should be recalled at this point that in contrast to δZ_e and δc_{W}^2 the counterterms for the masses are not gauge-independent. This can be traced to the mechanism of spontaneous symmetry breaking. The non-vanishing vacuum expectation value of the Higgs field, which generates the mass terms, is clearly not invariant under gauge transformations. Whereas the renormalized value $v = 2s_{\mathrm{W}}M_{\mathrm{W}}/e$ is gauge-independent, the bare quantity v_0 and the corresponding counterterm δv are not [66]. As a consequence, the bare masses which depend on v_0 are gauge-dependent. Thus, the counterterms δM_{W}^2, δM_{Z}^2, δM_{H}^2, δm_f and δt are also gauge-dependent. The physical masses, however, are determined by the pole positions of the propagators, i.e. the zeros of $k^2 - M^2 - \delta M^2 + C\delta t/M_{\mathrm{H}}^2 + \Sigma(k^2) + CT^{\hat{H}}/M_{\mathrm{H}}^2$, where C denotes the coupling of the fields to the Higgs field. The linear combination $\delta M^2 - C\delta t/M_{\mathrm{H}}^2$ of the mass and tadpole counterterm is independent of δv and thus gauge-independent.

The relations (5.30) – (5.32) reduce the number of independent renormalization constants considerably. One is left with the parameter renormalizations appearing in (5.27) and the fermion field renormalization constants δZ_f^{L}, $\delta Z_{f_u}^{\mathrm{R}}$ and $\delta Z_{f_d}^{\mathrm{R}}$. We choose on-shell renormalization conditions for the parameters as in Ref. [32][§] and express the renormalization constants in terms of unrenormalized self-energies and the tadpole

$$\begin{aligned}
\delta Z_e &= \frac{1}{2}\frac{\partial \Sigma_{\mathrm{T}}^{\hat{A}\hat{A}}(k^2)}{\partial k^2}\bigg|_{k^2=0}, \\
\delta M_{\mathrm{W}}^2 &= \mathrm{Re}\left(\Sigma_{\mathrm{T}}^{\hat{W}\hat{W}}(M_{\mathrm{W}}^2)\right), \\
\delta M_{\mathrm{Z}}^2 &= \mathrm{Re}\left(\Sigma_{\mathrm{T}}^{\hat{Z}\hat{Z}}(M_{\mathrm{Z}}^2)\right), \\
\delta M_{\mathrm{H}}^2 &= \mathrm{Re}\left(\Sigma^{\hat{H}\hat{H}}(M_{\mathrm{H}}^2)\right), \\
\delta m_f &= \frac{1}{2} m_f \,\mathrm{Re}\Big[\Sigma_{\mathrm{L}}^{\bar{f}f}(m_f^2) + \Sigma_{\mathrm{R}}^{\bar{f}f}(m_f^2) + 2\Sigma_{\mathrm{S}}^{\bar{f}f}(m_f^2)\Big], \\
\delta t &= -T^{\hat{H}}.
\end{aligned} \tag{5.34}$$

The fermion field renormalization constants can be fixed as follows

$$\begin{aligned}
\delta Z_f^{\mathrm{L}} &= -\mathrm{Re}\,\Sigma_{\mathrm{L}}^{\bar{f}_d f_d}(m_{f_d}^2) - m_{f_d}^2 \frac{\partial}{\partial k^2}\mathrm{Re}\left(\Sigma_{\mathrm{L}}^{\bar{f}_d f_d}(k^2) + \Sigma_{\mathrm{R}}^{\bar{f}_d f_d}(k^2) + 2\Sigma_{\mathrm{S}}^{\bar{f}_d f_d}(k^2)\right)\bigg|_{k^2=m_{f_d}^2}, \\
\delta Z_{f_u}^{\mathrm{R}} &= -\mathrm{Re}\,\Sigma_{\mathrm{R}}^{\bar{f}_u f_u}(m_{f_u}^2) - m_{f_u}^2 \frac{\partial}{\partial k^2}\mathrm{Re}\left(\Sigma_{\mathrm{L}}^{\bar{f}_u f_u}(k^2) + \Sigma_{\mathrm{R}}^{\bar{f}_u f_u}(k^2) + 2\Sigma_{\mathrm{S}}^{\bar{f}_u f_u}(k^2)\right)\bigg|_{k^2=m_{f_u}^2}, \\
\delta Z_{f_d}^{\mathrm{R}} &= -\mathrm{Re}\,\Sigma_{\mathrm{R}}^{\bar{f}_d f_d}(m_{f_d}^2) - m_{f_d}^2 \frac{\partial}{\partial k^2}\mathrm{Re}\left(\Sigma_{\mathrm{L}}^{\bar{f}_d f_d}(k^2) + \Sigma_{\mathrm{R}}^{\bar{f}_d f_d}(k^2) + 2\Sigma_{\mathrm{S}}^{\bar{f}_d f_d}(k^2)\right)\bigg|_{k^2=m_{f_d}^2}.
\end{aligned} \tag{5.35}$$

Although there is no freedom to choose the field renormalizations of the gauge bosons, scalars and left-handed up-type fermions in the BFM, the specified set of renormalization constants is still sufficient to render all background-field vertex functions finite[¶]. This is evident since the divergences of the vertex functions are subject to the same restriction as the counterterms. In order to illustrate this fact at one-loop

[§]The charge renormalization condition formulated in Ref. [32] assumes that the residue of the renormalized photon propagator equals unity and that the photon–Z-boson mixing vanishes for on-shell photons. Owing to the Ward identities, these conditions are fulfilled and we can use the same condition in the BFM.

[¶]Beyond one-loop order one needs in addition a renormalization of the quantum gauge parameters [21]. At the one-loop level these counterterms do not enter the background-field vertex functions because ξ_Q does not appear in pure background-field vertices. Clearly, the renormalization of gauge parameters is irrelevant for S-matrix elements at any order as these are gauge-independent.

order we list the divergent part of the self-energies in the BFM using dimensional regularization and writing the dimension as $D = 4 - \epsilon$,

$$
\begin{aligned}
\left(\Sigma_{\mathrm{T}}^{\hat{A}\hat{A}}(k^2)\right)^{\mathrm{div}} &= \frac{e^2}{16\pi^2} k^2 \left(\frac{32}{9}n - 7\right)\frac{2}{\epsilon}\,, \\
\left(\Sigma_{\mathrm{T}}^{\hat{A}\hat{Z}}(k^2)\right)^{\mathrm{div}} &= \frac{e^2}{16\pi^2} k^2 \left(\frac{32 s_{\mathrm{W}}^2 - 12}{9 c_{\mathrm{W}} s_{\mathrm{W}}} n + \frac{42 c_{\mathrm{W}}^2 + 1}{6 c_{\mathrm{W}} s_{\mathrm{W}}}\right)\frac{2}{\epsilon}\,, \\
\left(\Sigma_{\mathrm{T}}^{\hat{Z}\hat{Z}}(k^2)\right)^{\mathrm{div}} &= \frac{e^2}{16\pi^2}\Bigg[k^2 \left(\frac{32 s_{\mathrm{W}}^4 - 12(2 s_{\mathrm{W}}^2 - 1)}{9 c_{\mathrm{W}}^2 s_{\mathrm{W}}^2} n - \frac{42 c_{\mathrm{W}}^4 + 2 c_{\mathrm{W}}^2 - 1}{6 c_{\mathrm{W}}^2 s_{\mathrm{W}}^2}\right) \\
&\qquad - \sum_f \frac{m_f^2}{2 c_{\mathrm{W}}^2 s_{\mathrm{W}}^2} + \frac{2 M_{\mathrm{W}}^2 + M_{\mathrm{Z}}^2}{4 c_{\mathrm{W}}^2 s_{\mathrm{W}}^2}(\xi_Q + 3)\Bigg]\frac{2}{\epsilon}\,, \\
\left(\Sigma_{\mathrm{T}}^{\hat{W}\hat{W}}(k^2)\right)^{\mathrm{div}} &= \frac{e^2}{16\pi^2}\Bigg[k^2 \left(\frac{4}{3 s_{\mathrm{W}}^2} n - \frac{43}{6 s_{\mathrm{W}}^2}\right) - \sum_f \frac{m_f^2}{2 s_{\mathrm{W}}^2} + \frac{2 M_{\mathrm{W}}^2 + M_{\mathrm{Z}}^2}{4 s_{\mathrm{W}}^2}(\xi_Q + 3)\Bigg]\frac{2}{\epsilon}\,, \\
\left(\Sigma^{\hat{H}\hat{H}}(k^2)\right)^{\mathrm{div}} &= \frac{e^2}{16\pi^2}\Bigg[k^2 \left(\sum_f \frac{m_f^2}{2 M_{\mathrm{W}}^2 s_{\mathrm{W}}^2} - \frac{2 c_{\mathrm{W}}^2 + 1}{4 c_{\mathrm{W}}^2 s_{\mathrm{W}}^2}(\xi_Q + 3)\right) - \sum_f \frac{3 m_f^4}{M_{\mathrm{W}}^2 s_{\mathrm{W}}^2} \\
&\qquad + \frac{3(5 M_{\mathrm{H}}^4 + 12 M_{\mathrm{W}}^4 + 6 M_{\mathrm{Z}}^4) + 5 M_{\mathrm{H}}^2 (2 M_{\mathrm{W}}^2 + M_{\mathrm{Z}}^2)\xi_Q}{8 M_{\mathrm{W}}^2 s_{\mathrm{W}}^2}\Bigg]\frac{2}{\epsilon}\,, \\
\left(\Sigma^{\hat{\chi}\hat{\chi}}(k^2)\right)^{\mathrm{div}} &= \left(\Sigma^{\hat{\phi}\hat{\phi}}(k^2)\right)^{\mathrm{div}} \\
&= \frac{e^2}{16\pi^2}\Bigg[k^2 \left(\sum_f \frac{m_f^2}{2 M_{\mathrm{W}}^2 s_{\mathrm{W}}^2} - \frac{2 c_{\mathrm{W}}^2 + 1}{4 c_{\mathrm{W}}^2 s_{\mathrm{W}}^2}(\xi_Q + 3)\right)\Bigg]\frac{2}{\epsilon} + \frac{e}{2 M_{\mathrm{W}} s_{\mathrm{W}}}\left(T^{\hat{H}}\right)^{\mathrm{div}}, \\
\left(\Sigma_{\mathrm{L}}^{\bar{f}f}(k^2)\right)^{\mathrm{div}} &= \frac{e^2}{16\pi^2}\left(\frac{m_f^2 + m_{f'}^2}{4 M_{\mathrm{W}}^2 s_{\mathrm{W}}^2} + \frac{4 s_{\mathrm{W}}^2 Q_f^2 - 8 I_{\mathrm{W},f}^3 Q_f s_{\mathrm{W}}^2 + 2 c_{\mathrm{W}}^2 + 1}{4 c_{\mathrm{W}}^2 s_{\mathrm{W}}^2}\xi_Q\right)\frac{2}{\epsilon}\,, \\
\left(\Sigma_{\mathrm{R}}^{\bar{f}f}(k^2)\right)^{\mathrm{div}} &= \frac{e^2}{16\pi^2}\left(\frac{m_f^2}{2 M_{\mathrm{W}}^2 s_{\mathrm{W}}^2} + \frac{Q_f^2}{c_{\mathrm{W}}^2}\xi_Q\right)\frac{2}{\epsilon}\,, \\
\left(\Sigma_{\mathrm{S}}^{\bar{f}f}(k^2)\right)^{\mathrm{div}} &= -\frac{e^2}{16\pi^2}\left(\frac{m_{f'}^2}{2 M_{\mathrm{W}}^2 s_{\mathrm{W}}^2} + \frac{(Q_f - I_{\mathrm{W},f}^3) Q_f}{c_{\mathrm{W}}^2}(\xi_Q + 3)\right)\frac{2}{\epsilon}\,, \\
\left(T^{\hat{H}}\right)^{\mathrm{div}} &= \frac{e^2}{16\pi^2}\Bigg(-\sum_f \frac{2 m_f^4}{e M_{\mathrm{W}} s_{\mathrm{W}}} \\
&\qquad + \frac{3(M_{\mathrm{H}}^4 + 4 M_{\mathrm{W}}^4 + 2 M_{\mathrm{Z}}^4) + M_{\mathrm{H}}^2 (2 M_{\mathrm{W}}^2 + M_{\mathrm{Z}}^2)\xi_Q}{4 e M_{\mathrm{W}} s_{\mathrm{W}}}\Bigg)\frac{2}{\epsilon}\,, \qquad (5.36)
\end{aligned}
$$

where f' is the isospin partner of fermion f, n denotes the number of fermion generations and the summations run over all fermion flavors and colors. The fermion self-energies and the fermionic contributions to the gauge-boson and scalar self-energies are included for completeness. They have the same form as in the conventional formalism.

Using (5.34) and (5.35) we obtain the divergent parts of the renormalization constants

$$
\begin{aligned}
(\delta Z_e)^{\mathrm{div}} &= \frac{e^2}{16\pi^2}\frac{1}{2}\left(\frac{32}{9}n-7\right)\frac{2}{\epsilon}\,, \\
\left(\frac{\delta M_{\mathrm{W}}^2}{M_{\mathrm{W}}^2}\right)^{\mathrm{div}} &= \frac{e^2}{16\pi^2}\left(\frac{4}{3s_{\mathrm{W}}^2}n-\frac{43}{6s_{\mathrm{W}}^2}-\sum_f\frac{m_f^2}{2M_{\mathrm{W}}^2s_{\mathrm{W}}^2}+\frac{2c_{\mathrm{W}}^2+1}{4c_{\mathrm{W}}^2s_{\mathrm{W}}^2}(\xi_Q+3)\right)\frac{2}{\epsilon}\,, \\
\left(\frac{\delta c_{\mathrm{W}}^2}{c_{\mathrm{W}}^2}\right)^{\mathrm{div}} &= \frac{e^2}{16\pi^2}\left(\frac{-32s_{\mathrm{W}}^2+12}{9c_{\mathrm{W}}^2}n-7-\frac{1}{6c_{\mathrm{W}}^2}\right)\frac{2}{\epsilon}\,, \\
\left(\delta M_{\mathrm{H}}^2\right)^{\mathrm{div}} &= \frac{e^2}{16\pi^2}\left(\sum_f\frac{m_f^2(M_{\mathrm{H}}^2-6m_f^2)}{2M_{\mathrm{W}}^2s_{\mathrm{W}}^2}\right. \\
&\quad\left.+\frac{3[5M_{\mathrm{H}}^4+12M_{\mathrm{W}}^4+6M_{\mathrm{Z}}^4+M_{\mathrm{H}}^2(2M_{\mathrm{W}}^2+M_{\mathrm{Z}}^2)(\xi_Q-2)]}{8M_{\mathrm{W}}^2s_{\mathrm{W}}^2}\right)\frac{2}{\epsilon}\,, \\
(\delta m_f)^{\mathrm{div}} &= \frac{e^2}{16\pi^2}m_f\left(\frac{3(m_f^2-m_{f'}^2)}{8M_{\mathrm{W}}^2s_{\mathrm{W}}^2}-\frac{3Q_f(Q_f-I_{\mathrm{W},3}^f)}{c_{\mathrm{W}}^2}+\frac{(2c_{\mathrm{W}}^2+1)\xi_Q}{8c_{\mathrm{W}}^2s_{\mathrm{W}}^2}\right)\frac{2}{\epsilon}\,, \\
(\delta t)^{\mathrm{div}} &= -\left(T^{\hat{H}}\right)^{\mathrm{div}}, \\
\left(\delta Z_f^{\mathrm{L}}\right)^{\mathrm{div}} &= -\left(\Sigma_{\mathrm{L}}^{\bar{f}_uf_u}(k^2)\right)^{\mathrm{div}} = -\left(\Sigma_{\mathrm{L}}^{\bar{f}_df_d}(k^2)\right)^{\mathrm{div}}, \\
\left(\delta Z_{f_u}^{\mathrm{R}}\right)^{\mathrm{div}} &= -\left(\Sigma_{\mathrm{R}}^{\bar{f}_uf_u}(k^2)\right)^{\mathrm{div}}, \\
\left(\delta Z_{f_d}^{\mathrm{R}}\right)^{\mathrm{div}} &= -\left(\Sigma_{\mathrm{R}}^{\bar{f}_df_d}(k^2)\right)^{\mathrm{div}}.
\end{aligned}
\tag{5.37}
$$

According to (5.31) and (5.32), this also fixes the divergent parts of the gauge-boson and scalar field renormalization constants yielding

$$
\begin{aligned}
(\delta Z_{\hat{A}\hat{A}})^{\mathrm{div}} &= -\frac{e^2}{16\pi^2}\left(\frac{32}{9}n-7\right)\frac{2}{\epsilon}\,, \\
(\delta Z_{\hat{A}\hat{Z}})^{\mathrm{div}} &= -\frac{e^2}{16\pi^2}2\left(\frac{32s_{\mathrm{W}}^2-12}{9c_{\mathrm{W}}s_{\mathrm{W}}}n+\frac{42c_{\mathrm{W}}^2+1}{6c_{\mathrm{W}}s_{\mathrm{W}}}\right)\frac{2}{\epsilon}\,, \\
(\delta Z_{\hat{Z}\hat{Z}})^{\mathrm{div}} &= -\frac{e^2}{16\pi^2}\left(\frac{32s_{\mathrm{W}}^4-12(2s_{\mathrm{W}}^2-1)}{9c_{\mathrm{W}}^2s_{\mathrm{W}}^2}n-\frac{42c_{\mathrm{W}}^4+2c_{\mathrm{W}}^2-1}{6c_{\mathrm{W}}^2s_{\mathrm{W}}^2}\right)\frac{2}{\epsilon}\,, \\
(\delta Z_{\hat{W}})^{\mathrm{div}} &= -\frac{e^2}{16\pi^2}\left(\frac{4}{3s_{\mathrm{W}}^2}n-\frac{43}{6s_{\mathrm{W}}^2}\right)\frac{2}{\epsilon}\,, \\
(\delta Z_{\hat{H}})^{\mathrm{div}} &= (\delta Z_{\hat{\chi}})^{\mathrm{div}} = \left(\delta Z_{\hat{\phi}}\right)^{\mathrm{div}} \\
&= -\frac{e^2}{16\pi^2}\left(\sum_f\frac{m_f^2}{2M_{\mathrm{W}}^2s_{\mathrm{W}}^2}-\frac{2c_{\mathrm{W}}^2+1}{4c_{\mathrm{W}}^2s_{\mathrm{W}}^2}(\xi_Q+3)\right)\frac{2}{\epsilon}\,.
\end{aligned}
\tag{5.38}
$$

The divergent parts of the gauge-boson field renormalization constants are independent of ξ_Q in accordance with the general discussion given above.

The renormalized self-energies are obtained by adding the counterterms specified in (A.3) – (A.10) to the unrenormalized self-energies. It is evident from (5.36) – (5.38) that although the field renormalization constants cannot be chosen freely in the BFM, all renormalized self-energies are nevertheless finite. Whereas in the conventional formalism the field renormalization constants are adjusted in order to obtain finite self-energies, this happens automatically in the BFM as a consequence of the Ward identities. The finiteness of the longitudinal parts of the gauge-boson self-energies and of the gauge-boson–scalar mixing energies follows directly from the finiteness of the renormalized tadpole and scalar self-energies and the Ward identities (5.5) – (5.9).

A renormalization based on the on-shell definition of all parameters can therefore consistently be used in the BFM. It renders all vertex functions finite while respecting the full gauge symmetry of the BFM.

Since the divergent parts of the unrenormalized self-energies fulfill the Ward identities by themselves, it is obvious that renormalization in the minimal-subtraction scheme also preserves the symmetry of the BFM.

As mentioned above, the on-shell renormalization in the BFM fixes the residue of the photon propagator to unity. The propagators of the other gauge bosons, scalars and left-handed up-type fermions acquire residues different from unity. This is similar to the minimal on-shell scheme of the conventional formalism and has to be corrected in the S-matrix elements by a UV-finite wave-function renormalization.

The renormalization constants introduced in (5.27) and (5.28) correspond to the physical fields, i.e. the mass and electric charge eigenstates $\hat{A}, \hat{Z}, \hat{W}^{\pm}$. Alternatively, one can introduce renormalization constants in the symmetric formulation (see e.g. Ref. [35]) resulting in the minimal on-shell scheme. In the bosonic sector these renormalization constants are given by

$$
\begin{aligned}
\hat{W}_0^a &= (Z_2^{\hat{W}})^{1/2} \hat{W}^a, \\
\hat{B}_0 &= (Z_2^{\hat{B}})^{1/2} \hat{B}, \\
\hat{\Phi}_0 &= (Z^{\hat{\Phi}})^{1/2} \hat{\Phi}, \\
g_{2,0} &= Z_1^{\hat{W}} (Z_2^{\hat{W}})^{-3/2} g_2 = Z_{g_2} g_2, \\
g_{1,0} &= Z_1^{\hat{B}} (Z_2^{\hat{B}})^{-3/2} g_1 = Z_{g_1} g_1, \\
v_0 &= (Z^{\hat{\Phi}})^{1/2} (v - \delta v), \\
\mu_0^2 &= (Z^{\hat{\Phi}})^{-1} (\mu^2 - \delta \mu^2), \\
\lambda_0 &= Z^{\lambda} (Z^{\hat{\Phi}})^{-2} \lambda.
\end{aligned} \tag{5.39}
$$

In this formulation, the gauge symmetry of the BFM implies in addition to $Z_1^{\hat{B}} = Z_2^{\hat{B}}$

$$
\begin{aligned}
Z_1^{\hat{W}} &= Z_2^{\hat{W}}, \\
\delta v &= 0.
\end{aligned} \tag{5.40}
$$

Thus, for both the isotriplet fields of $\mathrm{SU(2)_W}$ and the isosinglet field of $\mathrm{U(1)_Y}$ a QED-like relation between coupling constant and field renormalization holds, and there is no renormalization of the vacuum expectation value other than the one owing to the Higgs-field renormalization. The other restrictions following from (5.30) – (5.33) are already taken into account in the ansatz (5.39) for the field renormalization. It is clear that also in this on-shell scheme the field renormalizations of the gauge bosons are gauge-independent. With the restrictions imposed by the BFM, the two renormalization schemes become in fact equivalent, i.e. both schemes yield identical Green functions.

Recently, the renormalization of the SM in the BFM was considered using the minimal scheme specified in (5.39) and omitting fermions [67]. As in this reference the residue of the Higgs field is required to be unity, in contrast to our result (5.40) a nonzero (but nevertheless finite) correction δv to the vacuum expectation value of the Higgs field is needed. This violates the naïve Ward identities and is cured in Ref. [67] by including δv into the renormalized Ward identities. Since the renormalization in the BFM necessarily involves fields whose residues differ from unity we find it preferable to carry it out in such a way that the explicit gauge invariance and correspondingly the form of the Ward identities is retained. Furthermore, we disagree with the conclusion of Ref. [67] that the Landau gauge would be enforced for the background fields. In fact, we do not find any reason that would require this restriction.

We have derived the relations between the renormalization constants from the background-field Ward identities given in the last section. As the gauge invariance of the effective action is closely related to the gauge invariance of the classical Lagrangian [21], those relations can also be inferred directly from the Lagrangian. One can check that the relations listed above are precisely those required to render the renormalized classical Lagrangian $\mathcal{L}_{\mathrm{C}}$ gauge-invariant.

As a consequence of the relations (5.30) – (5.33), the counterterm vertices of the background fields have a much simpler structure than the ones in the conventional formalism (see e.g. Ref. [32]). Their explicit form is given in the appendix. Moreover, all realizations of a generic vertex are renormalized in the same way (see appendix).

In the appendix we have listed the counterterms for all vertices involving only background fields. These counterterms are sufficient for the renormalization of all one-loop processes. Through the parameter renormalizations and the renormalizations of the background fields also the vertices containing both quantum and background fields and the pure quantum-field vertices acquire counterterms. These become relevant in higher orders. Their explicit form can easily be obtained using (5.27), (5.28) and the Feynman rules given in the appendix.

5.4 Properties of background-field vertex functions

As indicated above, the gauge invariance present in the BFM gives rise to vertex functions possessing improved theoretical properties compared to their conventional counterparts. In this section it will be illustrated for some examples how such properties can be inferred from the relations between field and parameter renormalization constants derived in Section 5.3.

We first consider the fermion–gauge-boson vertex functions. As mentioned in Section 5.2.2, the one-loop fermion–gauge-boson "vertex functions" of the PT are UV-finite when the fermion field renormalization has been added [18]. This has been found by explicit calculation. In the BFM, this fact is an example of the simplifications arising in the renormalization as a consequence of gauge invariance. As follows from (A.25), the counterterm for the $\hat{V}\bar{F}F$-vertex is solely given by the fermion field renormalization. Adding it to the vertex function evidently cancels the UV divergence. Obviously, this fact holds for all values of the quantum gauge parameter ξ_Q. From the counterterm structure given in the appendix similar conclusions can be drawn for other vertex functions. In particular, the $\hat{V}\hat{V}'\hat{W}\hat{W}$ and $\hat{V}\hat{W}\hat{W}$ vertices become UV-finite after adding the field renormalization of two $\hat{W}$ fields as can be read from (A.11) and (A.13). In Ref. [18] it was also noted that the one-loop fermion–photon vertex functions including fermion field renormalization vanish at zero momentum transfer of the photon. In the BFM, the inclusion of the fermion field renormalization amounts to the complete renormalization of this vertex. The renormalized vertex correction vanishes owing to the renormalization condition for the electric charge.

Next, we investigate the asymptotic behavior of the gauge-boson self-energies in the BFM for $|q^2| \to \infty$. In Ref. [61], the explicit one-loop result for the leading logarithms of the bosonic contributions to the gauge-boson self-energies in the BFM has been given showing that their coefficients are independent of ξ_Q. However, this feature can also be deduced from the Ward identities as follows. In dimensional regularization the unrenormalized one-loop self-energies obey ($\hat{V}, \hat{V}' = \hat{A}, \hat{Z}, \hat{W}$)

$$\Pi^{\hat{V}\hat{V}'}(q^2) = \frac{\Sigma_{\mathrm{T}}^{\hat{V}\hat{V}'}(q^2) - \Sigma_{\mathrm{T}}^{\hat{V}\hat{V}'}(0)}{q^2} = g^2_{\hat{V}\hat{V}'}\mu^{\epsilon}\left(-c_{\hat{V}\hat{V}'}\frac{2}{\epsilon} + \text{UV-finite terms}\right), \tag{5.41}$$

where $c_{\hat{V}\hat{V}'}$ is a q^2-independent coefficient, which can be read off from (5.36), $g_{\hat{A}\hat{A}} = e$, $g_{\hat{W}\hat{W}} = g_2 = e/s_{\mathrm{W}}$, $g_{\hat{Z}\hat{Z}} = e/(c_{\mathrm{W}}s_{\mathrm{W}})$, $g_{\hat{A}\hat{Z}} = e/\sqrt{c_{\mathrm{W}}s_{\mathrm{W}}}$, and μ is a mass parameter necessary to keep $g^2_{\hat{V}\hat{V}'}$ dimensionless. In the limit $|q^2| \to \infty$ all masses can be neglected and on dimensional grounds the self-energies behave as

$$\Pi^{\hat{V}\hat{V}'}(q^2) \underset{|q^2|\to\infty}{\sim} g^2_{\hat{V}\hat{V}'}\left(-c_{\hat{V}\hat{V}'}\frac{2}{\epsilon} + c_{\hat{V}\hat{V}'}\log\frac{|q^2|}{\mu^2} + \text{UV-finite constant}\right). \tag{5.42}$$

Using the identities

$$(\delta Z_{\hat{V}\hat{V}})^{\mathrm{div}} = -\left(\Pi^{\hat{V}\hat{V}}(q^2)\right)^{\mathrm{div}}, \quad (\delta Z_{\hat{A}\hat{Z}})^{\mathrm{div}} = -2\left(\Pi^{\hat{A}\hat{Z}}(q^2)\right)^{\mathrm{div}} \tag{5.43}$$

we can identify the divergent parts of $\delta Z_{\hat{V}\hat{V}}$ and $\delta Z_{\hat{A}\hat{Z}}$ as

$$(\delta Z_{\hat{V}\hat{V}})^{\mathrm{div}} = g^2_{\hat{V}\hat{V}}c_{\hat{V}\hat{V}}\frac{2}{\epsilon}, \quad (\delta Z_{\hat{A}\hat{Z}})^{\mathrm{div}} = 2g^2_{\hat{A}\hat{Z}}c_{\hat{A}\hat{Z}}\frac{2}{\epsilon}. \tag{5.44}$$

We found in Section 5.3 that the field renormalization constants for the gauge bosons and thus $(\delta Z_{\hat{V}\hat{V}'})^{\mathrm{div}}$ are gauge-independent. As a consequence, also the coefficients $c_{\hat{V}\hat{V}'}$ of the leading logarithms of $\Sigma_{\mathrm{T}}^{\hat{V}\hat{V}'}(q^2)$ are independent of ξ_Q.

The gauge invariance of the BFM gives rise to QED-like relations for the β-functions of the gauge couplings in QCD (see Section 5.2.1) and the SM. They are directly related to the anomalous dimensions and thus to the field renormalization constants of the gauge bosons. For the β-function associated with the electromagnetic coupling in the minimal-subtraction scheme this can be inferred from the identity $Z_e = Z_{\hat{A}\hat{A}}^{-1/2}$ yielding

$$\beta_e(e) = -c_{\hat{A}\hat{A}} e^3 + \mathcal{O}(e^5) \tag{5.45}$$

Analogously, from the relation $Z_{g_2} = (Z_2^{\hat{W}})^{-1/2} = Z_{\hat{W}}^{-1/2}$, which follows from (5.32) and (5.40), we obtain for the charged-current coupling

$$\beta_{g_2}(g_2) = -c_{\hat{W}\hat{W}} g_2^3 + \mathcal{O}(g_2^5). \tag{5.46}$$

The fact that the coefficients of the leading logarithms of the self-energies equal the coefficients of the β-functions implies that the asymptotic behavior of effective coupling constants $e^2(q^2)$ and $g_2^2(q^2)$ defined via Dyson summation of self-energies (see e.g. Refs. [5, 6, 18]) is automatically governed by the renormalization group. Accordingly, we can introduce running couplings as follows

$$\begin{aligned} e^2(q^2) &= \frac{e_0^2}{1+\mathrm{Re}\,\Pi^{\hat{A}\hat{A}}(q^2)} = \frac{e^2}{1+\mathrm{Re}\,\Pi^{\hat{A}\hat{A},\mathrm{ren}}(q^2)}, \\ g_2^2(q^2) &= \frac{g_{2,0}^2}{1+\mathrm{Re}\,\Pi^{\hat{W}\hat{W}}(q^2)} = \frac{g_2^2}{1+\mathrm{Re}\,\Pi^{\hat{W}\hat{W},\mathrm{ren}}(q^2)}, \end{aligned} \tag{5.47}$$

where the quantities on the right-hand side are the renormalized ones and the second equality holds because of $Z_e = Z_{\hat{A}\hat{A}}^{-1/2}$ and $Z_{g_2} = Z_{\hat{W}}^{-1/2}$, respectively. As these running couplings can be expressed in terms of bare quantities, they are manifestly renormalization-scheme independent in the BFM .

The asymptotic behavior of $e^2(q^2)$ and $g_2^2(q^2)$ is independent of the quantum gauge parameter. They coincide in this region with the running couplings defined in Refs. [5, 6, 18]. For finite values of q^2, however, there are differences[‖] and the couplings (5.47) depend on ξ_Q. This indicates that any definition of running couplings via Dyson summation of self-energies that take into account mass effects is not unique but a matter of convention. This arbitrariness is made transparent in the BFM and has to be taken into account in treatments based on running couplings.

We can define a running $s_\mathrm{W}(q^2)$ as the ratio of the electromagnetic and charged-current running coupling constants

$$s_\mathrm{W}^2(q^2) = \frac{e^2(q^2)}{g_2^2(q^2)} = \frac{e^2}{g_2^2}\frac{1+\mathrm{Re}\,\Pi^{\hat{W}\hat{W},\mathrm{ren}}(q^2)}{1+\mathrm{Re}\,\Pi^{\hat{A}\hat{A},\mathrm{ren}}(q^2)}, \tag{5.48}$$

[‖]Those differences also exist between the different formulations of the previous treatments.

which is obviously renormalization-scheme independent. In the leading-logarithmic approximation this can be written as

$$s_{\mathrm{W}}^2(q^2) \underset{|q^2|\to\infty}{\widetilde{}} s_{\mathrm{W}}^2 \left(1 - \frac{c_{\mathrm{W}}}{s_{\mathrm{W}}} \operatorname{Re} \Pi^{\hat{A}\hat{Z},\mathrm{ren}}(q^2)\right) + \mathcal{O}(e^4). \tag{5.49}$$

This resembles the definition of a running $s_{\mathrm{W}}^2(q^2)$ used for example in Ref. [68].

Chapter 6

Conclusion

The Glashow–Salam–Weinberg model, also known as the electroweak Standard Model (SM), gives the at present most comprehensive description of the electroweak phenomena being theoretically consistent and in extraordinary agreement with the experimental data. Nevertheless, the SM is by no means tested in its full scope. In particular, very little is known about the Higgs sector being responsible for generating the masses of the particles. The precision tests of the SM provide indirect information about its as yet not experimentally verified ingredients. Furthermore, these investigations allow to narrow down the possible size of effects caused by new physics, or — in case of a deviation — may eventually signal the need for a more fundamental theory going beyond the SM.

These precision tests require a very careful study of the radiative corrections entering the SM predictions. Whereas the techniques for performing one-loop calculations in massive gauge theories are well established, so far no complete treatment of irreducible two-loop corrections has been achieved. A further problem in the investigation of radiative corrections is due to the gauge dependence of Green functions, e.g. when a resummation of higher-order corrections is considered.

On the way towards complete two-loop calculations in the SM the two-loop self-energies are of particular interest as these universal building blocks enter every two-loop calculation. Up to now, only some limiting cases of heavy top mass or heavy Higgs mass have been treated. In view of a realistic scenario, however, the evaluation of two-loop self-energies in the most general case is needed.

In this thesis, it has been shown that all two-loop self-energies in massive gauge theories can be reduced to standard scalar integrals. An algebraic method for the tensor reduction of general two-loop self-energy integrals has been worked out. Making use of the symmetries of the scalar integrals and certain integral relations, the two-loop self-energies can be expressed in terms of a minimal set of basic scalar integrals.

The algebraic approach presented here is suited for a fully automated treatment and has been implemented into the computer-algebra program *TwoCalc*. The results are valid for arbitrary values of the invariant momentum, the particle masses, the space–time dimension and the gauge parameters. They directly display the gauge dependence of the considered quantities and properties like transversality or relations between the self-energies. This has been demonstrated by explicitly checking Slavnov–Taylor identities involving several thousands of Feynman diagrams.

As an application, the two-loop photon self-energy in QED and the light-fermion contributions to the gauge-boson self-energies in the SM have been treated. Using a general R_ξ gauge in the conventional formalism and in the background-field method, the gauge dependence of these contributions has been discussed.

With the algebraic method worked out here, the task of evaluating any two-loop self-energy involving in general a huge number of Feynman amplitudes is reduced to the problem of calculating only three different types of two-loop scalar integrals. Since meanwhile suitable results have been obtained for these integrals in the literature, the necessary tools are now at hand to study the effects resulting from two-loop self-energies in the most general case.

The problems caused by the gauge dependence of Green functions have recently given rise to a number of proposals for modifying the conventional Green functions in order to eliminate their gauge-parameter dependence and to construct new "Green functions" with certain desirable theoretical properties. In particular, the pinch technique (PT) was found to yield gauge-parameter-independent quantities at one-loop order fulfilling simple Ward identities and possessing other desirable properties like a decent asymptotic behavior.

In this work, an extension of the PT approach to the two-loop level has been discussed for the example of light-fermion contributions to the two-loop gauge-boson self-energies. A gauge-parameter-independent two-loop "self-energy" has been constructed. On the other hand, this example has made apparent the problems inherent in the PT treatment. The elimination of the gauge parameters can only be achieved in the PT at the cost of an involved rearrangement between different Green functions and between one-particle irreducible and reducible contributions. Since the field-theoretical meaning of this procedure is not clear, the theoretical interpretation of the PT quantities remains obscure. In particular, their process independence has not been proven. As a further serious shortcoming, the PT offers no explanation for the desirable properties fulfilled by its "Green functions". These properties have only been noted by explicitly computing specific one-loop examples.

In order to provide some insight into these problems, the gauge dependence of Green functions has been studied using a completely different approach, namely the background-field method (BFM). Application of the BFM yields a gauge-invariant effective action for QCD and the SM. The Green functions derived from the gauge-invariant effective action can directly be calculated from Feynman rules in all orders of perturbation theory and have a well-defined field-theoretical interpretation. It has been shown that the gauge invariance gives rise to QED-like Ward identities in the BFM being of much simpler form than the Slavnov–Taylor identities of the conventional formalism.

A connection between the PT and the BFM has been established by showing that the PT results obtained in the literature for QCD and the SM coincide with the special case of the BFM results where the quantum gauge parameter ξ_Q is equal to one. In addition, the desirable properties noted for the PT "Green functions" have been shown to be a direct consequence of the Ward identities in the BFM. These Ward identities and the desirable features of Green functions hold in the BFM for all values of ξ_Q. As an example, this has been illustrated for running couplings directly defined via Dyson summation. As derived from gauge invariance, their asymptotic behavior is independent of ξ_Q and automatically governed by the renormalization group.

These results are of importance for a better understanding of the problems related to the gauge dependence of Green functions. In particular, it has been shown that not the requirement of gauge-parameter independence is the crucial criterion leading to Green functions with suitable properties but the Ward identities following from gauge invariance. It has become apparent that the desirable properties noted within the PT do not single out any distinguished form of a Green function. Instead, the BFM yields an infinite set of vertex functions parametrized by ξ_Q which fulfill the same properties. This ambiguity, made transparent in the BFM, is also inherent in the PT and similar treatments where it corresponds to the ambiguity in choosing different prescriptions for eliminating the gauge-parameter dependence. It must be taken into account when considering applications involving off-shell Green functions.

Besides its use for studying the properties of off-shell Green functions, the BFM is also a well-suited tool for practical calculations of S-matrix elements. Such calculations simplify considerably in the BFM. The freedom to choose an appropriate gauge for the background fields, e.g. the unitary gauge, independently of the quantum gauge fixing allows to reduce the number of contributing Feynman diagrams drastically. In addition, also the evaluation of loop diagrams simplifies. This holds in particular in the 't Hooft–Feynman gauge for the quantum fields.

For calculations within the BFM a consistent renormalization of the SM in accordance with the BFM gauge invariance is needed. This has explicitly been worked out starting from two different renormalization schemes, a complete and a minimal on-shell scheme. The gauge symmetry imposes relations between field renormalization constants and the renormalization constants of the SM parameters, i.e. electric charge and particle masses. It was pointed out that even with this reduced set of independent renormalization constants all Green functions of the SM become finite. This has been verified explicitly at one-loop order by calculating the relevant quantities. The renormalization constants of the physical parameters are still independent of each other so that all on-shell parameter renormalization conditions can be maintained. Thus, the on-shell scheme is compatible with the symmetries of the BFM.

As a consequence of gauge invariance, the renormalization in the BFM drastically simplifies compared to the conventional formalism both technically and conceptually. The counterterms in the BFM possess a much simpler structure. In particular, all realizations of a generic vertex have one single universal counterterm. If charge and particle masses are identified with their physical values, the field renormalizations of all gauge bosons become gauge-independent. A complete set of BFM Feynman rules for the SM has been listed including all counterterms necessary for one-loop calculations.

The methods developed in this thesis should be useful for an improved investigation of higher order corrections in massive gauge theories. In view of the precise measurement of the W-boson mass expected from LEP2, these corrections will in particular be of interest for further exploring the correlation between the vector boson masses and the other electroweak precision parameters, i.e. the electromagnetic fine structure constant and the Fermi constant determined from muon decay.

Appendix A

Feynman rules in the background-field method

In this appendix we list the Feynman rules of the SM in the BFM for an arbitrary quantum gauge parameter $\xi_Q = \xi_Q^W = \xi_Q^B$. We write down generic Feynman rules for all vertices and give the possible actual insertions. We use here the shorthand notation

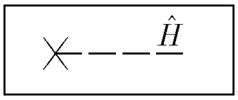
$$c = c_{\mathrm{W}}, \qquad s = s_{\mathrm{W}}. \tag{A.1}$$

From the Feynman rules given here, the vertex functions corresponding to the gauge-invariant effective action of the BFM can be calculated. No gauge-fixing term is included for the background fields. Such a term is only relevant for the construction of connected Green functions and S-matrix elements from the vertex functions. It can be chosen independently from the gauge-fixing of the quantum fields. If a linear gauge is used, only the propagators of the background fields are affected. In a background R_ξ gauge, the background-field propagators take the same form as the quantum-field propagators given below with ξ_Q replaced by the background gauge parameter ξ_B. Note, however, that it is preferable to use a more convenient gauge for the background fields like the unitary gauge.

We first list the vertices containing only background fields including counterterms. In lowest order, these vertices are identical to the ones in the conventional formalism (see e.g. Ref. [32]). Their counterterms, however, have a much simpler structure. Apart from the two-point functions, each generic vertex in the BFM has a universal counterterm. As mentioned above, these counterterms are sufficient for the renormalization of all one-loop processes.

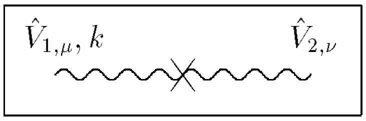
In the vertices all momenta and fields are considered as incoming.

- tadpole:



$$= i\delta t. \tag{A.2}$$

- $\hat{V}\hat{V}$ counterterm:



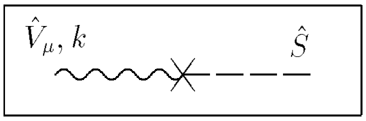
$$= i\Big[(-g_{\mu\nu}k^2 + k_\mu k_\nu)C_1 + g_{\mu\nu}C_2\Big] \tag{A.3}$$

with the actual values of $\hat{V}_1$, $\hat{V}_2$ and C_1, C_2

$\hat{V}_1\hat{V}_2$	$\hat{W}^+\hat{W}^-$	$\hat{Z}\hat{Z}$	$\hat{A}\hat{Z}$	$\hat{A}\hat{A}$
C_1	$\delta Z_{\hat{W}}$	$\delta Z_{\hat{Z}\hat{Z}}$	$\frac{1}{2}\delta Z_{\hat{A}\hat{Z}}$	$\delta Z_{\hat{A}\hat{A}}$
C_2	$M_\mathrm{W}^2\delta Z_{\hat{W}} + \delta M_\mathrm{W}^2$	$M_\mathrm{Z}^2\delta Z_{\hat{Z}\hat{Z}} + \delta M_\mathrm{Z}^2$	0	0

(A.4)

- $\hat{V}\hat{S}$ counterterm:



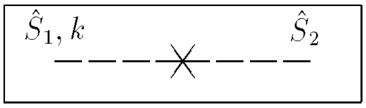
$$= ik_\mu C\delta Z_{\hat{H}} \tag{A.5}$$

with the actual values of $\hat{V}$, $\hat{S}$ and C

$\hat{V}\hat{S}$	$\hat{W}^\pm\hat{\phi}^\mp$	$\hat{Z}\hat{\chi}$
C	$\pm M_\mathrm{W}$	iM_Z

(A.6)

- $\hat{S}\hat{S}$ counterterm:



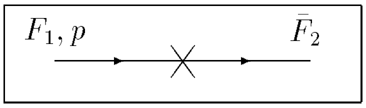
$$= i\Big[\delta Z_{\hat{H}}k^2 - C\Big] \tag{A.7}$$

with the actual values of $\hat{S}_1$, $\hat{S}_2$ and C_1, C_2

$\hat{S}_1\hat{S}_2$	$\hat{H}\hat{H}$	$\hat{\chi}\hat{\chi}, \hat{\phi}\hat{\phi}$
C	$M_\mathrm{H}^2\delta Z_{\hat{H}} + \delta M_\mathrm{H}^2$	$-\frac{e}{2s}\frac{\delta t}{M_\mathrm{W}}$

(A.8)

- $F\bar{F}$ counterterm:



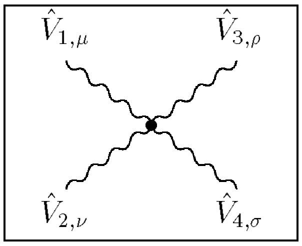
$$= i\Big[C_\mathrm{L}\not{p}\,\omega_- + C_\mathrm{R}\not{p}\,\omega_+ - C_\mathrm{S}\Big] \tag{A.9}$$

with the actual values of F_1, $\bar{F}_2$ and C_L, C_R, C_S

$F_1\bar{F}_2$	$f\bar{f}$
C_L	δZ_f^L
C_R	δZ_f^R
C_S	$m_f\frac{1}{2}\left(\delta Z_f^\mathrm{L} + \delta Z_f^\mathrm{R}\right) + \delta m_f$

(A.10)

- $\hat{V}\hat{V}\hat{V}\hat{V}$ coupling:



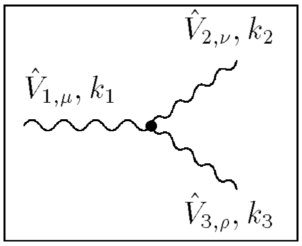
$$= ie^2C\Big[2g_{\mu\nu}g_{\sigma\rho} - g_{\nu\rho}g_{\mu\sigma} - g_{\rho\mu}g_{\nu\sigma}\Big](1+\delta Z_{\hat{W}}) \qquad \text{(A.11)}$$

with the actual values of $\hat{V}_1$, $\hat{V}_2$, $\hat{V}_3$, $\hat{V}_4$ and C

$\hat{V}_1\hat{V}_2\hat{V}_3\hat{V}_4$	$\hat{W}^+\hat{W}^+\hat{W}^-\hat{W}^-$	$\hat{W}^+\hat{W}^-\hat{Z}\hat{Z}$	$\hat{W}^+\hat{W}^-\hat{A}\hat{Z}$	$\hat{W}^+\hat{W}^-\hat{A}\hat{A}$
C	$\frac{1}{s^2}$	$-\frac{c^2}{s^2}$	$\frac{c}{s}$	-1

(A.12)

- $\hat{V}\hat{V}\hat{V}$ coupling:



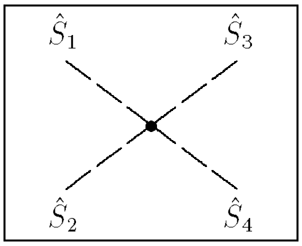
$$\begin{aligned} = -ieC\Big[&g_{\mu\nu}(k_2-k_1)_\rho + g_{\nu\rho}(k_3-k_2)_\mu \\ &+ g_{\rho\mu}(k_1-k_3)_\nu\Big](1+\delta Z_{\hat{W}}) \end{aligned} \qquad \text{(A.13)}$$

with the actual values of $\hat{V}_1$, $\hat{V}_2$, $\hat{V}_3$ and C

$\hat{V}_1\hat{V}_2\hat{V}_3$	$\hat{A}\hat{W}^+\hat{W}^-$	$\hat{Z}\hat{W}^+\hat{W}^-$
C	1	$-\frac{c}{s}$

(A.14)

- $\hat{S}\hat{S}\hat{S}\hat{S}$ coupling:



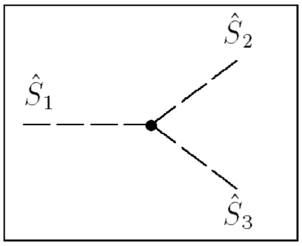
$$= ie^2C\Big[1 + \frac{\delta M_{\mathrm{H}}^2}{M_{\mathrm{H}}^2} + \frac{e}{2s}\frac{\delta t}{M_{\mathrm{W}}M_{\mathrm{H}}^2} + \delta Z_{\hat{H}}\Big] \qquad \text{(A.15)}$$

with the actual values of $\hat{S}_1$, $\hat{S}_2$, $\hat{S}_3$, $\hat{S}_4$ and C

$\hat{S}_1\hat{S}_2\hat{S}_3\hat{S}_4$	$\hat{H}\hat{H}\hat{H}\hat{H}$, $\hat{\chi}\hat{\chi}\hat{\chi}\hat{\chi}$	$\hat{H}\hat{H}\hat{\chi}\hat{\chi}$, $\hat{H}\hat{H}\hat{\phi}^+\hat{\phi}^-$, $\hat{\chi}\hat{\chi}\hat{\phi}^+\hat{\phi}^-$	$\hat{\phi}^+\hat{\phi}^-\hat{\phi}^+\hat{\phi}^-$
C	$-\frac{3}{4s^2}\frac{M_{\mathrm{H}}^2}{M_{\mathrm{W}}^2}$	$-\frac{1}{4s^2}\frac{M_{\mathrm{H}}^2}{M_{\mathrm{W}}^2}$	$-\frac{1}{2s^2}\frac{M_{\mathrm{H}}^2}{M_{\mathrm{W}}^2}$

(A.16)

- $\hat{S}\hat{S}\hat{S}$ coupling:



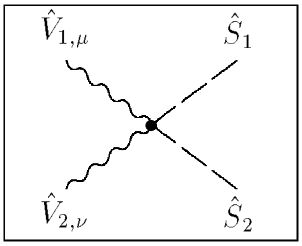
$$= ieC\Big[1 + \frac{\delta M_{\mathrm{H}}^2}{M_{\mathrm{H}}^2} + \frac{e}{2s}\frac{\delta t}{M_{\mathrm{W}} M_{\mathrm{H}}^2} + \delta Z_{\hat{H}}\Big] \tag{A.17}$$

with the actual values of $\hat{S}_1$, $\hat{S}_2$, $\hat{S}_3$ and C

$\hat{S}_1\hat{S}_2\hat{S}_3$	$\hat{H}\hat{H}\hat{H}$	$\hat{H}\hat{\chi}\hat{\chi}, \hat{H}\hat{\phi}^+\hat{\phi}^-$
C	$-\frac{3}{2s}\frac{M_{\mathrm{H}}^2}{M_{\mathrm{W}}}$	$-\frac{1}{2s}\frac{M_{\mathrm{H}}^2}{M_{\mathrm{W}}}$

(A.18)

- $\hat{V}\hat{V}\hat{S}\hat{S}$ coupling:



$$= ie^2 g_{\mu\nu} C (1 + \delta Z_{\hat{H}}) \tag{A.19}$$

with the actual values of $\hat{V}_1$, $\hat{V}_2$, $\hat{S}_1$, $\hat{S}_2$ and C

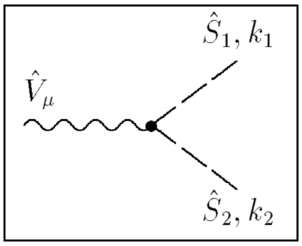
$\hat{V}_1\hat{V}_2\hat{S}_1\hat{S}_2$	$\hat{Z}\hat{Z}\hat{H}\hat{H}$ $\hat{Z}\hat{Z}\hat{\chi}\hat{\chi}$	$\hat{W}^+\hat{W}^-\hat{H}\hat{H}, \hat{W}^+\hat{W}^-\hat{\phi}^+\hat{\phi}^-$ $\hat{W}^+\hat{W}^-\hat{\chi}\hat{\chi}$	$\hat{A}\hat{A}\hat{\phi}^+\hat{\phi}^-$	$\hat{Z}\hat{A}\hat{\phi}^+\hat{\phi}^-$	$\hat{Z}\hat{Z}\hat{\phi}^+\hat{\phi}^-$
C	$\frac{1}{2c^2s^2}$	$\frac{1}{2s^2}$	2	$-\frac{c^2-s^2}{cs}$	$\frac{(c^2-s^2)^2}{2c^2s^2}$

and

$\hat{V}_1\hat{V}_2\hat{S}_1\hat{S}_2$	$\hat{W}^\pm\hat{A}\hat{\phi}^\mp\hat{H}$	$\hat{W}^\pm\hat{A}\hat{\phi}^\mp\hat{\chi}$	$\hat{W}^\pm\hat{Z}\hat{\phi}^\mp\hat{H}$	$\hat{W}^\pm\hat{Z}\hat{\phi}^\mp\hat{\chi}$
C	$-\frac{1}{2s}$	$\mp\frac{i}{2s}$	$-\frac{1}{2c}$	$\mp\frac{i}{2c}$

(A.20)

- $\hat{V}\hat{S}\hat{S}$ coupling:



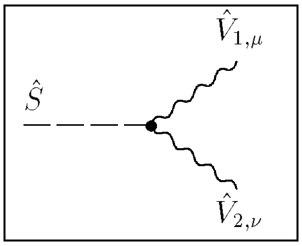
$$= ieC(k_1 - k_2)_\mu(1 + \delta Z_{\hat{H}}) \tag{A.21}$$

with the actual values of $\hat{V}$, $\hat{S}_1$, $\hat{S}_2$ and C

$\hat{V}\hat{S}_1\hat{S}_2$	$\hat{Z}\hat{\chi}\hat{H}$	$\hat{A}\hat{\phi}^+\hat{\phi}^-$	$\hat{Z}\hat{\phi}^+\hat{\phi}^-$	$\hat{W}^\pm\hat{\phi}^\mp\hat{H}$	$\hat{W}^\pm\hat{\phi}^\mp\hat{\chi}$
C	$-\frac{i}{2cs}$	-1	$\frac{c^2-s^2}{2cs}$	$\mp\frac{1}{2s}$	$-\frac{i}{2s}$

(A.22)

- $\hat{S}\hat{V}\hat{V}$ coupling:



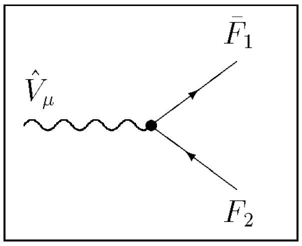
$$= ieg_{\mu\nu}C(1 + \delta Z_{\hat{H}}) \tag{A.23}$$

with the actual values of $\hat{S}$, $\hat{V}_1$, $\hat{V}_2$ and C

$\hat{S}\hat{V}_1\hat{V}_2$	$\hat{H}\hat{Z}\hat{Z}$	$\hat{H}\hat{W}^+\hat{W}^-$	$\hat{\phi}^\pm\hat{W}^\mp\hat{A}$	$\hat{\phi}^\pm\hat{W}^\mp\hat{Z}$
C	$\frac{1}{c^2 s}M_\mathrm{W}$	$\frac{1}{s}M_\mathrm{W}$	$-M_\mathrm{W}$	$-\frac{s}{c}M_\mathrm{W}$

(A.24)

- $\hat{V}\bar{F}F$ coupling:



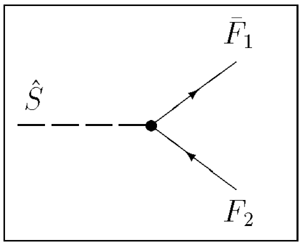
$$= ie\gamma_\mu\Big[C_\mathrm{L}\omega_-(1 + \delta Z^\mathrm{L}_{F_1}) + C_\mathrm{R}\omega_+\Big(1 + \tfrac{1}{2}(\delta Z^\mathrm{R}_{F_1} + \delta Z^\mathrm{R}_{F_2})\Big)\Big] \tag{A.25}$$

with the actual values of $\hat{V}$, $\bar{F}_1$, F_2 and C_R, C_L

$\hat{V}\bar{F}_1F_2$	$\hat{A}\bar{f}f$	$\hat{Z}\bar{f}f$	$\hat{W}^+\bar{f}_uf_d, \hat{W}^-\bar{f}_df_u$
C_L	$-Q_f$	$\frac{I^3_{\mathrm{W},f}-s^2Q_f}{cs}$	$\frac{1}{\sqrt{2}s}$
C_R	$-Q_f$	$-\frac{s}{c}Q_f$	0

(A.26)

- $\hat{S}\bar{F}F$ coupling:



$$= ie\Big[C_{\mathrm{L}}\omega_-\Big(1 + \frac{\delta m_{F_1}}{m_{F_1}} + \frac{1}{2}\delta Z^{\mathrm{L}}_{F_1} + \frac{1}{2}\delta Z^{\mathrm{R}}_{F_1}\Big) + C_{\mathrm{R}}\omega_+\Big(1 + \frac{\delta m_{F_2}}{m_{F_2}} + \frac{1}{2}\delta Z^{\mathrm{L}}_{F_1} + \frac{1}{2}\delta Z^{\mathrm{R}}_{F_2}\Big)\Big] \qquad \text{(A.27)}$$

with the actual values of $\hat{S}$, $\bar{F}_1$, F_2 and C_{R}, C_{L}

$\hat{S}\bar{F}_1F_2$	$\hat{H}\bar{f}f$	$\hat{\chi}\bar{f}f$	$\hat{\phi}^+\bar{f}_uf_d$	$\hat{\phi}^-\bar{f}_df_u$
C_{L}	$-\frac{1}{2s}\frac{m_f}{M_{\mathrm{W}}}$	$-i\frac{1}{2s}2I^3_{\mathrm{W},f}\frac{m_f}{M_{\mathrm{W}}}$	$+\frac{1}{\sqrt{2}s}\frac{m_{f_u}}{M_{\mathrm{W}}}$	$-\frac{1}{\sqrt{2}s}\frac{m_{f_d}}{M_{\mathrm{W}}}$
C_{R}	$-\frac{1}{2s}\frac{m_f}{M_{\mathrm{W}}}$	$i\frac{1}{2s}2I^3_{\mathrm{W},f}\frac{m_f}{M_{\mathrm{W}}}$	$-\frac{1}{\sqrt{2}s}\frac{m_{f_d}}{M_{\mathrm{W}}}$	$+\frac{1}{\sqrt{2}s}\frac{m_{f_u}}{M_{\mathrm{W}}}$

(A.28)

Note that in contrast to the conventional formalism no counterterms are needed for the $\hat{Z}\hat{A}\hat{H}\hat{H}$, $\hat{Z}\hat{A}\hat{\chi}\hat{\chi}$, $\hat{A}\hat{\chi}\hat{H}$ and $\hat{H}\hat{Z}\hat{A}$ couplings.

We now consider the Feynman rules for vertices containing quantum fields. We treat these vertices in lowest order, i.e. we do not list the counterterms explicitly. As mentioned above, all lowest-order vertices involving fermions have the usual form. Since the gauge-fixing term (2.39) is quadratic in the quantum fields, apart from vertices involving ghost fields only vertices containing exactly two quantum fields differ from the conventional ones. Thus, the other vertices involving quantum fields have in lowest order the same form as the pure background-field vertices given above. Their insertions can be obtained from the ones listed for the pure background-field vertices by forming all possible combinations of quantum and background fields, e.g. one infers $\hat{W}^+W^-AZ, W^+\hat{W}^-AZ, W^+W^-\hat{A}Z$ and $W^+W^-A\hat{Z}$ as the possible insertions for the $\hat{V}VVV$ coupling corresponding to $\hat{W}^+\hat{W}^-\hat{A}\hat{Z}$.

Some of the vertices containing two quantum fields also have the usual Feynman rules. These are $\hat{V}\hat{V}SS$, $\hat{S}\hat{S}VV$, $\hat{V}SS$ and $\hat{S}VV$. In the following we list those couplings for which the generic form or actual insertion differs from the ones in the conventional formalism. Note that some of the insertions appearing in the conventional couplings have no counterpart here. We list only the non-vanishing insertions.

- $\hat{V}\hat{V}VV$ coupling:
  The $\hat{V}\hat{V}VV$ coupling has two generic forms depending on the actual insertions,

$$\hat{V}_{1,\mu}\hat{V}_{2,\nu}V_{3,\rho}V_{4,\sigma} : ie^2C\Big[2g_{\mu\nu}g_{\rho\sigma} - g_{\mu\sigma}g_{\nu\rho}(1 - \frac{1}{\xi_Q}) - g_{\mu\rho}g_{\nu\sigma}(1 - \frac{1}{\xi_Q})\Big] \qquad \text{(A.29)}$$

for the insertions

$$\begin{array}{|r||c|c|c|c|}\hline \hat V_1\hat V_2V_3V_4 & \hat W^\pm\hat W^\pm W^\mp W^\mp & \hat Z\hat Z W^+W^- & \hat A\hat Z W^+W^- & \hat A\hat A W^+W^- \\ & & \hat W^+\hat W^- ZZ & \hat W^+\hat W^- AZ & \hat W^+\hat W^- AA \\ \hline C & \frac{1}{s^2} & -\frac{c^2}{s^2} & \frac{c}{s} & -1 \\ \hline \end{array} \tag{A.30}$$

and

$$\hat V_{1,\mu}\hat V_{2,\nu}V_{3,\rho}V_{4,\sigma} : ie^2C\Big[2g_{\mu\rho}g_{\nu\sigma} - g_{\mu\nu}g_{\rho\sigma} - g_{\mu\sigma}g_{\nu\rho}(1+\frac{1}{\xi_Q})\Big] \tag{A.31}$$

for the insertions

$$\begin{array}{|r||c|c|c|c|}\hline \hat V_1\hat V_2V_3V_4 & \hat W^+\hat W^- W^+W^- & \hat W^\pm\hat Z W^\mp Z & \hat W^\pm\hat A W^\mp Z & \hat W^\pm\hat A W^\mp A \\ & & & \hat W^\pm\hat Z W^\mp A & \\ \hline C & \frac{1}{s^2} & -\frac{c^2}{s^2} & \frac{c}{s} & -1 \\ \hline \end{array} \tag{A.32}$$

- $\hat V VV$ coupling:

$$\begin{aligned} &\hat V_{1,\mu}(k_1)V_{2,\nu}(k_2)V_{3,\rho}(k_3) : \\ &\quad -ieC\Big[g_{\nu\rho}(k_3-k_2)_\mu + g_{\mu\nu}(k_2-k_1+\frac{k_3}{\xi_Q})_\rho + g_{\rho\mu}(k_1-k_3-\frac{k_2}{\xi_Q})_\nu\Big] \end{aligned} \tag{A.33}$$

with the actual values of $\hat V_1$, V_2, V_3 and C

$$\begin{array}{|r||c|c|}\hline \hat V_1V_2V_3 & \hat A W^+W^-, \hat W^+W^-A & \hat Z W^+W^-, \hat W^+W^-Z \\ & \hat W^- A W^+ & \hat W^- Z W^+ \\ \hline C & 1 & -\frac{c}{s} \\ \hline \end{array} \tag{A.34}$$

- $\hat S\hat S SS$ coupling:

$$\hat S_1\hat S_2S_3S_4 : ie^2C \tag{A.35}$$

with the actual values of $\hat S_1$, $\hat S_2$, S_3, S_4 and C

$$\begin{array}{|r||c|c|c|c|}\hline \hat S_1\hat S_2S_3S_4 & \hat H\hat H HH & \hat H\hat H\chi\chi & \hat H\hat\chi H\chi & \hat\phi^+\hat\phi^- HH, \hat H\hat H\phi^+\phi^- \\ & \hat\chi\hat\chi\chi\chi & \hat\chi\hat\chi HH & & \hat\phi^+\hat\phi^-\chi\chi, \hat\chi\hat\chi\phi^+\phi^- \\ \hline C & -\frac{3}{4s^2}\frac{M_{\mathrm{H}}^2}{M_{\mathrm{W}}^2} & -\frac{1}{4s^2}\frac{M_{\mathrm{H}}^2}{M_{\mathrm{W}}^2} - \frac{\xi_Q}{2c^2s^2} & -\frac{1}{4s^2}\frac{M_{\mathrm{H}}^2}{M_{\mathrm{W}}^2} + \frac{\xi_Q}{4c^2s^2} & -\frac{1}{4s^2}\frac{M_{\mathrm{H}}^2}{M_{\mathrm{W}}^2} - \frac{\xi_Q}{2s^2} \\ \hline \end{array}$$

and

$\hat S_1 \hat S_2 S_3 S_4$	$\hat\phi^\pm \hat H \phi^\mp H$ $\hat\phi^\pm \hat\chi \phi^\mp \chi$	$\hat\phi^+ \hat\phi^- \phi^+ \phi^-$	$\hat\phi^\pm \hat\phi^\pm \phi^\mp \phi^\mp$	$\hat\phi^\pm \hat H \phi^\mp \chi$ $\hat\phi^\mp \hat\chi \phi^\pm H$
C	$-\frac{1}{4s^2}\frac{M_\mathrm{H}^2}{M_\mathrm{W}^2} + \frac{\xi_Q}{4s^2}$	$-\frac{1}{2s^2}\frac{M_\mathrm{H}^2}{M_\mathrm{W}^2} - \frac{\xi_Q}{4c^2 s^2}$	$-\frac{1}{2s^2}\frac{M_\mathrm{H}^2}{M_\mathrm{W}^2} + \frac{\xi_Q}{2c^2 s^2}$	$\mp \frac{i\xi_Q}{4c^2}$

(A.36)

- $\hat S S S$ coupling:

$$\hat S_1 S_2 S_3 : ieC \tag{A.37}$$

with the actual values of $\hat S_1$, S_2, S_3 and C

$\hat S_1 S_2 S_3$	$\hat H H H$	$\hat H \chi\chi$	$\hat\chi H \chi$	$\hat H \phi^+ \phi^-$
C	$-\frac{3}{2s}\frac{M_\mathrm{H}^2}{M_\mathrm{W}}$	$-\frac{1}{2s}\frac{M_\mathrm{H}^2}{M_\mathrm{W}} - \xi_Q \frac{M_\mathrm{W}}{c^2 s}$	$-\frac{1}{2s}\frac{M_\mathrm{H}^2}{M_\mathrm{W}} + \xi_Q \frac{M_\mathrm{W}}{2c^2 s}$	$-\frac{1}{2s}\frac{M_\mathrm{H}^2}{M_\mathrm{W}} - \xi_Q \frac{M_\mathrm{W}}{s}$

and

$\hat S_1 S_2 S_3$	$\hat\phi^\pm \phi^\mp H$	$\hat\phi^\pm \phi^\mp \chi$
C	$-\frac{1}{2s}\frac{M_\mathrm{H}^2}{M_\mathrm{W}} + \xi_Q \frac{M_\mathrm{W}}{2s}$	$\mp i\xi_Q M_\mathrm{W} \frac{s}{2c^2}$

(A.38)

- $\hat V V \hat S S$ coupling:

$$\hat V_{1,\mu} V_{2,\nu} \hat S_1 S_2 : ie^2 g_{\mu\nu} C \tag{A.39}$$

with the actual values of $\hat V_1$, V_2, $\hat S_1$, S_2 and C

$\hat V_1 V_2 \hat S_1 S_2$	$\hat Z Z \hat H H$ $\hat Z Z \hat\chi \chi$	$\hat W^\pm W^\mp \hat H H$ $\hat W^\pm W^\mp \hat\chi \chi$	$\hat W^\pm W^\mp \hat\phi^\mp \phi^\pm$	$\hat A A \hat\phi^\pm \phi^\mp$	$\hat Z A \hat\phi^\pm \phi^\mp$ $\hat A Z \hat\phi^\pm \phi^\mp$	$\hat Z Z \hat\phi^\pm \phi^\mp$
C	$\frac{1}{2c^2 s^2}$	$\frac{1}{2s^2}$	$\frac{1}{s^2}$	2	$-\frac{c^2-s^2}{cs}$	$\frac{(c^2-s^2)^2}{2c^2 s^2}$

and

$\hat V_1 V_2 \hat S_1 S_2$	$\hat W^\pm A \hat H \phi^\mp$ $\hat A W^\pm \hat\phi^\mp H$	$\hat W^\pm A \hat\chi \phi^\mp$ $\hat A W^\pm \hat\phi^\mp \chi$	$\hat W^\pm Z \hat\phi^\mp H$ $\hat Z W^\pm \hat H \phi^\mp$	$\hat W^\pm Z \hat\phi^\mp \chi$ $\hat Z W^\pm \hat\chi \phi^\mp$	$\hat W^\pm Z \hat H \phi^\mp$ $\hat Z W^\pm \hat\phi^\mp H$	$\hat W^\pm Z \hat\chi \phi^\mp$ $\hat Z W^\pm \hat\phi^\mp \chi$
C	$-\frac{1}{s}$	$\mp\frac{i}{s}$	$-\frac{1}{2cs^2}$	$\mp\frac{i}{2cs^2}$	$\frac{c^2-s^2}{2cs^2}$	$\pm i\frac{c^2-s^2}{2cs^2}$

and

$\hat V_1 V_2 \hat S_1 S_2$	$\hat W^\pm W^\mp \hat\chi H$ $\hat W^\mp W^\pm \hat H \chi$
C	$\pm\frac{i}{2s^2}$

(A.40)

- $V\hat{S}S$ coupling:

$$V_\mu \hat{S}_1(k_1) S_2(k_2) : ieC2k_{1,\mu} \tag{A.41}$$

with the actual values of V, $\hat{S}_1$, S_2 and C

$V\hat{S}_1S_2$	$Z\hat{\chi}H$	$Z\hat{H}\chi$	$A\hat{\phi}^\pm\phi^\mp$	$Z\hat{\phi}^\pm\phi^\mp$	$W^\pm\hat{\phi}^\mp H, W^\mp\hat{H}\phi^\pm$	$W^\pm\hat{\phi}^\mp\chi$	$W^\pm\hat{\chi}\phi^\mp$
C	$-\frac{i}{2cs}$	$\frac{i}{2cs}$	∓ 1	$\pm\frac{c^2-s^2}{2cs}$	$\mp\frac{1}{2s}$	$-\frac{i}{2s}$	$\frac{i}{2s}$

(A.42)

- $S\hat{V}V$ coupling:

$$S\hat{V}_{1,\mu}V_{2,\nu} : ieg_{\mu\nu}C \tag{A.43}$$

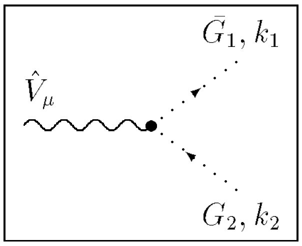
with the actual values of S, $\hat{V}_1$, V_2 and C

$S\hat{V}_1V_2$	$H\hat{Z}Z$	$H\hat{W}^\pm W^\mp$	$\chi\hat{W}^\pm W^\mp$	$\phi^\pm\hat{W}^\mp A$	$\phi^\pm\hat{W}^\mp Z$	$\phi^\pm\hat{Z}W^\mp$
C	$\frac{1}{c^2 s}M_\mathrm{W}$	$\frac{1}{s}M_\mathrm{W}$	$\mp\frac{i}{s}M_\mathrm{W}$	$-2M_\mathrm{W}$	$\frac{c^2-s^2}{cs}M_\mathrm{W}$	$-\frac{1}{cs}M_\mathrm{W}$

(A.44)

Next, we list the Feynman rules for couplings involving ghost fields. As above, pure quantum-field vertices have the usual Feynman rules.

- $\hat{V}\bar{G}G$ coupling:



$$= ie(k_1 - k_2)_\mu C \tag{A.45}$$

with the actual values of $\hat{V}$, $\bar{G}_1$, G_2 and C

$\hat{V}\bar{G}_1G_2$	$\hat{A}\bar{u}^\pm u^\pm, \hat{W}^\pm\bar{u}^A u^\mp, \hat{W}^\mp\bar{u}^\mp u^A$	$\hat{Z}\bar{u}^\pm u^\pm, \hat{W}^\pm\bar{u}^Z u^\mp, \hat{W}^\mp\bar{u}^\mp u^Z$
C	± 1	$\mp\frac{c}{s}$

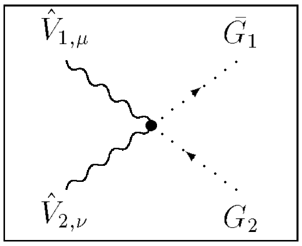
(A.46)

- $V\bar{G}G$ coupling:

$$V_\mu \bar{G}_1 G_2 : iek_{1,\mu}C \tag{A.47}$$

with the actual values of V, $\bar{G}_1$, G_2 and C as given in (A.46).

- $\hat{V}\hat{V}\bar{G}G$ coupling:



$$= ie^2 g_{\mu\nu} C \tag{A.48}$$

with the actual values of $\hat{V}_1$, $\hat{V}_2$, $\bar{G}_1$, G_2 and C

$\hat{V}_1\hat{V}_2\bar{G}_1G_2$	$\hat{W}^\pm\hat{W}^\pm\bar{u}^\pm u^\mp$	$\hat{W}^+\hat{W}^-\bar{u}^A u^A$ $\hat{A}\hat{A}\bar{u}^\pm u^\pm$	$\hat{W}^+\hat{W}^-\bar{u}^A u^Z$ $\hat{W}^+\hat{W}^-\bar{u}^Z u^A$ $\hat{A}\hat{Z}\bar{u}^\pm u^\pm$	$\hat{W}^+\hat{W}^-\bar{u}^Z u^Z$ $\hat{Z}\hat{Z}\bar{u}^\pm u^\pm$
C	$-\frac{2}{s^2}$	2	$-2\frac{c}{s}$	$2\frac{c^2}{s^2}$

(A.49a)

and

$\hat{V}_1\hat{V}_2\bar{G}_1G_2$	$\hat{W}^+\hat{W}^-\bar{u}^\pm u^\pm$	$\hat{A}\hat{W}^\pm\bar{u}^\pm u^A$ $\hat{A}\hat{W}^\pm\bar{u}^A u^\mp$	$\hat{Z}\hat{W}^\pm\bar{u}^\pm u^A$, $\hat{A}\hat{W}^\pm\bar{u}^\pm u^Z$ $\hat{Z}\hat{W}^\pm\bar{u}^A u^\mp$, $\hat{A}\hat{W}^\pm\bar{u}^Z u^\mp$	$\hat{Z}\hat{W}^\pm\bar{u}^\pm u^Z$ $\hat{Z}\hat{W}^\pm\bar{u}^Z u^\mp$
C	$\frac{1}{s^2}$	-1	$\frac{c}{s}$	$-\frac{c^2}{s^2}$

(A.49b)

- $\hat{V}V\bar{G}G$ coupling:

$$\hat{V}_{1,\mu}V_{2,\nu}\bar{G}_1G_2 : ie^2 g_{\mu\nu} C \tag{A.50}$$

with the actual values of $\hat{V}_1$, $\hat{V}_2$, $\bar{G}_1$, G_2 and C

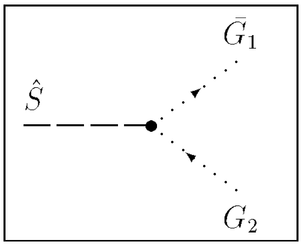
$\hat{V}_1V_2\bar{G}_1G_2$	$\hat{W}^\pm W^\pm\bar{u}^\pm u^\mp$	$\hat{W}^\pm W^\mp\bar{u}^A u^A$ $\hat{A}A\bar{u}^\pm u^\pm$	$\hat{W}^\pm W^\mp\bar{u}^A u^Z$, $\hat{A}Z\bar{u}^\pm u^\pm$ $\hat{W}^\pm\hat{W}^\mp\bar{u}^Z u^A$, $\hat{Z}A\bar{u}^\pm u^\pm$	$\hat{W}^\pm W^\mp\bar{u}^Z u^Z$ $\hat{Z}Z\bar{u}^\pm u^\pm$
C	$-\frac{1}{s^2}$	1	$-\frac{c}{s}$	$\frac{c^2}{s^2}$

and

$\hat{V}_1V_2\bar{G}_1G_2$	$\hat{W}^\pm W^\mp\bar{u}^\pm u^\pm$	$\hat{A}W^\pm\bar{u}^\pm u^A$ $\hat{W}^\pm A\bar{u}^A u^\mp$	$\hat{Z}W^\pm\bar{u}^\pm u^A$, $\hat{A}W^\pm\bar{u}^\pm u^Z$ $\hat{W}^\pm Z\bar{u}^A u^\mp$, $\hat{W}^\pm A\bar{u}^Z u^\mp$	$\hat{Z}W^\pm\bar{u}^\pm u^Z$ $\hat{W}^\pm Z\bar{u}^Z u^\mp$
C	$\frac{1}{s^2}$	-1	$\frac{c}{s}$	$-\frac{c^2}{s^2}$

(A.51)

- $\hat{S}\bar{G}G$ coupling:



$$= ieC\xi_Q \tag{A.52}$$

with the actual values of $\hat{S}$, $\bar{G}_1$, G_2 and C

$\hat{S}\bar{G}_1G_2$	$\hat{H}\bar{u}^Zu^Z$	$\hat{H}\bar{u}^\pm u^\pm$	$\hat{\phi}^\pm\bar{u}^\pm u^A$, $\hat{\phi}^\pm\bar{u}^A u^\mp$	$\hat{\phi}^\pm\bar{u}^\pm u^Z$, $\hat{\phi}^\pm\bar{u}^Z u^\mp$
C	$-\frac{1}{c^2s}M_\mathrm{W}$	$-\frac{1}{s}M_\mathrm{W}$	M_W	$\frac{s}{c}M_\mathrm{W}$

(A.53)

- $S\bar{G}G$ coupling:

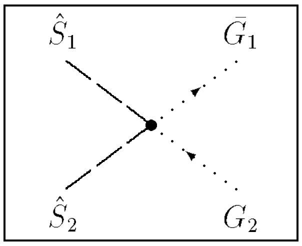
$$S\bar{G}_1G_2 : ieC\xi_Q \tag{A.54}$$

with the actual values of S, $\bar{G}_1$, G_2 and C

$S\bar{G}_1G_2$	$H\bar{u}^Zu^Z$	$H\bar{u}^\pm u^\pm$	$\chi\bar{u}^\pm u^\pm$	$\phi^\pm\bar{u}^\pm u^A$	$\phi^\pm\bar{u}^\pm u^Z$	$\phi^\pm\bar{u}^Z u^\mp$
C	$-\frac{1}{2c^2s}M_\mathrm{W}$	$-\frac{1}{2s}M_\mathrm{W}$	$\mp\frac{i}{2s}M_\mathrm{W}$	M_W	$-\frac{c^2-s^2}{2cs}M_\mathrm{W}$	$\frac{1}{2cs}M_\mathrm{W}$

(A.55)

- $\hat{S}\hat{S}\bar{G}G$ coupling:



$$= ie^2C\xi_Q \tag{A.56}$$

with the actual values of $\hat{S}_1$, $\hat{S}_2$, $\bar{G}_1$, G_2 and C

$\hat{S}_1\hat{S}_2\bar{G}_1G_2$	$\hat{H}\hat{H}\bar{u}^Zu^Z$ $\hat{\chi}\hat{\chi}\bar{u}^Zu^Z$	$\hat{H}\hat{H}\bar{u}^\pm u^\pm$, $\hat{\chi}\hat{\chi}\bar{u}^\pm u^\pm$ $\hat{\phi}^+\hat{\phi}^-\bar{u}^\pm u^\pm$	$\hat{\phi}^+\hat{\phi}^-\bar{u}^Au^A$	$\hat{\phi}^+\hat{\phi}^-\bar{u}^Au^Z$ $\hat{\phi}^+\hat{\phi}^-\bar{u}^Zu^A$
C	$-\frac{1}{2c^2s^2}$	$-\frac{1}{2s^2}$	-2	$\frac{c^2-s^2}{cs}$

and

$\hat{S}_1\hat{S}_2\bar{G}_1G_2$	$\hat{\phi}^+\hat{\phi}^-\bar{u}^Zu^Z$	$\hat{H}\hat{\phi}^\pm\bar{u}^\pm u^A$ $\hat{\phi}^\pm\hat{H}\bar{u}^Au^\mp$	$\hat{\chi}\hat{\phi}^\pm\bar{u}^\pm u^A$ $\hat{\phi}^\pm\hat{\chi}\bar{u}^Au^\mp$	$\hat{H}\hat{\phi}^\pm\bar{u}^\pm u^Z$ $\hat{\phi}^\pm\hat{H}\bar{u}^Zu^\mp$	$\hat{\chi}\hat{\phi}^\pm\bar{u}^\pm u^Z$ $\hat{\phi}^\pm\chi\bar{u}^Zu^\mp$
C	$-\frac{(c^2-s^2)^2}{2c^2s^2}$	$\frac{1}{2s}$	$\mp\frac{i}{2s}$	$\frac{1}{2c}$	$\mp i\frac{1}{2c}$

(A.57)

- $\hat{S}S\bar{G}G$ coupling:

$$\hat{S}_1 S_2 \bar{G}_1 G_2 : ie^2 C \xi_Q \tag{A.58}$$

with the actual values of $\hat{S}_1$, S_2, $\bar{G}_1$, G_2 and C

$\hat{S}_1S_2\bar{G}_1G_2$	$\hat{H}H\bar{u}^Z u^Z$ $\hat{\chi}\chi\bar{u}^Z u^Z$	$\hat{H}H\bar{u}^\pm u^\pm$ $\hat{\chi}\chi\bar{u}^\pm u^\pm$	$\hat{H}\chi\bar{u}^\pm u^\pm$ $\hat{\chi}H\bar{u}^\mp u^\mp$	$\hat{\phi}^\pm\phi^\mp\bar{u}^A u^A$	$\hat{\phi}^\pm\phi^\mp\bar{u}^A u^Z$ $\hat{\phi}^\pm\phi^\mp\bar{u}^Z u^A$	$\hat{\phi}^\pm\phi^\mp\bar{u}^Z u^Z$
C	$-\frac{1}{4c^2s^2}$	$-\frac{1}{4s^2}$	$\mp\frac{i}{4s^2}$	-1	$\frac{c^2-s^2}{2cs}$	$-\frac{(c^2-s^2)^2}{4c^2s^2}$

and

$\hat{S}_1S_2\bar{G}_1G_2$	$\hat{H}\phi^\pm\bar{u}^\pm u^A$ $\hat{\phi}^\pm H\bar{u}^A u^\mp$	$\hat{\chi}\phi^\pm\bar{u}^\pm u^A$ $\hat{\phi}^\pm\chi\bar{u}^A u^\mp$	$\hat{H}\phi^\pm\bar{u}^\pm u^Z$ $\hat{\phi}^\pm H\bar{u}^Z u^\mp$	$\hat{H}\phi^\pm\bar{u}^Z u^\mp$ $\hat{\phi}^\pm H\bar{u}^\pm u^Z$	$\hat{\chi}\phi^\pm\bar{u}^\pm u^Z$ $\hat{\phi}^\pm\chi\bar{u}^Z u^\mp$	$\hat{\chi}\phi^\pm\bar{u}^Z u^\mp$ $\hat{\phi}^\pm\chi\bar{u}^\pm u^Z$
C	$\frac{1}{2s}$	$\mp\frac{i}{2s}$	$-\frac{c^2-s^2}{4cs^2}$	$\frac{1}{4cs^2}$	$\pm i\frac{c^2-s^2}{4cs^2}$	$\mp\frac{i}{4cs^2}$

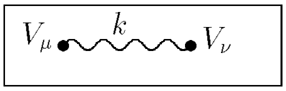
and

$\hat{S}_1S_2\bar{G}_1G_2$	$\hat{\phi}^\pm\phi^\mp\bar{u}^\pm u^\pm$
C	$-\frac{1}{2s^2}$

(A.59)

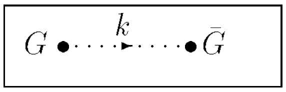
Finally, we give the quantum-field propagators:

- gauge bosons $V = A,\ Z,\ W$ $(M_\gamma^2 = 0)$



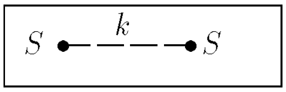
$$= -i\left[\frac{g_{\mu\nu}}{k^2 - M_V^2} - \frac{(1-\xi_Q)k_\mu k_\nu}{(k^2 - M_V^2)(k^2 - \xi_Q M_V^2)}\right] \tag{A.60}$$

- Faddeev–Popov ghosts $G = u^A,\ u^Z,\ u^\pm$ $(M_{u^A}^2 = 0,\ M_{u^Z}^2 = \xi_Q M_\mathrm{Z}^2,\ M_{u^\pm}^2 = \xi_Q M_\mathrm{W}^2)$



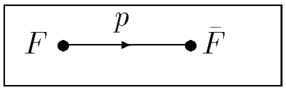
$$= \frac{i}{k^2 - M_G^2} \tag{A.61}$$

- scalar fields $S = H,\ \chi,\ \phi$ $(M_\chi^2 = \xi_Q M_\mathrm{Z}^2,\ M_\phi^2 = \xi_Q M_\mathrm{W}^2)$



$$= \frac{i}{k^2 - M_S^2} \tag{A.62}$$

- fermion fields $F = f$



$$= \frac{i(\not{p} + m_F)}{p^2 - m_F^2} \tag{A.63}$$

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List of publications

research papers

- Computeralgebraic calculation of 2-loop self-energies in the electroweak Standard Model,
  G. Weiglein, R. Mertig, R. Scharf and M. Böhm,
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Zusammenfassung

Das Glashow–Salam–Weinberg (GSW)-Modell der elektroschwachen Wechselwirkung erlaubt eine umfassende vereinheitlichte Beschreibung der elektromagnetischen und der schwachen Wechselwirkung. Es basiert auf dem Prinzip der Eichinvarianz, das die Form der Wechselwirkung mit den inneren Symmetrien verknüpft, gemäß derer sich die Elementarteilchen einordnen lassen. Das GSW-Modell ist in seiner theoretischen Formulierung konsistent und innerhalb der Meßgenauigkeit in Übereinstimmung mit allen relevanten experimentellen Ergebnissen. Aus diesem Grund wird es auch als elektroschwaches Standardmodell bezeichnet.

Trotz dieser spektakulären Erfolge ist das Standardmodell keineswegs in allen seinen Bestandteilen experimentell verifiziert. Während es mittlerweile erste experimentelle Hinweise auf das im Standardmodell postulierte top-Quark gibt, ist weiterhin nur sehr wenig über die Struktur des Higgs-Sektors bekannt, der die Erzeugung der Teilchenmassen im Standardmodell gewährleistet und für die innere Konsistenz des Modells von entscheidender Bedeutung ist.

Einen indirekten Zugang zu den experimentell noch nicht gesicherten Bestandteilen des Modells erlauben die Präzisionstests des Standardmodells, die gegenwärtig vor allem am LEP-Beschleuniger des CERN durchgeführt werden. Diese Experimente sind sensitiv für die Quantenkorrekturen der Theorie, in welche die gesamte Struktur des Modells eingeht. Aus der bisher beobachteten Übereinstimmung der Präzisionsexperimente mit den Vorhersagen des Standardmodells ergeben sich Einschränkungen für mögliche physikalische Phänomene jenseits des Standardmodells. Falls in zukünftigen Experimenten allerdings eine signifikante Abweichung festgestellt wird, könnte dies einen Hinweis darauf liefern, daß das Standardmodell nur den Grenzfall einer zugrundeliegenden allgemeineren Theorie darstellt.

Die Präzisionstests des Standardmodells erfordern eine sehr genaue Auswertung der Quantenkorrekturen des Modells. Diese sogenannten Strahlungskorrekturen werden durch störungstheoretische Auswertung des Modells gewonnen. Während die Techniken für die Berechnung von Ein-Schleifen-Korrekturen gut ausgearbeitet sind, gelang bisher noch keine vollständige Behandlung von irreduziblen Zwei-Schleifen-Korrekturen.

Ein weiteres Problem bei der Auswertung von Strahlungskorrekturen ist die Eichabhängigkeit der Greenschen Funktionen. Diese rührt daher, daß die Quantisierung der Theorie eine Fixierung der Eichung und somit eine Brechung der Eichinvarianz verlangt. Die Eichinvarianz kann erst am Ende der Berechnung von S-Matrixelementen durch Projektion auf physikalische Freiheitsgrade wiederhergestellt werden. In praktischen Rechnungen führt die Eichabhängigkeit der Greenschen Funktionen insbesondere dann zu Schwierigkeiten, wenn eine Aufsummation von Beiträgen höherer Ordnung vorgenommen wird oder wenn Formfaktoren untersucht werden, die nur einen Teil eines kompletten S-Matrixelements beinhalten.

Einen wichtigen Schritt hin zu vollständigen Berechnungen von Zwei-Schleifen-Korrekturen stellt die Auswertung der Zwei-Schleifen-Selbstenergien dar. Diese universellen Korrekturen sind ein zentraler Bestandteil aller Zwei-Schleifen-Rechnungen. Bisher gelang die Berechnung von Zwei-Schleifen-Selbstenergien im Standardmodell nur im Grenzfall schwerer Massen des top-Quarks oder des Higgs-Bosons. Da man für diese Teilchen allerdings Massen in der gleichen Größenordnung wie die der Eichbosonen erwartet, erfordert eine realistische Auswertung die Berechnung von Zwei-Schleifen-Selbstenergien ohne derartige Näherungen. Die Probleme einer solchen Rechnung liegen zum einen in der enormen Zahl der beitragenden Feynman-Diagramme begründet, zum anderen in der komplizierten Struktur der Zwei-Schleifen-Integrale.

In dieser Arbeit wurde gezeigt, daß alle Zwei-Schleifen-Selbstenergien in massiven Eichtheorien auf einen minimalen Satz von skalaren Standardintegralen zurückgeführt werden können. Zu diesem Zweck wurde eine algebraische Methode erarbeitet, welche die Tensorzerlegung aller Zwei-Schleifen-Integrale vom Selbstenergie-Typ gestattet. Zusätzlich wurden die Symmetrien der skalaren Integrale und bestimmte Integralrelationen verwendet.

Das vorgestellte algebraische Verfahren ist algorithmisch und erlaubt den Einsatz von Computeralgebra. Hierfür wurde das Programm *TwoCalc* erstellt, das die algebraische Berechnung von Zwei-Schleifen-Selbstenergien vollautomatisch durchführt. Die Resultate gelten für beliebige Werte des invarianten Impulses, aller Teilchenmassen, der Raum-Zeit-Dimension und der Eichparameter. An den algebraischen Resultaten lassen sich unmittelbar wichtige Eigenschaften der betrachteten Größe ablesen, wie deren Eichabhängigkeit oder die Gültigkeit von Beziehungen zwischen verschiedenen Selbstenergie-Beiträgen. Um dies zu demonstrieren, wurden durch explizite Rechnung Slavnov–Taylor-Relationen verifiziert, die mehrere tausend Feynman-Diagramme miteinander verknüpfen.

Als weitere Anwendungen der algebraischen Methode wurden die Zwei-Schleifen-Selbstenergie des Photons in der QED und die Beiträge der leichten Fermionen zu den Eichboson-Selbstenergien im Standardmodell berechnet. Dies wurde in einer allgemeinen R_ξ-Eichung durchgeführt, wobei im Standardmodell sowohl der konventionelle Formalismus als auch die Background-Field-Methode verwendet wurde. Die Eichabhängigkeit der berechneten Größen wurde diskutiert.

Mit Hilfe des hier beschriebenen Verfahrens wird das Problem der Berechnung beliebiger Zwei-Schleifen-Selbstenergien, die im allgemeinen mehrere tausend Feynman-

Diagramme beinhalten, darauf reduziert, lediglich drei verschiedene Typen von skalaren Zwei-Schleifen-Integralen auszuarbeiten. Da mittlerweile geeignete Resultate für diese Integrale in der Literatur vorliegen, können nunmehr die Beiträge aller Zwei-Schleifen-Selbstenergien im Standardmodell analysiert werden.

Im Hinblick auf die Probleme, die durch die Eichabhängigkeit von Greenschen Funktionen verursacht werden, wurden in den letzten Jahren verschiedene Vorschläge ausgearbeitet, die auf eine Abänderung der konventionellen Greenschen Funktionen abzielen. Durch eine Elimination der Eichparameterabhängigkeit wurde versucht, neue "Greensche Funktionen" mit verbesserten theoretischen Eigenschaften zu konstruieren. Hierbei erwies sich insbesondere die Pinch-Technik als ein Verfahren, das in Ein-Schleifen-Ordnung eichparameterunabhängige Größen liefert, die einfache Ward-Identitäten erfüllen sowie eine Reihe weiterer günstiger Eigenschaften besitzen, wie beispielsweise ein verbessertes Hochenergieverhalten.

In der vorliegenden Arbeit wurde eine Erweiterung der Pinch-Technik auf das Zwei-Schleifen-Niveau diskutiert. Dies wurde am Beispiel der Beiträge leichter Fermionen zu den Eichboson-Selbstenergien durchgeführt, und es wurde gezeigt, daß eine eichparameterunabhängige "Zwei-Schleifen-Selbstenergie" konstruiert werden kann. An diesem Beispiel werden allerdings auch die Probleme deutlich, die das Verfahren der Pinch-Technik beinhaltet. Die Elimination der Eichparameter kann in der Pinch-Technik nur auf Kosten einer komplizierten Umverteilung von Beiträgen zwischen verschiedenen Greenschen Funktionen und zwischen irreduziblen und reduziblen Graphen erreicht werden. Da die feldtheoretische Bedeutung eines derartigen Verfahrens nicht geklärt ist, ist die theoretische Interpretation der in der Pinch-Technik konstruierten Größen äußerst problematisch. Insbesondere konnte deren Prozeßunabhängigkeit bisher nicht bewiesen werden. Ein weiteres schwerwiegendes Problem ist die Tatsache, daß die Pinch-Technik keine Erklärung für die günstigen Eigenschaften ihrer "Greenschen Funktionen" liefert. Diese wurden lediglich für bestimmte Beispiele durch explizite Berechnung in Ein-Schleifen-Ordnung verifiziert.

Es wurde daher in dieser Arbeit ein völlig anderer Zugang gewählt, um die Eichabhängigkeit von Greenschen Funktionen zu untersuchen. Es handelt sich hierbei um die oben bereits erwähnte Background-Field-Methode (BFM). Während in der Pinch-Technik und ähnlichen Verfahren die Eichparameterabhängigkeit von Greenschen Funktionen im Mittelpunkt steht, beschäftigt sich die BFM direkt mit der Brechung der Eichinvarianz in der Lagrangedichte der Theorie. Durch die Aufspaltung des üblichen Eichfelds in ein Quantenfeld und ein Background-Feld gelingt es, die für die Quantisierung erforderliche Eichfixierung vorzunehmen ohne dabei die Eichinvarianz der aus der Lagrangedichte abgeleiteten effektiven Wirkung zu brechen.

Die aus dieser effektiven Wirkung abgeleiteten Greenschen Funktionen besitzen eine wohldefinierte feldtheoretische Bedeutung und können in allen Ordnungen der Störungstheorie direkt aus Feynmanregeln berechnet werden. Aus der Eichinvarianz der effektiven Wirkung ergeben sich QED-ähnliche Ward-Identitäten für die Greenschen Funktionen. Diese Relationen besitzen eine wesentlich einfachere Form als die Slavnov–Taylor-Identitäten, die im konventionellen Formalismus gelten.

In dieser Arbeit wurde ein Zusammenhang zwischen der Pinch-Technik und der BFM aufgezeigt. Im Rahmen der BFM wurden diejenigen Größen in der QCD und im Standardmodell berechnet, für die in der Literatur ein entsprechendes Resultat der Pinch-Technik vorliegt. Dabei wurde nachgewiesen, daß die Ergebnisse der Pinch-Technik in allen Fällen mit demjenigen Spezialfall der BFM übereinstimmen, bei dem der Quanteneichparameter ξ_Q gleich Eins gesetzt ist. Weiterhin konnte gezeigt werden, daß die für die "Greenschen Funktionen" der Pinch-Technik festgestellten günstigen Eigenschaften in der BFM unmittelbar aus den Ward-Identitäten folgen, also eine direkte Konsequenz der in der BFM vorliegenden Eichinvarianz sind. Die Ward-Identitäten und folglich auch die erwähnten Eigenschaften der Greenschen Funktionen gelten in der BFM für alle Werte von ξ_Q. Um dies zu illustrieren, wurde das Beispiel laufender Kopplungen behandelt, die durch Dyson-Summation von Selbstenergie-Beiträgen definiert sind. Es läßt sich aus den Ward-Identitäten zeigen, daß das asymptotische Verhalten der so definierten Kopplungen unabhängig von ξ_Q ist und genau dem der laufenden Kopplungen in den Renormierungsgruppengleichungen entspricht.

Diese Ergebnisse sind von Bedeutung für ein besseres Verständnis der Probleme, die mit der Eichabhängigkeit Greenscher Funktionen zusammenhängen. Insbesondere wurde ersichtlich, daß das entscheidende Kriterium für günstige Eigenschaften Greenscher Funktionen nicht deren Eichparameterunabhängigkeit ist, sondern die Gültigkeit von Ward-Identitäten, die aus der Eichinvarianz folgen. Weiterhin wurde deutlich, daß die innerhalb der Pinch-Technik festgestellten günstigen Eigenschaften keine bestimmte Form der Greenschen Funktionen festlegen. Die BFM liefert vielmehr einen unendlich großen Satz von Greenschen Funktionen, der durch ξ_Q parametrisiert wird und die gleichen positiven Eigenschaften besitzt. Diese Vieldeutigkeit, die im Rahmen der BFM ersichtlich wird, ist implizit auch in der Pinch-Technik und ähnlichen Verfahren enthalten. Sie entspricht dort der Freiheit, verschiedene Vorschriften für die Elimination der Eichparameter zu wählen. Diese Vieldeutigkeit muß in Anwendungen berücksichtigt werden, in denen Greensche Funktionen anstelle kompletter S-Matrixelemente verwendet werden.

Die Benutzung der BFM ist nicht nur für die eben besprochenen Untersuchungen von Vorteil. Praktische Berechnungen von S-Matrixelementen vereinfachen sich in der BFM beträchtlich. Die Freiheit, unabhängig von der Eichfixierung der Quantenfelder eine geeignete Eichung für die Background-Felder wählen zu können, läßt sich dazu benutzen, die Anzahl der benötigten Feynman-Diagramme drastisch zu reduzieren. Eine mögliche Eichung für die Background-Felder ist beispielsweise die unitäre Eichung. Auch die Berechnung der Schleifendiagramme vereinfacht sich in der BFM. Dies gilt vor allem, wenn die 't Hooft–Feynman-Eichung für die Quantenfelder gewählt wird.

Um die BFM im Standardmodell anwenden zu können, ist eine konsistente Renormierung erforderlich, die mit der Eichinvarianz der BFM in Einklang steht. Eine solche Renormierungsvorschrift wurde explizit ausgearbeitet, wobei zwei verschiedene Renormierungsschemata als Ausgangspunkt gewählt wurden, ein vollständiges und ein minimales on-shell-Schema. Aus der Eichsymmetrie ergeben sich Relationen zwischen den Renormierungskonstanten der Felder und denen der Parameter des Standardmodells, d.h. der elektrischen Ladung und der Teilchenmassen. Es wurde nachgewiesen, daß auch mit diesem reduzierten Satz von unabhängigen Renormierungskonstanten alle Greenschen Funktionen des Standardmodells endlich werden. Dies wurde in Ein-Schleifen-Ordnung durch explizite Berechnung der relevanten Größen verifiziert. Die Renormierungskonstanten aller physikalischen Parameter bleiben auch in der BFM unabhängig voneinander. Daher können alle on-shell-Bedingungen für die Parameter verlangt werden. Das on-shell-Schema ist somit verträglich mit den Symmetrien der BFM.

Die Eichinvarianz der BFM bewirkt, daß sich die Renormierung gegenüber dem konventionellen Formalismus stark vereinfacht. Die Counterterme in der BFM besitzen eine wesentlich einfachere Struktur. Insbesondere werden die verschiedenen Realisierungen einer generischen Kopplung alle in gleicher Weise renormiert. Wenn die Parameter der elektrischen Ladung und der Massen der Teilchen mit den entsprechenden physikalischen Größen identifiziert werden, erhält man für alle Eichbosonfelder eichunabhängige Renormierungskonstanten.

Im Anhang dieser Arbeit wurde ein vollständiger Satz von Feynmanregeln der BFM für das Standardmodell aufgelistet. Er enthält alle Counterterme, die für Ein-Schleifen-Rechnungen benötigt werden.

Die in dieser Arbeit entwickelten Methoden stellen einen Schritt hin zu einer verbesserten Analyse der Beiträge höherer Ordnung in der störungstheoretischen Behandlung massiver Eichtheorien dar. In den kommenden Jahren ist von den Messungen am Beschleuniger LEP2 eine deutliche Verringerung des Fehlers in der Bestimmung der Masse des W-Bosons zu erwarten. Die Auswertung von Korrekturen höherer Ordnung ist daher von besonderer Bedeutung für eine genauere Untersuchung der Beziehung, welche die Massen des W- und des Z-Bosons mit der elektromagnetischen Feinstrukturkonstante und der Fermi-Konstante verknüpft, die beide mit äußerst hoher Genauigkeit bekannt sind.