The Standard Model of Particle Physics

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Exercise sheet

Exercise 1: SU(N) **representation**

The generators T^a of the fundamental representation of the SU(N) are given by

$$T^a_{ij}\,, \qquad a = 1, \ldots, N^2-1\,, \qquad i,j = 1,\ldots,N\,.$$

They are Hermitian, $T^{a\dagger} = T^a$, traceless, $\mathrm{Tr}(T^a) = 0$, and normalized through

$$\mathrm{Tr}\left(T^a T^b\right) = \frac{1}{2}\delta^{ab}\,.$$

They satisfy the commutation and anti-commutation relation

$$\left[T^a, T^b\right] = i f_{abc} T^c\,, \tag{1}$$

$$\left\{T^a, T^b\right\} = \frac{1}{N}\,\delta^{ab}\,\mathbb{1}_{N\times N} + d_{abc}T^c\,, \tag{2}$$

which defines the total antisymmetric structure constants f_{abc} and the total symmetric symbols d_{abc} of the SU(N). The commutation relation, Eq. (1), is satisfied for all SU(N) representations, whereas Eq. (2) only holds for the fundamental representation.
Every complex $N \times N$ matrix M can be decomposed into a linear combination of these N^2-1 generators, with coefficients c_0, c_a, as follows:

$$M = c_0 \mathbb{1}_{N\times N} + \sum_{a=1}^{N^2-1} c_a T^a\,. \tag{3}$$

(a) Show that the Fierz identity of the SU(N),

$$T^a_{ij}T^a_{kl} \equiv \sum_{a=1}^{N^2-1} T^a_{ij}T^a_{kl} = \frac{1}{2}\delta_{il}\delta_{jk} - \frac{1}{2N}\delta_{ij}\delta_{kl}\,, \tag{4}$$

is a result of combining the completeness relation, eq. (3), with the tracelessness of T^a.

(b) Show that, independent of any specific representation of SU(N), that

$$C_2 = T^aT^a \equiv \sum_{a=1}^{N^2-1} T^aT^a$$

is a Casimir invariant, i.e. that $[C_2, T^a] = 0$ holds for all generators T^a.

(c) By using the hermiticity of the generators, show that f_{abc} and d_{abc} are real.

(d) Calculate the value of C_2 in the fundamental representation.

Exercise 2: Transformation of the covariant derivative

The covariant derivative

$$D_\mu = \partial_\mu + igA_\mu = \partial_\mu + igA_\mu^a T^a \ ,$$

is explicitly dependent on the chosen representation of the generators T^a of the gauge group. In this exercise, we consider the transformation of the covariant derivative and of the gauge field,

$$D'_\mu = UD_\mu U^{-1} \,, \qquad\qquad A'_\mu = UA_\mu U^{-1} - \frac{i}{g} U(\partial_\mu U^{-1})$$

where representation matrices $U = \exp(i\vartheta^a T^a)$ are given in the *fundamental* representation. With this, prove that the covariant derivative transforms like

$$D'_\mu = VD_\mu V^{-1}$$

for *any arbitray* representation V and calculate the transformation explicitly.
Hint: Use the Baker–Hausdorff formula

$$\mathrm{e}^B A \mathrm{e}^{-B} = \sum_{n=0}^{\infty} \frac{1}{n!} A_n \ ,$$

where $A_n = [B, A_{n-1}]$, $A_0 = A$ *and* $A = A_\mu$ *respectively* $A = \partial_\mu$ *and* $B = i\vartheta^a T^a$.

By starting with

$$D'_\mu = V\left(\partial_\mu + igA_\mu\right)V^{-1} = \partial_\mu + V(\partial_\mu V^{-1}) + igVA_\mu V^{-1} \ ,$$

you can transform the right-hand side in such a way that the transformation of the gauge fields A_μ *in the adjoint representation, with* $(T^a_{adj})_{bc} = (-if^a)_{bc}$*, appears explicitly. The structure constants* f^{abc} *are the same is introduced in exercise 3.*

Exercise 3: Symmetry breaking

We first analyze the Lagrangian of φ^4 theory for real fields $\varphi(x)$, given by

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\varphi\right)\left(\partial^\mu\varphi\right) - V(\varphi) \ , \quad V(\varphi) = -\frac{1}{2}\mu^2\varphi^2 + \frac{\lambda}{4}\varphi^4 \ .$$

where μ^2 and λ are constants of the potential and $\lambda > 0$.

(a) Find the trivial extremum $\langle\varphi\rangle_1$ and the non-trivial extremum $v \equiv \langle\varphi\rangle_2$ of the potential $V(\varphi)$ with respect to the field φ. The non-trivial extremum v is called the *vacuum expectation value* of the field φ. What condition must μ^2 fulfill that the non-trivial extremum v actually exists in the potential $V(\varphi)$? In this case, is it a global minimum or maximum of the potential?

(b) The Lagrangian is invariant under the *discrete* symmetry $\varphi \to -\varphi$. We assume that the non-trivial minimum v exists. In this case, the field can condensate into this new minimum and it can be expanded about the vacuum expectation value as

$$\varphi(x) = v + \sigma(x) \ ,$$

where $\sigma(x)$ is a small perturbation of the field near v. Rewrite the Lagrangian in terms of v, λ and $\sigma(x)$. Express the mass of the field $\sigma(x)$ through the original parameters of the potential and then rewrite the Lagrangian in terms of this mass and v, only.

We now consider a complex scalar field $\varphi(x)$ which couples both to itself and to a vector field $A_\mu(x)$, described by the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\varphi)\,(D^\mu\varphi)^* - V(\varphi)\ , \quad V(\varphi) = -\mu^2\varphi^*\varphi + \frac{\lambda}{2}\,(\varphi^*\varphi)^2 \ ,$$

where $D_\mu = \partial_\mu + igA_\mu$.

(c) Analogous to part (a), find the trivial and non-trivial extrema of the potential. Show that the Lagrangian is invariant under the local *continuous* $U(1)$ gauge transformation

$$\varphi(x) \longrightarrow e^{i\alpha(x)}\varphi(x)\ , \quad A_\mu(x) \longrightarrow A_\mu(x) - \frac{1}{g}\partial_\mu\alpha(x)\ .$$

What would happen with this symmetry if we would naively add a mass term $m_A^2 A_\mu A^\mu$ for the gauge field by hand to the Lagrangian? Interpret this finding.

(d) Expand the field φ about its non-trivial extremum v,

$$\varphi(x) = v + \frac{h(x) + iG(x)}{\sqrt{2}} \ ,$$

where $h(x)$ and $G(x)$ are perturbations of the field near v. Why do we need two fields now in contrast to part (b), where one field $h(x)$ was sufficient in the expansion? Express the potential $V(\varphi)$ in terms of v, λ, $h(x)$ and $G(x)$ and identify the masses of the fields $h(x)$ and $G(x)$.

(e) Rewrite the kinetic term $(D_\mu\varphi)\,(D^\mu\varphi)^\dagger$ of the Lagrangian in terms of the new fields, where you can omit (mixing) terms cubic and quartic in all fields $A_\mu(x)$, $h(x)$ and $G(x)$. Identify the effective mass term of the gauge field $A_\mu(x)$. Interpret your findings by comparing this to the result you found in part (c).

(f) Use the gauge freedom of the field $\varphi(x)$ as given in part (c) to remove the massless Goldstone field $G(x)$ from the Lagrangian. This special choice of gauge is called *unitary gauge*. Express the Lagrangian in this special case. Interpret your findings. *Hint:* Rewrite the expansion of the field $\varphi(x)$ from part (d) in such a way that the field perturbation $G(x)$ appears in the argument of an exponential function and use the freedom of choice of $\alpha(x)$ to formally remove $G(x)$. After that, insert the new expansion (without $G(x)$) into the Lagrangian.

Exercise 4: Range of Forces

In particle physics, forces (interactions) between two particles are described as constant exchange of bosons. These interaction bosons can be massive in general, and their rest mass has to be created temporarily, at least during the exchange. This is not possible in classical physics, but the Heisenberg uncertainty relation offers a way out here.

(a) Calculate the typical range of the force, assuming that the exchange/interaction particle moves at the speed of light for

  (i) the electromagnetic interaction

  (ii) the weak interaction
  (exchange particles: W and Z with $M_W = 80.398 \frac{\mathrm{GeV}}{\mathrm{c}^2}$, $M_Z = 91.1876 \frac{\mathrm{GeV}}{\mathrm{c}^2}$).

(b) The force between nucleons is typically of the order of 1 fm, and is transmitted through pions (bound quark-antiquark states). What is the implication for the mass of the pions?

Exercise 5: Polarization of a massive vector boson

We consider a vector boson with mass $M \neq 0$ and polarization vectors $\varepsilon_\lambda^\mu(k)$, where k^μ is its four-vector and λ denotes the three physical degrees of freedom for the polarization of the massive vector boson. The polarization vectors are normalized through the following relations:

$$\begin{aligned} k \cdot \varepsilon_\lambda(k) &= 0\,, \\ \varepsilon_\lambda(k) \cdot \varepsilon^*_{\lambda'}(k) &= -\delta_{\lambda\lambda'}\,. \end{aligned} \tag{1}$$

(a) Boost into the rest frame of the vector boson. By using the relations from Eq. (1), determine the form of the three polarization vectors under the assumption that the vector boson is linearly polarized in one longitudinal and two transversal modes.

(b) By using again the relations from Eq. (1) in the rest frame of the vector boson, determine the form of the polarization vectors if we now consider the vector boson to be longitudinally polarized in the z direction, but circularly polarized in the $x-y$ plane.

(c) By using Lorentz covariance, guess the form of the *completeness relation* $\Sigma_\lambda \varepsilon^\mu_\lambda(k)\varepsilon^{*\nu}_\lambda(k)$ of the massive vector boson and use Eq. (1) to determine the correct form of the polarization sum as given in the lecture.
Hint: Which tensors and four-vector combinations are compatible with $\varepsilon^\mu_\lambda(k)\varepsilon^{*\nu}_\lambda(k)$ to preserve Lorentz covariance? Express the completeness relation as a linear combination of these possible components and use Eq. (1) to determine the coefficients of these components in the rest frame of the vector boson.

(d) Show that the circularly polarized vector boson from part (b) fulfills the completeness relation from part (c) by inserting the polarization vectors explicitly for all μ and ν. You can again work in the rest frame of the vector boson.

Exercise 6: Lagrangian of a massive vector field

The Lagrangian of a massive free vector field $V^\mu(x)$ is given by

$$\mathcal{L}_V = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m_V^2}{2}V_\mu V^\mu$$

where $m_V \neq 0$ denotes the mass of the vector particle and $F^{\mu\nu} = \partial^\mu V^\nu - \partial^\nu V^\mu$ denotes the field-strength tensor.

(a) Calculate the equations of motion for V^μ, the so-called Proca equations.

(b) Using the equations of motion, prove that

$$\partial_\mu V^\mu = 0$$

(c) Use the results from (a) and (b) to show that all components of V^μ satisfy the Klein-Gordon equation separately and explain the physical meaning of this result.

A new Lagrangian $\mathcal{L} = \mathcal{L}_V + \mathcal{L}_D$ is given by adding a Dirac term

$$\mathcal{L}_D = \overline{\psi}(x)\left(i\not{D} - m_D\right)\psi(x)\,,$$

where the *covariant derivative* $D_\mu = \partial_\mu + iqV_\mu$ yields a coupling between the spinor ψ and the vectorfield V_μ.

(d) Consider ψ, $\overline{\psi}$ and V_μ as independent fields and calculate the new equations of motion for all three of them, separately.

(e) The vector current j^μ and axial vector current $j^{\mu 5}$ can be defined as

$$j^\mu = \overline{\psi}\gamma^\mu\psi \qquad j^{\mu 5} = \overline{\psi}\gamma^\mu\gamma^5\psi.$$

Consider the special case of $q = 0$, i.e. the fermion decouples from the vector boson. By using the equations of motion, prove that j^μ is a conserved quantity, whereas $j^{5\mu}$ is not conserved in general. In which special case is $j^{\mu 5}$ conserved, as well?