Exercise 1: Einstein equations (10 points)

a) Write down the Einstein equations with a cosmological constant including all functional dependence, e.g. of spacetime coordinates.

b) Now consider the case that $\Lambda$ is not a cosmological constant, but spacetime dependent. Why is this problematic? What do you have to require to be consistent? What does that imply for the relation between matter and the cosmological constant?

c) Now consider the case that both, the cosmological constant $\Lambda$ and the gravitational constant $G$, are not constant but spacetime dependent. Again, find a consistency relation.

Exercise 2: Modified gravity (10 points)

The Einstein equations can be obtained by a variational principle from the action

$$ S = \int d^4x \sqrt{-\det g_{\mu\nu}} \mathcal{L} $$

with the Einstein-Hilbert Lagrange density

$$ \mathcal{L} = -\frac{1}{16\pi G} (R - 2\Lambda). $$

Consider the case of a modified gravitational action with $\mathcal{L} = f(R)/(16\pi G)$, where $f(R)$ is some arbitrary, sufficiently well-behaved function of the Ricci scalar.

a) Derive the equations of motion for this theory to obtain the modified Einstein equations. Check that you obtain the known result in the Einstein-Hilbert case.

b) Take the trace of the modified Einstein equations and consider the case of constant curvature. What kind of $f(R)$-theory do you obtain?