General Relativity II – Exercise Sheet 5

Exercise 1: Luminosity distance versus redshift (6 points)

The most common method of determining distances in cosmology is the measurement of the apparent luminosity of objects of known absolute luminosity. The absolute luminosity $L$ is the energy emitted per second, and the apparent luminosity $l$ is the energy received per second per square centimeter of receiving area. In the case of isotropic emission, in flat space one would obtain

$$l = \frac{L}{4\pi d^2}$$

at distance $d$. To obtain the modified formula in an expanding FRW Universe ($c = 1$) with $ds^2 = -dt^2 + a(t)^2 \left( \frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \right)$, one has to take into account that the proper area of a sphere around the luminous object is different, that the received photons will be redshifted until arrival by a factor $1/(1+z)$, which means that they are received at a lower rate than emitted, and that the energy of the individual photons received on Earth is reduced by the same factor. Show that then the apparent luminosity can be expressed in a similar way as

$$l = \frac{L}{4\pi d_L^2}$$

where $d_L$ depends on the scale factor and the redshift.

Exercise 2: Age of the FRW universe (14 points)

Consider a FRW universe with $ds^2 = -dt^2 + a(t)^2 \left( \frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \right)$ and the Friedmann equations

$$H^2 = \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho - \frac{k}{a^2}$$
$$\dot{\rho}_i = -3H(\rho_i + p_i)$$

with different types of matter (which do not interact with each other) with total energy density $\rho = \sum_i \rho_i$ and constant equation of state parameter $w_i = p_i/\rho_i$.

a) Consider matter types with $w_i = 0, 1/3, -1$. What types of matter are those?

b) Considering each matter type separately, obtain the relations $\rho_i(a), a_i(t), \rho_i(t)$.

c) Calculate the integral $\int_0^t \frac{dt}{a(t)}$ and conclude on the existence of a particle horizon.
d) Rewrite the Friedmann equations in terms of the solutions for the different energy densities and of variables $\Omega_i = \frac{8\pi G \rho_i}{3 H_0^2}$, where $H_0$ is the Hubble parameter measured at present. Use variables $x = \frac{a}{a_0} = \frac{1}{1+z}$, where $z$ is the redshift, and integrate to obtain the time

$$t(z) = \frac{1}{H_0} \int_0^{\frac{1+z}{1+z}} \frac{dx}{x \sqrt{\Omega_A + \Omega_k x^{-2} + \Omega_m x^{-3} + \Omega_r x^{-4}}}$$

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at which light was emitted which reaches us with redshift $z$.

e) Calculate the age of the universe $t(0)$. Assume (nearly) realistic values $\Omega_A = 0.7, \Omega_m = 0.3, \Omega_r = 0.0, \Omega_k = 0$.

f) Calculate the radial distance of the source emitting light from c) (you first have to solve the integral $\int_0^r \frac{dr}{\sqrt{1-kx^2}}$ for the cases $k = -1, 0, 1$).