Exercise 1: Black-body radiation (10 points)

The number density of photons in equilibrium with matter at temperature $T$ and for photon frequency between $\nu$ and $\nu + d\nu$ is given by the black-body spectrum

$$n_T(\nu)d\nu = \frac{8\pi\nu^2 d\nu}{\exp(h\nu/k_B T) - 1}$$  \hspace{1cm} (1)

where $h$ is Planck’s constant and $k_B$ is Boltzmann’s constant.

a) Where is the maximum for a temperature of $T = 2.725$ K? What is the corresponding wavelength?

b) Since the time of last scattering, when photons began free expansion, this spectrum has always kept the same form and the effect of the expansion of the universe can be taken into account by the temperature changing with the scale factor. Find how the temperature changes with the expansion by using how the number density and the frequency change with the scale factor.

c) Obtain the total energy density contained in the CMB radiation by integrating the energy $h\nu$ of individual photons over all frequencies weighted by Planck’s distribution function to obtain the Stefan-Boltzmann law. What is the ratio $\Omega_{R0}$ of this total energy density at $T = 2.725$ and the critical density $\rho_{0\text{crit}} = \frac{3H_0^2}{8\pi G} = 1.878 \times 10^{-29} h^2 g/cm^3$?

d) Calculate the present number density $n_{\gamma0}$ of photons by integrating (1) over all frequencies. Compare this to the number density of nucleons given by $n_{B0} = \frac{3\Omega_{B0}H_0^2}{8\pi G m_N}$ with the average nucleon mass $m_N$. How do $n_{\gamma}$ and $n_B$ change with the expansion of the universe?

e) Use the obtained results to estimate the temperature when the energy densities of matter and radiation were equal.

Exercise 2: CMB dipole (10 points)

Because of the motion of the earth, the CMB shows a dipole anisotropy. In this exercise we want to determine the velocity of the earth compared to the CMB background.

a) Obtain the density $N_\gamma(p)$ of photons in phase space defined by specifying that there are $N_\gamma(p)d^3p$ photons of each polarisation per unit spatial volume in a momentum-space volume $d^3p$ centered at $p$. Obtain this expression from (1) for $|p| = h\nu$ dividing by the momentum-space volume between frequencies $\nu$ and $d\nu$ and averaging over the photon polarisations.
b) \( N_\gamma(p) \) is a Lorentz-scalar as the phase space volume and the number of photons are Lorentz invariant. Calculate the temperature \( T' \) which therefore results from a Lorentz boost with \( |\mathbf{p}| = \gamma(1 + \beta \cos \vartheta)|\mathbf{p}'| \), where \( \vartheta \) is the angle between \( \mathbf{p}' \) and the \( z \)-axis, and \( \gamma = (1 - \beta^2)^{-1/2} \).

c) Expand \( \Delta T = T' - T \) in powers of \( \beta \). You should obtain Legendre polynomials as coefficients.

d) The temperature difference is greatest for \( \cos \vartheta = -1 \) and has been measured as \( \Delta T = 3.34 \text{ mK} \). Determine the velocity of the earth with respect to the CMB.