Decomposition of the quantized Klein Gordon field into plane waves

$$\phi(x) = \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_k} \left(e^{-ik\cdot x} a(k) + e^{ik\cdot x} a^+(k) \right)$$

with $\omega_k \equiv k^0 \equiv +\sqrt{\vec{k}^2 + m^2}$

positive energy solution:

$$\phi^+(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} a(k)\, e^{-ik\cdot x}$$

negative energy solution:

$$\phi^-(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} a^+(k)\, e^{+ik\cdot x}$$

Inverse Fourier transform yields a and a^+:

$$e^{-i\omega_p t} a(\vec{p}) = \int d^3x\, e^{-i\vec{p}\cdot\vec{x}} \left(\omega_p \phi(t,\vec{x}) + i\,\Pi(t,\vec{x}) \right)$$

$$e^{i\omega_p t} a^+(\vec{p}) = \int d^3x\, e^{+i\vec{p}\cdot\vec{x}} \left(\omega_p \phi(t,\vec{x}) - i\,\Pi(t,\vec{x}) \right)$$

Commutation relations of a and a^+

$$[a(\vec{p}), a^+(\vec{k})]$$

$$= e^{i(\omega_p - \omega_k)t} \int d^3x\, d^3y\, e^{-i\vec{p}\cdot\vec{x}} e^{+i\vec{k}\cdot\vec{y}} \cdot$$

$$\underbrace{[\omega_p \phi(t,\vec{x}) + i\pi(t,\vec{x}), \omega_k \phi(t,\vec{y}) - i\pi(t,\vec{y})]}_{\underbrace{\omega_p(-i)\, i\delta^3(\vec{x}-\vec{y}) + i\omega_k(-i\delta^3(\vec{x}-\vec{y}))}_{\delta^3(x-y)(\omega_p+\omega_k)}}$$

$$= e^{i(\omega_p - \omega_k)t} \underbrace{\int d^3\vec{x}\, e^{-i(\vec{p}-\vec{k})\cdot\vec{x}}}_{(2\pi)^3 \delta^3(\vec{p}-\vec{k})} (\omega_p + \omega_k)$$

$$= e^{i(\omega_p - \omega_p)t} (2\pi)^3 \delta^3(\vec{p}-\vec{k}) (\omega_p + \omega_p)$$

i.e.

$$[a(\vec{p}), a^+(\vec{k})] = (2\pi)^3\, 2\omega_p\, \delta^3(\vec{p}-\vec{k})$$

Commutation relations of a and $a^\dagger$

$$[a(\vec p),\ a^{(\dagger)}(\vec k)]$$

$$= e^{i(\omega_p \mp \omega_k)t} \int d^3x\, d^3y\, e^{-i\vec p\cdot\vec x}\, e^{\pm i\vec k\cdot\vec y} \cdot$$

$$\underbrace{[\omega_p \phi(t,\vec x) + i\pi(t,\vec x),\ \omega_k \phi(t,\vec y) \mp i\pi(t,\vec y)]}_{\underbrace{\omega_p(\mp i)\, i\delta^3(\vec x-\vec y) + i\omega_k(-i\delta^3(\vec x - \vec y))}_{\delta^3(x-y)(\omega_p \pm \omega_k)}}$$

$$= e^{i(\omega_p \mp \omega_k)t} \underbrace{\int d^3\vec x\, e^{-i(\vec p \mp \vec k)\cdot\vec x}}_{(2\pi)^3 \delta^3(\vec p \mp \vec k)} (\omega_p \pm \omega_k)$$

$$= e^{i(\omega_p \mp \omega_p)t} (2\pi)^3 \delta^3(\vec p \mp \vec k)(\omega_p \pm \omega_p)$$

i.e.

$$[a(\vec p), a^\dagger(\vec k)] = (2\pi)^3\, 2\omega_p\, \delta^3(\vec p - \vec k)$$

$$[a(\vec p), a(\vec k)] = 0 = [a^\dagger(\vec p), a^\dagger(\vec k)]$$

Quantized Klein-Gordon Hamiltonian

$$H = \frac{1}{2}\int d^3\vec{x}\left(\dot{\phi}^2(\vec{x},t) + \left(\vec{\nabla}\phi(\vec{x},t)\right)^2 + m^2\phi^2\right)$$

with

$$\phi(\vec{x},t) = \int d\tilde{k}\left[a(\vec{k})e^{-i\omega_k t} + a^\dagger(-\vec{k})e^{i\omega_k t}\right]e^{i\vec{k}\cdot\vec{x}}$$

$$\dot{\phi}(\vec{x},t) = \int d\tilde{k}\, i\omega_k\left[-a(\vec{k})e^{-i\omega_k t} + a^\dagger(-\vec{k})e^{i\omega_k t}\right]e^{i\vec{k}\cdot\vec{x}}$$

Here $\quad d\tilde{k} = d^3\vec{k}/[(2\pi)^3 2\omega_k]$

$$\Rightarrow \quad H = \frac{1}{2}\int d\tilde{k}\, d\tilde{p}\Big[-\omega_k\omega_p\left(-a(\vec{k})e^{-i\omega_k t} + a^\dagger(-\vec{k})e^{i\omega_k t}\right)$$
$$\cdot\left(-a(\vec{p})e^{-i\omega_p t} + a^\dagger(-\vec{p})e^{i\omega_p t}\right)$$
$$+\left(-\vec{k}\cdot\vec{p} + m^2\right)\left(a(\vec{k})e^{-i\omega_k t} + a^\dagger(-\vec{k})e^{i\omega_k t}\right)$$
$$\cdot\left(a(\vec{p})e^{-i\omega_p t} + a^\dagger(-\vec{p})e^{i\omega_p t}\right)\Big]\underbrace{\int d^3\vec{x}\, e^{i\vec{k}\cdot\vec{x}}e^{i\vec{p}\cdot\vec{x}}}_{(2\pi)^3\,\delta^3(\vec{k}+\vec{p})}$$

With $\vec{k} = -\vec{p}$, i.e. $\omega_k = +\omega_p = \omega$

$$H = \frac{1}{2}\int d\tilde{p}\,\frac{1}{2\omega_p}\Big[-\omega_p^2\Big\{\underline{a(-\vec{p})a(\vec{p})\,e^{-2i\omega t}} - a(-\vec{p})a^\dagger(-\vec{p}) - a^\dagger(\vec{p})a(+\vec{p}) + \underline{a^\dagger(\vec{p})a^\dagger(-\vec{p})\,e^{2i\omega t}}\Big\}$$

$$+ \underbrace{(m^2+\vec{p}^{\,2})}_{\omega_p^2}\Big\{\underline{a(-\vec{p})a(\vec{p})\,e^{-2i\omega t}} + a(-\vec{p})a^\dagger(-\vec{p}) + a^\dagger(\vec{p})a(+\vec{p}) + \underline{a^\dagger(\vec{p})a^\dagger(-\vec{p})\,e^{2i\omega t}}\Big\}$$

$$= \int d\tilde{p}\,\frac{\omega_p}{2}\Big\{a(-\vec{p})a^\dagger(-\vec{p}) + a^\dagger(\vec{p})a(\vec{p})\Big\}$$

$$= \int d\tilde{p}\,\frac{\omega_p}{2}\Big\{a(\vec{p})a^\dagger(\vec{p}) + a^\dagger(\vec{p})a(\vec{p})\Big\}$$

$$= \int d\tilde{p}\,\omega_p\Big\{a^\dagger(\vec{p})a(\vec{p}) + \underbrace{\frac{1}{2}\big[a(\vec{p}),a^\dagger(\vec{p})\big]}_{(2\pi)^3\omega_p\,\delta^3(0)}\Big\}$$