Physics beyond the Standard Model

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Exercise Sheet 12

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Exercise 16: Pion as pseudo-Goldstone boson

To understand the mechanism how the Higgs emerges as a composite Nambu-Goldstone boson when breaking a global symmetry of strong dynamics, one can study the analogous example of pions in QCD. We consider QCD in the chiral limit, where the quarks are massless and left- and right-handed fields transform independently. Non-perturbative effects lead to the formation of a quark condensate with non-vanishing vev, which breaks the $SU(2)_L \times SU(2)_R \to SU(2)_V$ and the pseudoscalar pions appear as the three massless Goldstone bosons.
Let us now add the electromagnetic interaction as $U(1)_{\text{em}}$. Via loop effects with photons, this will generate a potential and a mass term for the charged pions, while the neutral one stays massless (and is also not eaten to form a massive photon).
To write down the effective action, we assume that $SU(2)_L \times SU(2)_R$ is gauged by external fields L_μ, R_μ, neglect the momentum dependence of the pion field Σ. Then, in quadratic order in the gauge fields,

$$\mathcal{L} = \frac{1}{2}(P_T)^{\mu\nu}\Big(\Pi_L(q^2)\,\mathrm{Tr}[L_\mu L_\nu] + \Pi_R(q^2)\,\mathrm{Tr}[R_\mu R_\nu] - \Pi_{LR}(q^2)\,\mathrm{Tr}[\Sigma^\dagger L_\mu \Sigma R_\nu]\Big)$$

with the transverse projector $(P_T)^{\mu\nu} = g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}$.

(a) Rewrite $\mathcal{L}$ for the vacuum, $\langle\Sigma\rangle = 1$, and using vector and axial gauge fields, which are linear combinations of the right- and left-handed ones in the usual way. Obtain values for the form factors at zero momentum transfer, using that $\Pi_A(q^2=0) = f_\pi^2$ and the other vanish (why?).

(b) Going back to the original version of $\mathcal{L}$, now switch off all external gauge fields but the photon: $L_\mu(x) = R_\mu(x) = T^3 v_\mu(x)$. Show that $\mathcal{L}$ reduces to

$$\mathcal{L} = \frac{1}{2}(P_T)^{\mu\nu} v_\mu v_\nu \Big(\Pi_V(q^2) + \Pi_{LR}(q^2)\frac{\sin^2(\pi/f_\pi)}{\pi^2}\left(\pi^+\pi^-\right)\Big)$$

with $\Sigma = \exp\left(i\frac{\sigma^a\pi^a}{f_\pi}\right)$, $\pi \equiv \sqrt{(\pi^a)^2}$, $\pi^\pm = \pi_1 \mp i\pi_2$ and the Pauli matrices $\sigma^a = 2T^a$.

Resummation of the corresponding 1-loop diagrams then leads to the potential

$$V(\pi) = \frac{3}{16\pi^2}\int_0^\infty \mathrm{d}Q^2\, Q^2 \log\left(1 + \frac{1}{2}\frac{\Pi_{LR}(Q^2)}{\Pi_V(Q^2)}\frac{\sin^2(\pi/f_\pi)}{\pi^2}\left(\pi^+\pi^-\right)\right)$$

To estimate the convergence of the integral, we need the momentum dependence of the form factors $\Pi_{LR}(Q^2)$, $\Pi_V(Q^2)$ at large Q^2. $\Pi_V(Q^2)$ can be estimated as Q^2/e^2. For $\Pi_{LR}(Q^2)$ we observe that the first possible contribution is a dimension-6 operator, and hence the leading term $Q^2 \cdot \frac{\delta}{Q^6}$ with a constant δ.
Using two further assumptions, the large-N_c limit and vector meson dominance, we can write the vector or axial form factor in terms of an infinite sum of resonances, so that

$$\Pi_{LR}(Q^2) = Q^2\left(\sum_n \frac{f_{a_n}^2}{Q^2 + m_{a_n}^2} + \frac{f_\pi^2}{Q^2}\right) - Q^2 \sum_n \frac{f_{\rho_n}^2}{Q^2 + m_{\rho_n}^2}\,.$$

(c) Taking into account the leading behaviour of $\Pi_{LR}(Q^2)$ we have mentioned beforehand, show that from evaluating $\lim_{Q^2\to\infty}\Pi_{LR}(Q^2)$ and $\lim_{Q^2\to\infty} Q^2\Pi_{LR}(Q^2)$, you can deduce two (Weinberg) sum rules on the spectrum of masses and decay constants

$$\sum_n \left(f_{\rho_n}^2 - f_{a_n}^2\right) = f_\pi^2$$

$$\sum_n \left(f_{\rho_n}^2 m_{\rho_n}^2 - f_{a_n}^2 m_{a_n}^2\right) = 0\,.$$

(d) Neglect all higher ρ and a resonances except the first one. Express $\Pi_{LR}(Q^2)$ by those, eliminating f_{ρ_1} and f_{a_1} by the sum rules. Insert the expression in $V(\pi)$ given above and expand the logarithm in first order to obtain

$$V(\pi) \simeq \frac{3}{8\pi}\alpha_{\mathrm{em}} f_\pi^2 \frac{m_{\rho_1}^2 m_{a_1}^2}{m_{a_1}^2 - m_{\rho_1}^2}\log\frac{m_{a_1}^2}{m_{\rho_1}^2}\frac{\sin^2(\pi/f_\pi)}{\pi^2}\left(\pi^+\pi^-\right)\,.$$