Physics beyond the Standard Model

L: Prof. Dr. M. Mühlleitner, T: Dr. M. Rauch

Exercise Sheet 8

Discussion: Fr, 09.01.15

Exercise 10: SUSY Oscillator

We examine a one-dimensional harmonic oscillator. Consider first the standard case from Advanced Quantum Mechanics, which we will label as bosonic oscillator in the following. Its Hamilton operator is given by

$$H_B = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{q}^2$$

with mass m and frequency ω. Additionally, we define the creation and annihilation operators

$$b^{\pm} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{q} \mp \frac{i\hat{p}}{m\omega}\right) ,$$

which are adjoint to each other.

(a) Calculate the commutators $[b^{\pm}, b^{\pm}]$ and $[b^{\pm}, b^{\mp}]$ as well as the number operator $N_B = b^+b^-$. Express the Hamilton operator by the number operator. What are the energy eingenvalues?

As next step we consider the Fermi oscillator, which is given by the Hamilton function

$$H_F = i\omega\hat{\psi}\hat{\pi}$$

with the hermitian operators $\hat{\psi}$, $\hat{\pi}$ with dimension $\sqrt{[\text{action}]}$ and the following anti-commutator relations

$$\{\hat{\psi}, \hat{\pi}\} = 0 , \qquad \{\hat{\psi}, \hat{\psi}\} = \{\hat{\pi}, \hat{\pi}\} = \hbar .$$

In analogy to the Bose oscillator, we define creation and annilhilation operators adjoint to each other,

$$f^{\pm} = \sqrt{\frac{1}{2\hbar}}\left(\hat{\psi} \mp i\hat{\pi}\right) .$$

(b) Calculate the anti-commutators $\{f^{\pm}, f^{\pm}\}$ and $\{f^{\pm}, f^{\mp}\}$. Define $\hat{\psi}$ and $\hat{\pi}$ as function of $f^{\pm}$. Express the Hamilton operator by the number operator $N_F = f^+ f^-$ again. What are the energy eingevalues? How many are there?

Finally we consider the SUSY oscillator, given by $H_S = H_B + H_F$. The corresponding eigenstates are labelled by $|n_B, n_F\rangle = |n_B\rangle|n_F\rangle$. The SUSY operators are defined as $Q_+ = \sqrt{\hbar\omega}\, b^- f^+$ and $Q_- = \sqrt{\hbar\omega}\, b^+ f^-$.

(c) Calculate the following quantities:
   (i) $Q_+|n_B, n_F\rangle$, $Q_-|n_B, n_F\rangle$;
   (ii) Q_+^2, Q_-^2, $\{Q_+, Q_-\}$;
   (iii) $Q_\pm^\dagger$;
   (iv) $[H_S, Q_\pm]$.

(d) We now define linear combinations as

$$Q_1 = Q_+ + Q_- \ , \qquad\qquad Q_2 = -i(Q_+ - Q_-) \ .$$

Calculate $\{Q_1, Q_2\}$ as well as Q_1^2 and Q_2^2. As which quantity can you identify the last two operators?

(e) How does the spectrum of H_S look like? What can you say about degenerate states?