Institute for Theoretical Physics (ITP)
Karlsruhe Institute of Technology (KIT)

General Relativity II
Lecturer: Prof. Dr. Frans R. Klinkhamer
Assistant: Dr. Viacheslav A. Emelyanov

- Handing-in: Monday, 01.05.2017; Discussion: Wednesday, 03.05.2017
- All up-to-date information related to the course can be found under the link:
https://www.itp.kit.edu/~slava/relativitaetstheorie_ss_17.html

Name: ______________________ Points: ________

Exercise Sheet 1

Exercise 1.1: Killing vectors (4 points)

These are vectors which satisfy the Killing equation:

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu \;=\; 0\,. \tag{1}$$

These vectors are generators of coordinate transformations which leave the metric unchanged. If the vector K satisfies

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu \;=\; \lambda g_{\mu\nu}\,, \tag{2}$$

it is called as a conformal Killing vector, where λ depends, in general, on a space-time point. These generate coordinate transformations leaving the metric unchanged up to a prefactor.

(a) Find all Killing vectors of the 2-dimensional sphere of unit radius.

(b) Show that $D = x^\mu \partial_\mu$ is a conformal Killing vector in Minkowski space. Determine how the Minkowski metric modifies under the coordinate transformation induced by $\exp(aD)$.

Exercise 1.2: Cosmic microwave background (8 points)

The Universe turns out to be pervaded by a black-body radiation of temperature $T_{\gamma,0} \approx 2.73\,\mathrm{K}$. If we consider a box with the thermal radiation in a local inertial frame, then the one-particle distribution function $f_\gamma(x,p)$ of photons in (local) thermodynamic equilibrium is given by

$$f_\gamma(x,p) \;=\; \frac{1}{(2\pi)^3}\frac{2}{e^{\beta p_0}-1}\,, \tag{3}$$

where we have set the fundamental constants to unity and $\beta \equiv 1/T_\gamma$ is the inverse temperature.

State 26. 04. 2017 at 09:06:51

(a) The Universe at cosmological scales is approximately described by the Friedmann-Robertson-Walker geometry. Let us assume that the above local inertial frame corresponds to a co-moving frame in this universe. Determine the generalised form of the distribution $f_\gamma(x,p)$ which is appropriate for FRW space.

Hint: Use the fact that the one-particle distribution is a tensor of rank 0.

(b) This distribution satisfies the relativistic Boltzmann equation:

$$\left(p^\lambda \frac{\partial}{\partial x^\lambda} - \Gamma^\lambda_{\mu\nu} p^\mu p^\nu \frac{\partial}{\partial p^\lambda}\right) f_\gamma(x,p) \;=\; 0\,. \tag{4}$$

Determine how the temperature T_γ of the photon gas depends on the cosmic time in an FRW universe.

(c) Calculate the CMB temperature when the Universe was 1100 times smaller (photon decoupling epoch) than nowadays.