Institute for Theoretical Physics (ITP)
Karlsruhe Institute of Technology (KIT)

General Relativity II
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- Handing-in: Monday, 03.07.2017; Discussion: Wednesday, 05.07.2017
- All up-to-date information related to the course can be found under the link: https://www.itp.kit.edu/~slava/relativitaetstheorie_ss_17.html

Name: ______________________________ Points: ________

Exercise Sheet 10

Exercise 10.1: Inflation and higher-curvature terms (12 points)

Quantum fields in curved spacetimes naturally generate higher-curvature terms in the gravitational theory. As an example, consider a possible extension of General Relativity which is based on the following action:

$$S \;=\; -\frac{1}{16\pi G}\int d^4x\,(-g)^{\frac{1}{2}}f(R)\,, \tag{1}$$

where R is the Ricci scalar. This theory reduces to General Relativity if we set $f(R) = R$.

(a) Show that the modified Einstein field equation reads

$$f' R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f + g_{\mu\nu}\Box f' - \nabla_\mu\nabla_\nu f' \;=\; 0\,, \tag{2}$$

where the prime stands for the derivative with respect to R.

(b) Show that the modified Einstein field equation can be brought to the form

$$\bar{R}^\mu_\nu - \frac{1}{2}\,\delta^\mu_\nu\bar{R} \;=\; 8\pi G\bar{T}^\mu_\nu(\Phi)\,, \tag{3}$$

by use of the following conformal transformation:

$$g_{\mu\nu} \;\to\; \bar{g}_{\mu\nu} \;=\; f' g_{\mu\nu}\,, \tag{4a}$$

$$R^\mu_\nu \;\to\; \bar{R}^\mu_\nu \;=\; \frac{1}{f'}\,R^\mu_\nu - \frac{1}{(f')^2}\,\nabla_\nu\nabla^\mu f' - \frac{1}{2(f')^2}\,\delta^\mu_\nu\Box f' + \frac{3}{2(f')^3}\,\nabla_\nu f'\nabla^\mu f'\,, \tag{4b}$$

$$R \;\to\; \bar{R} \;=\; \frac{1}{f'}\,R - \frac{3}{(f')^2}\,\Box f' + \frac{3}{2(f')^3}\,\nabla_\mu f'\nabla^\mu f'\,. \tag{4c}$$

Derive how the effective scalar field Φ depends on f' and compute its potential $V(\Phi)$.

Hint: The stress tensor of a scalar field ϕ has been found in Exercise 9.1.a and is given by

$$T_{\mu\nu}(\phi) = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}(\nabla\phi)^2 + V(\phi)g_{\mu\nu}\,. \tag{5}$$

Hint: Result is

$$\Phi = \left(\frac{3}{16\pi G}\right)^{\frac{1}{2}} \ln f' \quad \textit{and} \quad V(\Phi) = \frac{1}{16\pi G}\frac{f - Rf'}{(f')^2}\,. \tag{6}$$

(c) We now assume that

$$f(R) = R - \frac{1}{6M^2}R^2\,, \tag{7}$$

where M is a constant parameter. Study the inflationary solutions of this model, assuming that the space-time geometry is described by an FRW universe with $k = 0$. What is the physical meaning of the parameter M?

Hint: Use the effective scalar field Φ with the potential $V(\Phi)$ to study the inflationary solutions of this model.

Remark: This type of inflation was first discussed by Starobinsky in 1980.