Institute for Theoretical Physics (ITP)
Karlsruhe Institute of Technology (KIT)

General Relativity II

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- `Handing-in: Monday, 10.07.2017; Discussion: Wednesday, 12.07.2017`
- `All up-to-date information related to the course can be found under the link:`
  `https://www.itp.kit.edu/~slava/relativitaetstheorie_ss_17.html`

Name: __________________________________ Points: ________

Exercise Sheet 11

Exercise 11.1: Gibbons-Hawking temperature of de Sitter spacetime (12 points)

Consider a massless, real scalar field $\Phi(x)$ with the conformal coupling to gravity in flat de Sitter spacetime, i.e. the line element reads

$$ds^2 \;=\; a^2(\eta)\big(d\eta^2 - dx^2 - dy^2 - dz^2\big) \quad \text{with} \quad a(\eta) \;=\; -\frac{1}{H\eta}\,, \tag{1}$$

where H is the Hubble constant and $\eta \in (-\infty, 0)$ is a conformal time.

(a) A part of flat de Sitter spacetime can be parametrized by static coordinates, in which the line element reads

$$ds^2 \;=\; \big(1 - H^2r^2\big)dt^2 - \frac{dr^2}{1-H^2r^2} - r^2\big(d\theta^2 + \sin^2\theta d\phi^2\big)\,, \quad \text{for} \quad r \;\in\; [0, 1/H)\,. \tag{2}$$

Verify that the static and flat coordinates are related as follows:

$$\eta \;=\; -\frac{\exp(-Ht)}{(1-H^2r^2)^{\frac{1}{2}}}\,, \tag{3a}$$

$$x \;=\; \frac{\exp(-Ht)}{(1-H^2r^2)^{\frac{1}{2}}}\,Hr\sin\theta\sin\phi\,, \tag{3b}$$

$$y \;=\; \frac{\exp(-Ht)}{(1-H^2r^2)^{\frac{1}{2}}}\,Hr\sin\theta\cos\phi\,, \tag{3c}$$

$$z \;=\; \frac{\exp(-Ht)}{(1-H^2r^2)^{\frac{1}{2}}}\,Hr\cos\theta\,. \tag{3d}$$

(b) Consider a Wick rotation of the time coordinate t, namely $t_E = it$. This leads to Euclidean de Sitter spacetime with the line element

$$-ds_E^2 = \big(1 - H^2 r^2\big) dt_E^2 + \frac{dr^2}{1 - H^2 r^2} - r^2\big(d\theta^2 + \sin^2\theta d\phi^2\big)\,. \tag{4}$$

The Euclidean time t_E must be periodic in order to avoid a conical singularity at $r = 1/H$. Determine its period β.

Hint: Consider first the near-horizon geometry.

Remark: The Gibbons-Hawking temperature T_{GH} is equal to $1/\beta$ (Gibbons&Hawking, 1977).

(c) If we now rescale the scalar field $\Phi(x)$ as follows:

$$\Phi(x) = \frac{1}{a(\eta)}\,\phi(x)\,, \tag{5}$$

then the effective scalar field $\phi(x)$ satisfies the field equation $\Box\phi = 0$, where

$$\Box = \partial_\eta^2 - \partial_x^2 - \partial_y^2 - \partial_z^2\,. \tag{6}$$

Quantize the field $\phi(x)$ as if it were in Minkowski spacetime with the coordinates (η, x, y, z) and, then, derive the Wightman function $W(x, x') = \langle 0|\hat{\Phi}(x)\hat{\Phi}(x')|0\rangle$.

(d) Express the two-point function $W(x, x')$ in terms of the static coordinates (t, r, θ, ϕ), assuming that $r = r' = 0$, $\theta = \theta'$ and $\phi = \phi'$ (this world line corresponds to a fiducial observer freely moving in the static de Sitter spacetime).

Hint: Result is

$$W(\Delta t) = -\frac{H^2}{8\pi^2}\frac{1}{\cosh(H\Delta t - i0) - 1}\,, \quad \text{where} \quad \Delta t \equiv t - t'\,. \tag{7}$$

(e) Compute the Fourier transform of $W(\Delta t)$ with respect to $\tau \equiv \Delta t$, namely

$$W(\omega) = \int_{-\infty}^{+\infty} d\tau\, W(\tau)\, e^{-i\omega\tau}\,. \tag{8}$$

What is the physical meaning of your result?

(f) The renormalised stress-energy tensor of the quantum scalar field $\hat{\Phi}(x)$ reads

$$\langle 0|\hat{T}^\mu_\nu(x)|0\rangle = \frac{H^4}{960\pi^2}\,\delta^\mu_\nu\,. \tag{9}$$

Compare this with the stress tensor of a thermal isotropic gas at $T_{\rm GH}$. Interpret your result.

Remark: The non-vanishing value of $\langle 0|\hat{T}^\mu_\nu(x)|0\rangle$ is entirely due to the trace anomaly, which does not depend on the vacuum state.