Karlsruher Institut für Technologie

Institute for Theoretical Physics (ITP)
Karlsruhe Institute of Technology (KIT)

General Relativity II

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- Handing-in: Monday, 17.07.2017; Discussion: Wednesday, 19.07.2017
- All up-to-date information related to the course can be found under the link: https://www.itp.kit.edu/~slava/relativitaetstheorie_ss_17.html

Name: ______________________________ Points: __________

Exercise Sheet 12

Exercise 12.1: Quantum field theory and non-trivial space-time topology (12 points)

There appear new effects in quantum field theory in spacetimes with a boundary, non-trivial curvature or topology. As an example, consider a massless, real scalar field $\Phi(t,x)$ in 2-dimensional flat space with the topology $\mathbf{R}\times\mathbf{S}^1$, i.e. the spatial section is compactified to a circle. We assume that the circumference of $\mathbf{S}^1$ equals to L.

(a) We can expand the scalar field $\Phi(t,x)$ in the $\mathbf{R}\times\mathbf{S}^1$ universe with the plane-wave solutions $\Phi_k(t,x) \propto \exp(-i(\omega t - kx))$ as follows:

$$\Phi(x) \;=\; \sum_k \big(\Phi_k(x)a_k + \Phi_k^*(x)a_k^*\big) \quad \text{for} \quad \omega \;=\; |k|\,, \tag{1}$$

where a_k do not depend on the space-time point. Argue why the momentum k has to be discrete. Find its possible values for periodic boundary conditions: $\Phi_k(t,x) = \Phi_k(t,x+nL)$, where $n \in \mathbb{Z}$.

Remark: This is a simplified version of the Casimir set up. The Casimir effect is an experimentally observed prediction of quantum field theory that manifests itself as an attractive force between two uncharged conducting plates in vacuum.

(b) We promote the coefficients a_k in (1) to operators satisfying the following commutation relations:

$$\big[\hat{a}_k, \hat{a}_p^\dagger\big] \;=\; \delta_{kp} \quad \text{and} \quad \big[\hat{a}_k, \hat{a}_p\big] \;=\; 0\,. \tag{2}$$

Determine the normalisation of $\Phi_k(t,x)$ which leads to the correct canonical commutation relation between $\hat{\Phi}(t,x)$ and its conjugate field variable $\hat{\Pi}(t,x)$.

Hint: The Dirac delta function has the following representation through an infinite sum:

$$\delta(x-y) \;=\; \frac{1}{2\pi}\sum_{n=-\infty}^{+\infty} e^{in(x-y)}\,. \tag{3}$$

(c) We define the vacuum state, $|\text{vac}\rangle$, as the state which is annihilated by $\hat{a}_k$. Show that the vacuum expectation value of the tt-component of the stress-energy tensor $\hat{T}_{\mu\nu}$ is given by

$$\langle\text{vac}|\hat{T}_{tt}|\text{vac}\rangle \;=\; \frac{2\pi}{L^2}\sum_{n=0}^{+\infty} n\,. \tag{4}$$

(d) Propose a method how this sum can be regularised and, then, renormalise the sum by subtracting the vacuum energy density of 2-dimensional Minkowski space. Interpret your result.

Hint: Note that the cylinder $\mathbf{R}\times\mathbf{S}^1$ *goes over to Minkowski space* $\mathbf{R}\times\mathbf{R}$ *in the limit* $L\to\infty$.

(e) Repeat your computations for the scalar field satisfying antiperiodic boundary conditions, namely $\Phi_k(t,x) = (-1)^n\Phi_k(t,x+nL)$, where $n\in\mathbb{Z}$.