Institute for Theoretical Physics (ITP)
Karlsruhe Institute of Technology (KIT)

General Relativity II

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- `Handing-in: Monday, 05.06.2017; Discussion: Wednesday, 07.06.2017`
- `All up-to-date information related to the course can be found under the link:`
  `https://www.itp.kit.edu/~slava/relativitaetstheorie_ss_17.html`

Name: ________________ Points: ______

Exercise Sheet 6

Exercise 6.1: Helium and hydrogen recombination (6 points)

(a) Considering the reaction $\mathrm{He}^+ + e^- \rightleftarrows \mathrm{He} + \gamma$, derive the expression for the ratio of the number densities of He^+ and He. Estimate the temperature at which that ratio is of order unity.

(b) Consider now the equilibrium process $p + e^- \rightleftarrows \mathrm{H} + \gamma$. Estimate the temperature T_{rec} at which hydrogen recombination starts.

Exercise 6.2: Symmetry restoration at high temperatures (6 points)

We consider a toy model for the electroweak symmetry restoration at high temperatures. Let us take a real scalar field $\chi(x)$ with the potential

$$V(\chi) \;=\; \frac{\lambda}{4}\big(\chi^2 - \chi_0^2\big)^2\,, \tag{1}$$

where λ is a positive coupling constant and χ_0 a constant parameter.

(a) Find the ground state of the field by minimizing the potential $V(\chi)$ with respect to χ. Show that $\chi = 0$ is an unstable minimum of the potential, whereas $\chi = \pm\chi_0$ are stable.

(b) We want to find how the ground state changes under the influence of quantum fluctuations. Therefore, we assume that $\chi = \bar{\chi} + \phi$, where $\bar{\chi}$ and ϕ are, respectively, the mean field and the quantum fluctuation field. Show that the ground state is now determined by the equation

$$\lambda\bar{\chi}\big(\bar{\chi}^2 + 3\langle\hat{\phi}^2\rangle - \chi_0^2\big) \;=\; 0\,. \tag{2}$$

(c) Show that the state with thermal occupation number $n_{\mathbf{k}}(\beta)$ has

$$\langle\hat{\phi}^2\rangle \;=\; \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{\omega_{\mathbf{k}}}\Big(\frac{1}{2}+n_{\mathbf{k}}(\beta)\Big) \quad \text{with} \quad \omega_{\mathbf{k}} \;=\; \big(\mathbf{k}^2+\lambda\chi_0^2\big)^{\frac{1}{2}}. \tag{3}$$

Why is the UV divergence in the integral of Eq. (3) harmless?

(d) Show that, at temperatures $T \gg \lambda\chi_0^2$, the following expression holds:

$$\langle\hat{\phi}^2\rangle_T \;\equiv\; \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{\omega_{\mathbf{k}}}\, n_{\mathbf{k}}(\beta) \;=\; \frac{T^2}{12} + \mathrm{O}\big(\lambda\chi_0^2 T\big)\,. \tag{4}$$

(e) Find the effective potential $V_{\rm eff}(\bar{\chi})$ corresponding to Eq. (2) with $\langle\hat{\phi}^2\rangle_T$. Show that $\bar{\chi}=0$ is a stable minimum of the effective potential for temperatures $T > 2|\chi_0|$.