Institute for Theoretical Physics (ITP)
Karlsruhe Institute of Technology (KIT)

General Relativity II

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- Handing-in: Monday, 12.06.2017; Discussion: Wednesday, 14.06.2017
- All up-to-date information related to the course can be found under the link: https://www.itp.kit.edu/~slava/relativitaetstheorie_ss_17.html

Name: ______________________ Points: ________

Exercise Sheet 7

Exercise 7.1: Magnetic monopole (6 points)

Maxwell's equations are asymmetric, namely $\nabla\cdot\mathbf{B} = 0$ implies that magnetic charges do not exist. However, Maxwell theory can be generalised: Let us assume that $\nabla\cdot\mathbf{B} = 4\pi g\delta(\mathbf{r})$, where g is a magnetic charge. Then, it is straightforward to find that $\mathbf{B} = g\mathbf{r}/r^3$ holds.

(a) Show that this result can be obtained from $\mathbf{B} = \nabla\times\mathbf{A}$, where $\mathbf{A} = \mathbf{A}_+ \equiv (A_+^r, A_+^\theta, A_+^\phi)$ with

$$A_+^r = 0\,, \quad A_+^\theta = 0 \quad \text{and} \quad A_+^\phi = g\,\frac{1-\cos\theta}{r\sin\theta}\,. \tag{1}$$

(b) The vector potential $\mathbf{A}_+$ is, however, singular on the half-line with $\theta = \pi$ (called the Dirac string). Show that there exists a gauge function α, such that $\mathbf{A}_- = \mathbf{A}_+ + \nabla\alpha$ is regular at $\theta = \pi$ and the one-form $d\alpha$ is closed, but not exact.

Remark: This implies that there does not exist a non-singular global vector potential $\mathbf{A}$ *describing the magnetic field* $\mathbf{B}$ *of the monopole.*

(c) Consider a quantum-mechanical particle of charge e in the field of this magnetic monopole. Requiring that the wave function of this particle is single-valued, prove the Dirac quantization condition: $g = \hbar n/2e$, where $n \in \mathbb{Z}$.

Hint: We have found above that the singularity of the vector potential $\mathbf{A}_+$ *or the Dirac string can be shifted by performing an appropriate gauge transformation. Determine how the wave function changes under general gauge transformations. Find the condition under which the wave function is single-valued, independent of the location of the monopole's string.*

Exercise 7.2: Monopole problem (6 points)

Grand Unified Theories (GUTs) typically predict the existence of various topological defects. Estimate the energy density of magnetic monopoles in the Universe produced at $T_{\mathrm{GUT}} \sim 10^{15}\,\mathrm{GeV}$, assuming that there should be at least one monopole within the observable part of the Universe and that its mass M is of order T_{GUT}/α (where α is the fine-structure constant).