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Notes on Some Geometric and Algebraic Problems Solved by Origami

Wolfdieter L a n g [1]

Abstract

Details for known solutions of some geometric and algebraic problems with the help of origami are presented: two theorems of Haga, the general cubic equation, especially the heptagon equation, doubling the cube as well as the trisection of angles α, $\pi - \alpha$ and $\pi + \alpha$.

Introductory remarks

These notes give details on some geometric and algebraic problems related to cubic equations which are solved using origami (Japanese for folding paper). Seven axioms for origami can be found in [16].

These notes mostly start with a square of some given length, called R in some length unit. Given any (transparent) sheet larger than this square one can fold an $R \times R$ square, provided one can determine the distance R between two points P and Q on some line (crease). This assumes that one has some way to measure R, *e.g.*, a marked ruler. Then one starts with some crease, call it c_1, and folds perpendicular to this crease through some point, defined as the first corner A, another crease called c_2. A perpendicular folding with respect to some line c (crease) and a point P (not necessarily on c) can be accomplished (sometimes called axiom 4 or IV), but here it is useful to have a transparent sheet in order to see when the two parts of c fit together for this folding through P. Then one finds the next corner of the square, called D, at a given distance R from A on the crease c_1. Next, through D a crease c_3 perpendicular to c_1 is formed. The next crease c_4 is obtained by folding crease c_1 onto crease c_2 (point A will lie on both creases; guaranteed by axiom 3 or III). This will define the next corner C as the intersection point of c_4 with c_3. Finally a crease c_5 perpendicular to c_3 through point C is formed to find the last corner B as the intersection of c_5 with c_2. Alternatively one can fold crease c_1 onto crease c_3, with D on both creases, to find B as the intersection with crease c_2. This completes then the square A, B, C, D oriented in the positive sense (on one of the transparent paper's sides). In the following we will use the notation $\overline{B,C}$ to denote the straight line connecting points B and C, as well as the length of this line segment. The latter should be denoted by $|\,\overline{B,C}\,|$, but it should be clear what is meant in each case.

Problem 1: Haga's second Theorem

In the book of *Bellos* [2] one finds on p. 115 an origami leading to the "second theorem" of *Kazuo Haga*. For this one folds two neighboring corners of a square sheet of paper (length of the side R in some unit), say B and C, in turn on some point $B' = C'$ of the opposite side (bordered by the corners A and D). This is done by two intersecting creases (called f and g in *Figure 3*). The intersection point, called S in *Figure 3*, will always lie on the crease which arises if one folds C onto B (which gives one of the medians of the square). This happens independently of the position of the point on which the two corners have been folded. In addition, the three distances between the intersection point S and the chosen point C' and the two corners B and C coincide.

In order to analyse this consider first *Figure 1* where C is folded onto C' having distance $x\,R$ from the left upper corner A.

[1] wolfdieter.lang@partner.kit.edu, http://www-itp.particle.uni-karlsruhe.de/~wl

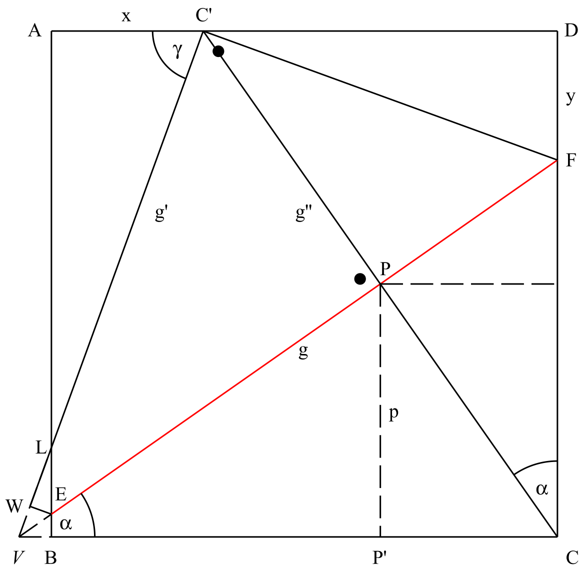


Figure 1: Folding C onto C'

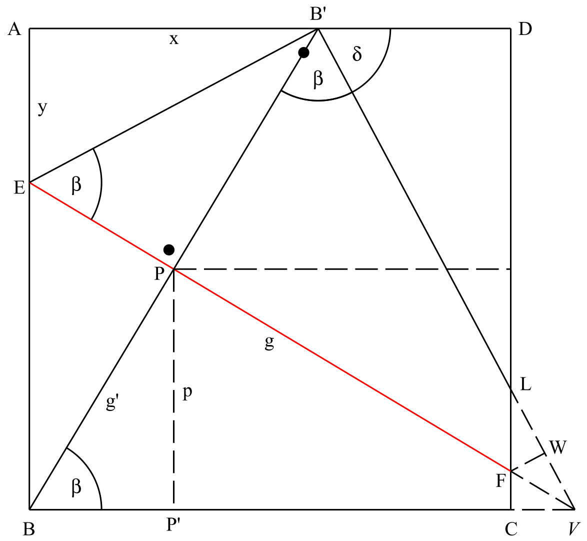


Figure 2: Folding B onto B'

Figure 1:
$\overline{A,B} = R = \overline{A,D}$, $\overline{A,C'} = x\,R$, $\overline{D,F} = y\,R$, $\overline{V,B} = \overline{V,W} = v\,R$, $\overline{W,C'} = \overline{B,C} = R$, $\overline{B,E} = \overline{W,E} = z\,R$, $\overline{P,C} = \overline{P,C'} = c\,R$, $\overline{P,P'} = p\,R$, $\angle(F,V,C) = \angle(C',V,F) = \angle(C',C,D) = \angle(P,C',F) = \alpha$, $\angle(V,P,C') = \frac{\pi}{2} = \angle(V,C',F)$ (indicated by the two bullets), $\angle(V,C',A) = \gamma = 2\,\alpha$.
P is the intersection point of two perpendicular straight lines, *viz* g'' connecting C' and C and g (the crease bringing C to C') connecting V and F.
The analytic data, depending on R (usually taken as 1 length unit) and x, is :
$y = \dfrac{x\,(x-2)}{2}$, $2\,c = \sqrt{1 + (1-x)^2}$, $\overline{P,F} = R\sqrt{(1-y)^2 - c^2}$, $\tan\alpha = \dfrac{\overline{F,C}}{\overline{V,C}} = \dfrac{z}{v} = \dfrac{\overline{P,F}}{c\,R} = \dfrac{\overline{C',D}}{R} = 1-x$, $\tan(2\,\alpha) = \dfrac{2\,(1-x)}{x\,(1-x)}$, $\sin\alpha = \dfrac{1-x}{\sqrt{1 + (1-x)^2}} = \dfrac{c}{1+v}$, $\cos\alpha = \dfrac{1}{\sqrt{1 + (1-x)^2}} = \dfrac{p}{c}$, $p = \dfrac{1}{2}$, $v = \dfrac{x^2}{2\,(1-x)}$, $\overline{F,C} = \overline{F,C'} = R\,(1 - y) = R\,\dfrac{2 - 2\,x - x^2}{2}$, $z = v\tan\alpha = \dfrac{x^2}{2}$, $\overline{V,E} = R\sqrt{z^2 + v^2} = R\,\dfrac{x^2\sqrt{1 + (1-x)^2}}{2\,(1-x)}$, $\overline{W,L} = R\,z\tan(2\,\alpha) = R\,\dfrac{x^2\,(1-x)}{\sqrt{x\,(2-x)}}$, $\overline{L,A} = R\,x\tan(2\,\alpha) = R\,\dfrac{2\,(1-x)}{2-x}$, $\overline{E,L} = \dfrac{z\,R}{\cos(2\,\alpha)} = R\,\dfrac{x\,(1 + (1-x)^2)}{2-x}$, $\overline{L,C'} = R\,\dfrac{x}{\sin(\frac{\pi}{2} - 2\,\alpha)} = R\,\dfrac{1 + (1-x)^2}{2-x}$, $\overline{L,B} = R\,v\tan(2\,\alpha) = R\,\dfrac{x}{2-x}$, $\overline{E,P} = \dfrac{p\,R}{sin\,\alpha} - \overline{V,E} = R\,\dfrac{(1+x)\sqrt{1 + (1-x)^2}}{2}$,
$\overline{B,P'} = R\,(1 - c\sin\alpha) = R\,\dfrac{1+x}{2}$, $\overline{P,C'} = R\,\dfrac{1-x}{2}$.
With the origin $O = B$ in the $(\hat{x},\hat{y})$-plane (no confusion with the above x and y should arise) the straight lines g, g' and g'' are given by:
$g:\ \hat{y} = (1 - x)\,(\hat{x} + v\,R)$, $g':\ \hat{y} = \dfrac{2\,(1-x)}{x\,(2-x)}\,(\hat{x} + v\,R)$, $g'':\ \hat{y} = -\dfrac{1}{1-x}\,(\hat{x} - R)\,.$

Observation 1: If x varies from 0 to R then P moves on the middle line $\hat{y} = \dfrac{R}{2}$ from $\hat{x} = \dfrac{R}{2}$ to R.

Next, the analysis is done for the case when the left lower corner B is folded onto B' on the side connecting the corners A and D, with distance $x\,R$ from A (not necessarily the same x as in *Figure 1*).

Figure 2:
$\overline{A,B} = R = \overline{A,D}$, $\overline{A,B'} = x\,R$, $\overline{A,E} = y\,R$, $\overline{V,C} = \overline{V,W} = v\,R$, $\overline{E,B} = \overline{E,B'} = (1-y)\,R$, $\overline{W,B'} = \overline{C,B} = R$, $\overline{C,F} = \overline{F,W} = z\,R$, $\overline{P,B} = \overline{P,B'} = b\,R$, $\overline{P,P'} = p\,R$, $\angle(P,B,V) = \angle(P,B',V) = \angle(A,B',B) = \angle(B',E,F) = \angle(P,E,B) = \angle(P',P,V) = \beta$, $\angle(L,B',D) = \delta = \pi - 2\,\beta$, $\angle(C,L,V) = \angle(B',L,D) = \angle(E,B',A) = 2\,\beta - \frac{\pi}{2}$.
P is the intersection point of the perpendicular straight lines g', with points B' and B, and g (the crease bringing B to B') with points V and E.
The analytic data which depends on R (usually taken as 1 length unit) and x is:
$y = \dfrac{1-x^2}{2}$, $2\,b = \sqrt{1+x^2}$, $\overline{P,E} = R\sqrt{(1-y)^2 - b^2}$, $\tan(\frac{\pi}{2} - \beta) = \dfrac{1}{\tan\beta} = \dfrac{\overline{E,B}}{\overline{B,V}} = \dfrac{\overline{E,B'}}{\overline{B',V}} = \dfrac{z}{v} = \dfrac{b\,R}{\overline{P,V}} = \dfrac{\overline{E,P}}{b\,R} = \dfrac{\overline{A,B'}}{\overline{A,B}} = x$, $\tan(2\,\beta - \frac{\pi}{2}) = -\dfrac{1}{\tan(2\,\beta)} = \dfrac{y}{x} = \dfrac{1-x^2}{2\,x}$, $\cos\beta = \dfrac{x}{\sqrt{1+x^2}} = \dfrac{\overline{B,P'}}{b\,R} = \dfrac{b}{1+v}$, $\overline{B,P'} = R\,\dfrac{x}{2}$, $v = \dfrac{(1-x)^2}{2\,x}$, $\sin\beta = \dfrac{1}{\sqrt{1+x^2}} = \dfrac{p}{b}$, $p = \dfrac{1}{2}$, $\overline{E,B} = \overline{E,B'} = R\,(1-y) = R\,\dfrac{1+x^2}{2}$, $z = v/\tan\beta = \dfrac{(1-x)^2}{2}$, $\overline{V,F} = R\sqrt{z^2 + v^2} = R\,\dfrac{(1-x)^2}{2\,x}\sqrt{1+x^2}$, $\overline{W,L} = R\,\dfrac{z}{\tan(2\,\beta - \frac{\pi}{2})} = R\,\dfrac{x\,(1-x)}{1+x}$, $\overline{L,D} = R\,\dfrac{1-x}{\tan(2\,\beta - \frac{\pi}{2})} = R\,\dfrac{2\,x}{1+x}$, $\overline{F,L} = \dfrac{z\,R}{\sin(2\,\beta - \frac{\pi}{2})} = z\,R\,\dfrac{1-y}{y} = R\,\dfrac{(1+x^2)\,(1-x)}{2\,(1+x)}$, $\overline{L,B'} = R\,\dfrac{1-x}{\sin(2\,\beta - \frac{\pi}{2})} = R\,\dfrac{1+x^2}{1+x}$, $\overline{L,C} = R\,\dfrac{v}{\tan(2\,\beta - \frac{\pi}{2})} = R\,\dfrac{1-x}{1+x}$, $\overline{F,P} = \dfrac{p\,R}{\cos\beta} - \overline{V,F} = R\,\dfrac{\sqrt{1+x^2}\,(2-x)}{2}$, $\overline{C,P'} = R\,(1 - b\cos\beta) = R\,\dfrac{2-x}{2}$.
With the origin $O = B$ in the $(\hat{x},\hat{y})$-plane the straight lines g and g' are given by:
$g:\ \hat{y} = -x\,(\hat{x} + R\,(1+v)) = -x\,\hat{x} - R\dfrac{1+x^2}{2}$, $g':\ \hat{y} = \dfrac{\hat{x}}{x}$.

Observation 1′: If x varies from 0 to R then P moves on the middle line $\hat{y} = \dfrac{R}{2}$ from $\hat{x} = 0$ to $\dfrac{R}{2}$.
Like depicted in *Figure 3*, one now folds the corners B *and* C onto the same point $C' = B'$ on the opposite side. The distance of C' from corner A is $x\,R$.

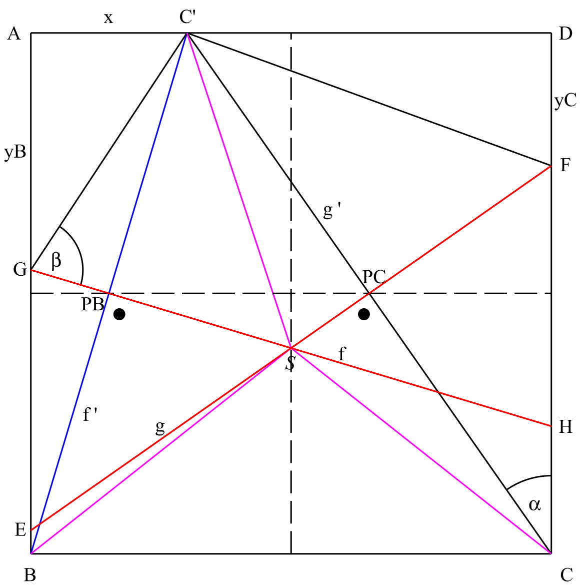


Figure 3: Folding C and B onto $C' = B'$

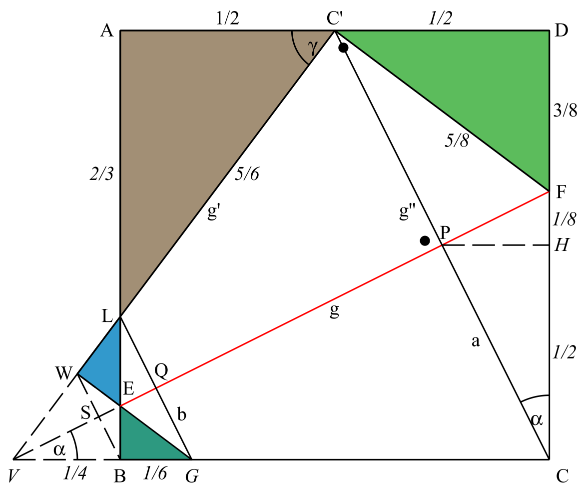


Figure 4: *Haga*'s triple of Egyptian triangles

Figure 3:
$\overline{A,B} = R = \overline{A,D}$, $\overline{A,C'} = R\,x$, $\overline{D,F} = R\,y_C$, $\overline{A,G} = R\,y_B$, $\overline{PC,C'} = \overline{PC,C} =: R\,c$, $\overline{PB,C'} = \overline{PB,B} =: R\,b$, $\angle(E,PC,C) = \frac{\pi}{2} = \angle(B,PB,H)$, $\angle(D,C,C') = \alpha = \angle(PC,C',F)$, $\angle(PB,G,C') = \beta = \angle(B,G,PB)$, $\angle(D,F,C') = 2\,\alpha$, $\angle(A,C',G) = 2\,\beta - \dfrac{\pi}{2}$, $\angle(B,C',C) = \alpha - \beta + \dfrac{\pi}{2}$.

The two creases (in red) are $\overline{E,F}$, on the straight line g, and $\overline{G,H}$, on the straight line f.

PC is the intersection point of the perpendicular straight lines g and g' connecting the points C' and C. PB is the intersection point of the perpendicular straight lines f and f' connecting the points C' and B. S is the intersection point of the two creases g and f.

The analytic data depending on R (usually taken as 1 length unit) and x, is (see the above data for *Figures 1* and *2*) :

$2\,c = \sqrt{1 + (1-x)^2}$, $2\,b = \sqrt{1 + x^2}$, $\overline{F,C'} = \overline{F,C} = R\,(1 - y_C) = R\,\dfrac{2-2\,x+x^2}{2}$, $\overline{G,C'} = \overline{G,B} = R\,(1 - y_B) = R\,\dfrac{1+x^2}{2}$..

In the $(\hat{x},\hat{y})$-plane with origin $O = B$ the coordinates of PC and PB are $\left[R\,\dfrac{1+x}{2}, R\,\dfrac{1}{2}\right]$ and $\left[R\,\dfrac{x}{2}, R\,\dfrac{1}{2}\right]$, respectively.

The straight lines of the two creases g and f are $\hat{y} = (1-x)\left(\hat{x} + R\,\dfrac{x^2}{2\,(1-x)}\right)$ and $\hat{y} = -x\left(\hat{x} + R\,\dfrac{1 + x^2}{2\,x}\right)$, respectively. This leads to the intersection point S with coordinates $\left[\dfrac{R}{2}, \dfrac{R}{2}\,(1 - x + x^2)\right]$.

Because S lies on the vertical median of the square one obtains $\overline{B,S} = \overline{C,S} = \dfrac{R}{2}\,\sqrt{1+(1-x+x^2)^2} = \dfrac{R}{2}\,\sqrt{2-x\,(1-x)\,(2-x\,(1-x))} = \dfrac{R}{2}\,\sqrt{(1 + (1-x)^2)\,(1 + x^2)}$. From crease g it is clear that $\overline{C,S} = \overline{C',S}$. This proves analytically the following second theorem of *Kazuo Haga* (named like this in [2], p. 115-116).

Theorem 1 (*K. Haga* see *Figure 3*)

1) The intersection point S of the two creases $\overline{E,F}$ and $\overline{G,H}$ lies on the vertical median of the square of length R, for each $x \in [0,R]$.

2) The length of the three lines $\overline{C',S}$, $\overline{B,S}$ and $\overline{C,S}$ are identical, namely $R\,\sqrt{2-x\,(1-x)\,(2-x\,(1-x))}$. Moreover,

3) The intersection points PC and PB lie on the horizontal median of the square of length R, and their distance is $\dfrac{R}{2}$, independently of $x \in [0,R]$.

The first part of **2)** follows immediately from the x-coordinate of S, *i.e.*, from **1)**. For the second part also the $y-$coordinate of S is needed.

Problem 2: Haga's Theorem on Egyptian Triangles

This is found in the book of *Bellos* [2] on p. 114, and has also been attributed to *Kazuo Haga*. Three Egyptian triangles (or scaled Pythagorean triangles) appear when folding a corner (vertex) of a square sheet of paper (here C) onto the midpoint of one of the non-adjacent sides (see *Figure 4* point C'). The crease is the straight line g. The right triangles are $T_1 = \triangle(F,D,C')$, $T_2 = \triangle(C',A,L)$ and $T_3 = \triangle(L,W,E)$. $T_3' = \triangle(G,B,E)$ folds onto T_3. Each of these right triangles has rational side lengths, and they are scaled $(3,4,5)-$Pythagorean triangles.

Theorem 2 (*K. Haga*)

If the length of the square R is taken as 1 length unit then the sides of the three right triangles of *Figure 4* have side lengths:

$T_1 : \left(\frac{3}{8}, \frac{1}{2}, \frac{5}{8}\right)$, $T_2 : \left(\frac{1}{2}, \frac{2}{3}, \frac{5}{6}\right)$, $T_3 : \left(\frac{1}{8}, \frac{1}{6}, \frac{5}{24}\right)$.

Therefore, if the length of the square R is chosen as 24 length units, these triangles become *Pythagorean* triangles $\widehat{T_1} : (9,\ 12,\ 15) = 3 * (3,\ 4,\ 5)$, $\widehat{T_2} : (12,\ 16,\ 20) = 4 * (3,\ 4,\ 5)$ and $\widehat{T_3} : (3,\ 4,\ 5)$.

Proof: The notation, with the length of the side of the square being R length units, is: C maps to C', B maps to W. L is the intersection point of $\overline{V, C'}$ (the straight line g' with segment $\overline{W, L}$) and $\overline{A, B}$. L maps to G. P is the midpoint of $\overline{C', C}$, Q is the midpoint of $\overline{L, G}$ and S is the midpoint of $\overline{W, B}$. The angle $\gamma = \angle(L, C', A)$ equals $2\,\alpha$ because $\angle(E, L, W) = \frac{\pi}{2} - 2\,\alpha = \angle(C', A, L)$

Similar right triangles with angle α are $\triangle(V, B, E)$, $\triangle(V, C, F)$, $\triangle(F, P, C)$, $\triangle(B, S, V)$, $\triangle(B, E, S)$, $\triangle(L, E, Q)$, and their mirrors obtained by folding along g. The four shaded right triangles with angle $\gamma = 2\,\alpha$ are also similar. $\tan\,\alpha = \frac{1}{2}$, $\sin\,\alpha = \frac{\tan\,\alpha}{\sqrt{1 + (\tan\,\alpha)^2}} = \frac{1}{5}\sqrt{5}$, $\cos\,\alpha = \frac{1}{\sqrt{1 + (\tan\,\alpha)^2}} = \frac{2}{5}\sqrt{5}$, $\tan(2\,\alpha) = \frac{2\,\tan\,\alpha}{1 - (\tan\,\alpha)^2} = \frac{4}{3}$, $\sin(2\,\alpha) = \frac{4}{5}$, $\cos(2\,\alpha) = \frac{3}{5}$.

The analytic data is: $\overline{A, C'} = \frac{1}{2} R = \overline{C', D}$, $\overline{C', P} = a = \overline{P, C} = R\,\frac{\sqrt{5}}{4}$, $\overline{L, Q} = b = \overline{Q, G} = R\,\frac{\sqrt{5}}{12}$, $\overline{W, E} = \overline{E, B} = R\,\frac{1}{8}$, $\overline{W, L} = \overline{B, G} = R\,\frac{1}{6}$, $\overline{L, E} = \overline{E, G} = R\,\frac{5}{24}$, $\overline{V, B} = \overline{V, W} = R\,\frac{1}{4}$, $\overline{V, F} = R\,\frac{5}{8}\sqrt{5}$, $\overline{V, E} = R\,\frac{1}{8}\sqrt{5}$, $\overline{E, Q} = R\,\frac{1}{24}\sqrt{5}$, $\overline{Q, P} = R\,\frac{1}{3}\sqrt{5}$, $\overline{V, S} = R\,\frac{1}{10}\sqrt{5}$, $\overline{S, B} = R\,\frac{1}{20}\sqrt{5}$, $\overline{S, E} = R\,\frac{1}{40}\sqrt{5}$. □

Problem 3: Solving Third 0rder Equations using Origami

It is well known that cubic (and higher) order equations cannot be solved geometrically using only an (unmarked) ruler and a compass (see e.g., Wantzel[15], Adler [1]. §36, pp. 188-195). It is also known that the general cubic equation can be solved geometrically with two right angles (at least one of them should have a scale in order to mark the absolute value of the coefficients) [6], p. 267, Abb. 150, adapted from [1], pp. 259-261, Fig. 156 (where we use the solving chain of straight lines A, X, Y, E with $\angle(B, A, X) = \hat{\omega}$ (not the ω of the figure), $x = \tan\,\hat{\omega} = \frac{\overline{B, X}}{a_0} = \frac{\overline{Y, C}}{\overline{X, C}} = \frac{\overline{D, E}}{\overline{Y, D}}$ and $\overline{X, C} = |a_1| - \overline{B, X} = |a_1| - a_0\,x$, $\overline{D, Y} = |a_2| + \overline{Y, C} = |a_2| + x\,\overline{X, C}$. With $\overline{D, E} = a_3 = x\,\overline{D, Y}$ this leads to the cubic equation $x^3 - |a_1|\,x^2 - |a_2|\,x + a_3 = 0$ if one takes $a_0 = 1$. Note that a_0 and a_1 are supposed to have opposite signs because, coming from $\overline{A, B}$, one takes a 90^o left turn at B to get to C. Similarly, a_1 and a_2 have like signs because, coming from $\overline{B, C}$, one takes a left turn at C to get to D. Then a_2 and a_3 have again opposite signs because of the left turn at D. The length of the lines are always positive. These sign rules are taken from [6], and in [1] a different solving path, namely A, F, G, H, has been chosen. This type of figure is also found in [5], p. 198, referring to *Fig. 3* on p. 207 (where the top vertex is A' which is connected to B' on line B. The distance between A' and I is x, and the distance between A and B' is y).

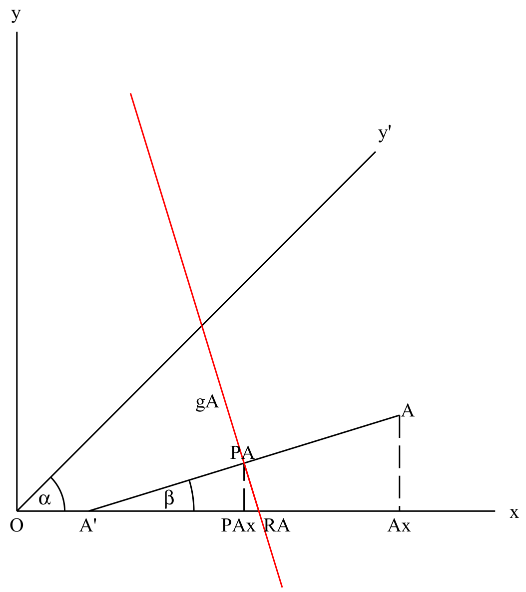


Figure 5: Folding A onto A'

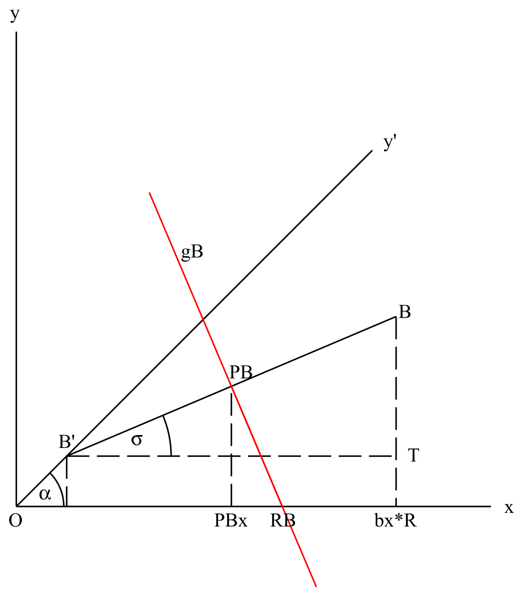


Figure 6: Folding B onto B'

In order to solve the general cubic equation $\mathbf{X^3 + a\,X^2 + b\,X - c = 0}$ with origami, following Huzita [5], p. 197 and Fig. 2, one first folds , like in *Figure 5*, a point A onto A' on the x-axis. The coordinates in the $[x, y]$-plane with length unit R are $A : [a_x\,R,\, a_y\,R]$ and $A' : [\tilde{x}\,R,\, 0]$. The y' axis with angle α has been added for later purposes, and is not relevant for this folding. g_A is the crease perpendicular to $\overline{A', A}$ with intersection point $P_A : [P_{A,x}, P_{A,y}]$. The crease hits the x-axis at point R_A. The inputs are a_x, a_y and $\tilde{x} = \frac{\overline{O,A'}}{R}$. Instead of $\tilde{x}$ we shall use $X := a_x - \tilde{x}$. The right triangles $\triangle(A', A_x, A)$ and $\triangle\,(P_A, P_{A,x}, R_A)$ are similar. The following analytic expressions are found immediately.

$\overline{A, P_A}/R =: a = \overline{P_A, A'}/R = \frac{1}{2}\,\frac{a_y}{\sin\,\beta} = \frac{1}{2}\,\frac{X}{\cos\,\beta}$. $\tan\,\beta = \frac{a_y}{X}$. $\overline{A', P_{A,x}} = \overline{P_{A,x}, A_x}$, $\overline{P_A, P_{A,x}} = \frac{a_y\,R}{2}$, $\overline{P_{A,x}, R_A} = \frac{R}{2}\,a_y \tan\beta$, $\overline{O, R_A} = \left(a_x - X + \frac{a_y}{2\tan\beta} + \frac{1}{2}\,a_y \tan\beta\right) R = \frac{1}{2}\left(\frac{a_y^2}{X} + 2\,a_x - X\right) R.$

The straight line g_A (the crease) satisfies $\frac{y}{R} = -\frac{X}{a_y}\left(\frac{x}{R} + \frac{1}{2}\,(X - \frac{a_y^2}{X} - 2\,a_x)\right)$.

Then a similar folding, shown in *Figure 6*, is done, in order to map a point B (different from A) onto a point B' on the y'-axis, forming some angle α from $\left(0, \frac{\pi}{2}\right)$ with the $x-$axis. The crease g_B is perpendicular to the line $\overline{B', B}$ with intersection at the midpoint P_B. It hits the $x-$axis at the point R_B. The (x, y) coordinates of B' are $[b'_x\,R,\, b'_y\,R]$. $\angle(B, B', T) = \sigma$. The inputs are $B : [b_x\,R, b_y\,R]$ and $\tilde{y} = \overline{0, B'}/R$. The right triangles $\triangle\,(B', T, B)$ and $\triangle\,(P_B, P_{B,x}, R_B)$ are similar. One finds:

$b'_y = \tilde{y}\,\sin\,\alpha$, $b'_x = \tilde{y}\,\cos\,\alpha$. $\tan\,\sigma = \frac{b_y - b'_y}{b_x - b'_x}$. $\overline{O, R_B} = R\,\frac{1}{2}\left(b_x + b'_x + \frac{(by - b'_y)^2}{b_x - b'_x} + 2\,b'_y\,\tan\,\sigma\right) = R\,\frac{1}{2}\,\frac{b_x^2 + b_y^2 - \tilde{y}^2}{b_x - b'_x}$.

The straight line g_B (the crease) satisfies $y = -\frac{1}{\tan\,\sigma}\,(x - \overline{O, R_B}) = -\frac{b_x - \tilde{y}\,\cos\,\alpha}{b_y - \tilde{y}\,\sin\,\alpha}\,x + R\,\frac{1}{2}\left(\frac{b_x^2 + b_y^2 - \tilde{y}^2}{b_y - \tilde{y}\,\sin\,\alpha}\right)$.

In [5] the coordinate axes y' and x' are used. The transformation between coordinates of a point P : $[p_x\,R,\, p_y\,R]$ and $[p_{x'}\,R,\, p_{y'}\,R]$ is $p_{x'} = p_x - \frac{p_y}{\tan\,\alpha}$ and $p_{y'} = \frac{p_y}{\sin\,\alpha} = \sqrt{1 + 1/(\tan\,\alpha)^2}\,p_y$.

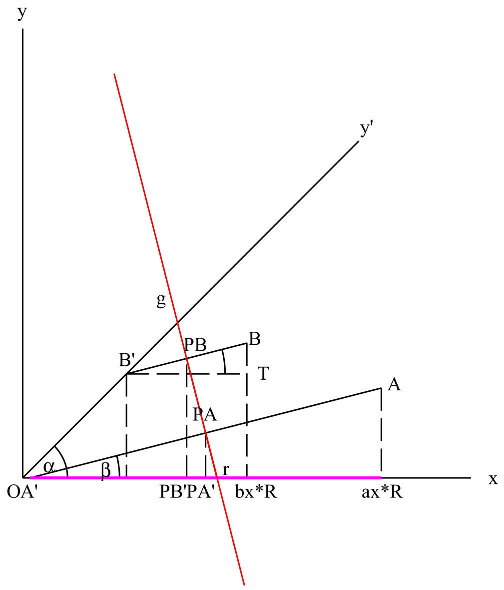


Figure 7: Third order eq. Folding A onto A', and B onto B', $\alpha = 45^o$

The typical origami which brings at the same time one point A onto A' on, say the x-axis, and another point B onto B' on some other axis (here y') is then shown to correspond to a cubic equation for a certain line segment (here $X = a_x - a'_x$). See *Figure 7* where $\alpha = 45^o$.

Theorem 3 (*Huzita* [5]): There exists a folding which brings $A : [a_x\,R,\, a_y\,R]$ onto $A' : [a'_x\,R,\, 0]$ and $B : [b_x\,R,\, b_y\,R]$,onto $B' : [b'_x\,R,\, b'_x\,\tan\alpha\,R]$ on the y' axis, which forms an angle $\alpha \in \left(0, \frac{\pi}{2}\right)$ with the $x-$axis. The solution for crease g is $y = -\dfrac{X}{a_y}(x - r\,R)$, with $r = \dfrac{1}{2}\left(\dfrac{a_y^2}{X} + 2\,a_x - X\right)$, and $X := a_x - a'_x$ is the real solution of the cubic equation

$$X^3 + \left(b_x - 2\,a_x + \frac{b_y - a_y}{\tan\alpha}\right) X^2 + a_y \left(2\,b_y - a_y + 2\,\frac{a_x - b_x}{\tan\alpha}\right) X - a_y^2 \left(b_x - \frac{a_y - b_y}{\tan\alpha}\right) = 0\,.$$

Before giving the proof a remark and an example are in order.

Remark: In [5] the components of A and B with respect to the axis X and Y (with $k = \cos\angle(X,Y)$) correspond to the above given $(a_{x'}, a_{y'})$ and $(b_{x'}, b_{y'})$, and $k = \cos\alpha$. Therefore z of eq. (1) on p. 197 is given by $z = a_{x'} - a'_{x'} = a_x - \dfrac{a_y}{\tan\alpha} - a'_x = X - \dfrac{a_y}{\tan\alpha}$ (with our $X = a_x - a'_x$ and $a'_y = 0$).

Example 1: In *Figure 7* we have chosen $\alpha = \dfrac{\pi}{4}$ and $R = 1$ length unit. With $A : [.8,\, .2]$ and $B : [.5,\, .3]$ one has the real solution of $X^3 - X^2 + 0.2\,X - 0.024 = 0$, which is $X \approx 0.7839279132$ (Maple 10 digits). This leads to $r = 0.4335485933$. X has been indicated by the fat (magenta) line in *Figure 7.* $a'_x = \tilde{x} = 0.8 - X \approx 0.0160720868$.

Proof: One combines the foldings of *Figure 5* and *Figure 6* with the constraint that the two creases, the straight lines g_A and g_B, coincide. This leads to two equations: (I) $\overline{O, R_A} = \overline{O, R_B}$ and (II) $\beta = \sigma$.

$$(\overline{O, R_A} - \overline{O, R_B})/R =$$
$$\frac{1}{2\,X\,(b_x - b'_x)}\left(X^2\,\tilde{y}^2 + \frac{X^2 - 2\,a_x\,X - a_y^2}{\sqrt{1 + (\tan\alpha)^2}}\,\tilde{y} - b_x\,X^2 + (2\,a_x\,b_x - b_x^2 - b_y^2)\,X + a_y^2\,b_x\right) = 0\,.$$

The pre-factor does not vanish and it is not divergent ($b_x \neq b'_x$), therefore the bracket term has to vanish. The other restriction is

$$\tan\sigma - \tan\beta = \frac{b_y - \tilde{y}\sin\alpha}{b_x - \tilde{y}\cos\alpha} - \frac{a_y}{X} = 0\,.$$

Solving for $\tilde{y}$ as a function of X yields

$$\tilde{y} = \sqrt{1 + (\tan\alpha)^2}\,\frac{a_y\,b_x - b_y\,X}{a_y - X\tan\alpha}\,.$$

Inserting this $\tilde{y}$ into the bracket term of $(\overline{O,R_A} - \overline{O,R_B})/R$ results in a factorized form given by

$$\frac{(b_y - b_x\tan\alpha)\,X}{(-a_y + \tan\alpha\,X)^2}\,\big[(\tan\alpha)\,X^3 + ((b_x - 2\,a_x)\tan\alpha + (b_y - a_y))\,X^2 +$$
$$+ (a_y\,(2\,b_y - a_y)\tan\alpha + 2\,(a_x - b_x))\,X + a_y^2\,(a_y - b_y - b_x\tan\alpha)\big] = 0\,.$$

Because B does not lie on the y' axis $b_y \neq b_x\tan\alpha$, and $\beta \neq \alpha$, hence $a_y \neq X\tan\alpha$. Therefore, the new pre-factor does neither vanish nor diverge, and the bracket term has to vanish.

Now two cases have to be considered:

i) $\tan\alpha \neq 0$ and $\neq \infty$, *i.e.*, $\alpha \in \left(0, \frac{\pi}{2}\right)$ and

ii) $\tan\alpha = \infty$, or $\alpha = 90^o$.

The case $\alpha = 0^o$ will later be treated separately.

i): This case leeds to the cubic equation for X given in *Theorem 3* after dividing $\tan\alpha$ out. The equation for the crease g is just given by g_A from above (see *Figure 5*), with $r = \overline{O,R_A}$ and $\beta = \sigma$ from condition (II). ☐

Special case ii) $\boldsymbol{\alpha = \frac{\pi}{2}}$ (see *Figures 8* and *9*)

Theorem 4

With the notation of *Theorem 3* and $\alpha = \dfrac{\pi}{2}$ the equation for $X = a_x - a'_x$ is

$$X^3 + (b_x - 2\,a_x)\,X^2 + a_y\,(2\,b_y - a_y)\,X - a_y^2\,b_x = 0\,.$$

Proof: Extract $\tan\alpha$ from the bracket term of$(\overline{O,R_A} - \overline{O,R_B})/R = 0$, factorized above, and observe that the original pre-factor $\dfrac{1}{2\,X\,(b_x - b'_x)}$ when multiplied with the factor $\dfrac{(b_y - b_x\tan\alpha)\,X}{(-a_y + \tan\alpha\,X)^2}$ and after extraction of the $\tan\alpha$ from the bracket term becomes, in the limit $\tan\alpha \to \infty$, $\dfrac{1}{2\,X^2}$, provided $b_x \neq 0$. This new factor does neither vanish nor diverge, and from the bracket term the claimed cubic equation for X is obtained.

This result can also be reached in the limit $\tan\alpha \to \infty$ from the above equation for X in *Theorem 3*, which, however, has been derived assuming $\tan\alpha \neq 0$. ☐

For the case $\alpha = \dfrac{\pi}{2}$ see *Figures 8* and *9*. The other quantities are found by first folding B onto B' on the $y-$axis. See *Figure 8*. The free parameter is $\tilde{y} = \overline{O,B'}$. $\delta = \dfrac{\pi}{2} - \gamma$, $b = \overline{B,PB} = \overline{PB,B'} = \dfrac{1}{2}\sqrt{(\tilde{y} - b_y)^2 + b_x^2}$, $\tan\gamma = \dfrac{\overline{O,PB_x} - \overline{O,R_B}}{b_y + b\sin\gamma}$, $PB_x : [R\,(b_x - \cos\gamma\,b), 0]$, $PB_y : [0, R\,(b_y + \sin\gamma\,b)]$, $RB : [PB_x[1] - \tan\gamma\,PB_y[2], 0]$. The crease is $g : y = \frac{1}{\tan\gamma}\,(x - RB[1])$.

Then also A is folded onto A' on the $x-$axis. This has been treated in connection with *Figure 5* (were the $y'-$axis was not important). The two constraints are $RA = RB$ and $\tan\delta = tan\,(\frac{\pi}{2} + \beta)$, *i.e.*,

$\tan\gamma = -\tan\beta$. One obtains a real solution of the cubic equation for $X = a_x - a'_x$ indicated by the thick (magenta) line segment in *Figure 9*, and $\tilde{y} = -b_x \dfrac{a_y}{X} + b_y$. $\square$

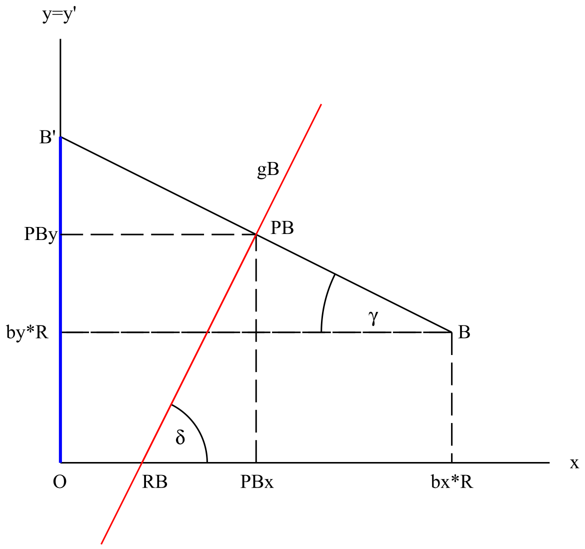
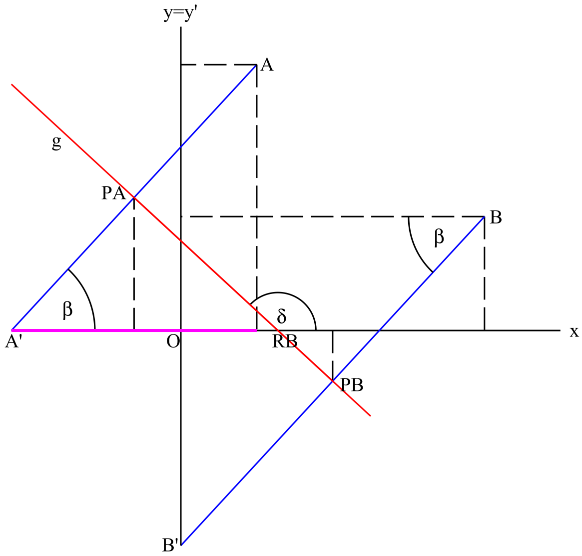


Figure 8: Folding B onto B'



Case $\alpha = \frac{\pi}{2}$

Figure 9: Folding A onto A' and B onto B'

Example 2: $R = 1$ length unit, $A:\ [.2, .7]$, $B:\ [.8, .3]$. $X = a_x - a'_x \approx .646416$, $\beta \approx 47,279^o$, $\delta = (90 + \beta)^o \approx 137.279$, $\overline{O, R_B} \approx .2558$, $P_A : [\approx -.123, .35]$, $P_B :\ [.4, \approx -.133]$.

Special case $\alpha = 0$

The special case $\alpha = 0$ is obtained from folding A onto A', discussed above (see *Figure 5*), and folding B onto B' (also on the $x-$axis). One uses the formulae given above in connection with *Figure 5* (with β changed into β_A) and replaces there A by B and A' by B' (with $\beta = \beta_B$). Then *Figure 10* is obtained by setting $R_A = R_B$ and $\beta_A = \beta_B$.

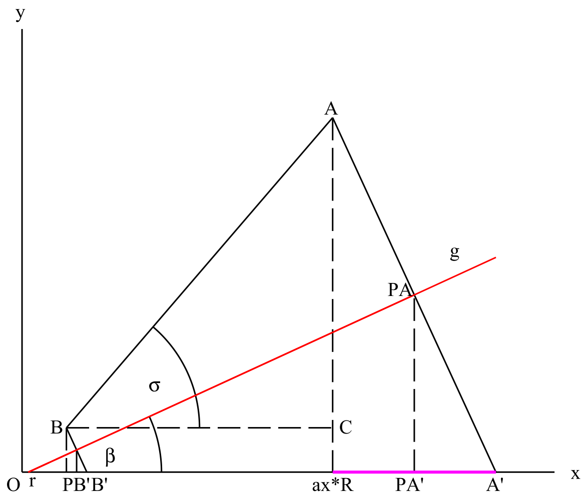


Figure 10: Folding A on A' and B on B' when $\alpha = 0$

The result of identifying both creases g_A and g_B, calling them g, leads to the following data: $X := a_x - a'_x$, $Y := b_x - b'_x$. Setting $\overline{O, R_A} - \overline{O, R_B} = 0$ leads to the equation involving X and Y.

$$- X + Y + a_y^2 \frac{1}{X} - b_y^2 \frac{1}{Y} + 2\,(a_x - b_x) = 0 \,.$$

The slope of g is given by $\tan \beta_A = \tan \beta_B$, which expresses Y in terms of X:

$$Y = \frac{b_y}{a_y} X \,.$$

The mid points of $\overline{A\,A'}$ and $\overline{B\,B'}$ are $P_A : \left[R\,\frac{2\,a_x - X}{2}, R\,\frac{a_y}{2}\right]$ and $P_B : \left[R\,\frac{2\,b_x - Y}{2}, R\,\frac{b_y}{2}\right]$, respectively. The intercept is $r = \overline{O, R_A} = \overline{O, R_B} = \frac{R}{2}\left(\frac{a_y^2}{X} + 2\,a_x - X\right)$.

For the following one assumes that $0 \neq a_y \neq b_y$. Plugging Y into the previous equation results in a quadratic equation for X:

$$X^2 - 2\,a_y \frac{1}{\tan\,\sigma} X - a_y^2 = 0 \,,$$

with $\tan\,\sigma = \frac{a_y - b_y}{a_x - b_x}$.

This equation can be obtained directly from the above given analysis for non-vanishing α before $\tan\,\alpha$ has been divided out. Just let there $\tan\,\alpha \to 0$, which eliminates the X^3 term, and divide by $b_y - a_y$. The relevant solution for X is then

$$X = a_x - a_{x'} = -a_y \frac{1 - \cos\,\sigma}{\sin\,\sigma} \,.$$

Example 3: $R = 1$ length unit, $A : [.7, .8]$, $B : [.1, .1]$. $\sigma \approx 49.399^o$, $-X \approx 0.368$, and $r \approx 0.014$.

If $a_y = b_y$, the equation for X becomes linear, in fact $X = 0$.

Degenerate case: Parallel lines with A′ and B′

As mentioned in [5], p. 197, the case of parallel lines, is also of interest. See *Figure 12*.

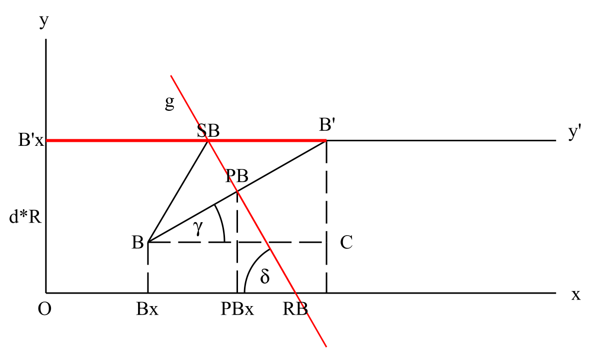


Figure 11: Folding B onto B'

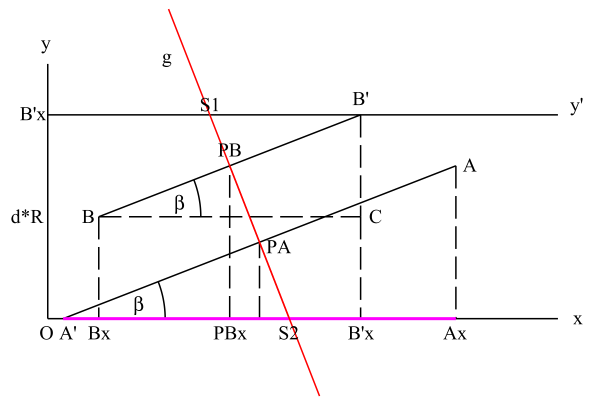


Figure 12: Folding A onto A' and B onto B'

First one folds B onto B' which lies on the horizontal y'-axis with $y = d\,R$, hence $B'_y = d\,R$, with some length scale R. This is shown in *Figure 11*.

Figure 11:

Given $d = b'_y$, A and B, the free parameter is $b'_x = \frac{B'_x}{R}$, the position of B' on the y'-axis. The straight line g is perpendicular to the line segment $\overline{B, B'}$ and passes through its midpoint P_B. This line g intersects the parallel x- and y'-axes at R_B and S_B, respectively. The angle δ equals $\frac{\pi}{2} - \gamma$. Half the distance between B and B' is $R\,b$.

The formulae are: $b = \frac{b'_x - b_x}{2\cos\gamma}$, $\tan\gamma = \frac{b'_y - b_y}{b'_x - b_x}$, $\cos\gamma = \frac{1}{\sqrt{1 + (\tan\gamma)^2}}$, $\sin\gamma = \frac{\tan\gamma}{\sqrt{1 + (\tan\gamma)^2}}$,

$PB:\ [R\,(b_x + b\cos\gamma),\, R\,(b_y + b\sin\gamma)]$, $R_B:\ \left[R\,\frac{{b'_x}^2 - b_x^2 + {b'_y}^2 - b_y^2}{2\,(b'_x - b_x)},\, 0\right]$,

$g:\ y = -\frac{1}{\tan\gamma}\,(x - R_{B,x})$, $S_B:\ [R_{B,x} - d\tan(\gamma)\,R,\, d\,R]$. If B lies on the y'-axis then $\gamma = 0$ and $\delta = \frac{\pi}{2}$, a simple special case.

Now A is folded onto A' on the x-axis at the same time as B is folded onto B'. See *Figure 12*. From the above results in connection with *Figure 5* one takes R_A and $\tan\beta$ in terms of $X := a_x - \tilde{x}$, where $\overline{O, A'} = R\,\tilde{x}$. The solution is obtained from $R_A = R_B$ and $\tan\beta = \tan\gamma$. The latter equation can be used to eliminate b'_x by (remember that $d = b'_y$)

$$b'_x = \frac{1}{a_y}\left((d - b_y)\,X + b_x\,a_y\right)\,,$$

provided $a_y \neq 0$, which we assume. If $a_y = 0$ then $b_y = b'_y = d$, $\beta = \gamma = 0$ and $\delta = \frac{\pi}{2}$, a simple special case. From $2\,\overline{O, R_A} = 2\,\overline{O, R_B}$ one obtains

$$X + \frac{a_y^2}{X} + 2\,a_x - \frac{{b'_x}^2 - b_x^2 + d^2 - b_y^2}{b'_x - b_x} = 0\,.$$

If b'_x is inserted one finds a quadratic equation for X, assuming that $a_y\,X$ does neither vanish nor diverge. Because $a_y \neq 0$ has been assumed, if $X = 0$ then $a_x = a'_x$, $b'_x = b_x$ $\gamma = \beta = \frac{\pi}{2}$, $\delta = 0$, another simple special case. With a_y also X will be finite.

$$(a_y - b_y + d)\,X^2 - 2\,a_y\,(a_x - b_x)\,X - a_y^2\,(a_y - b_y - d) = 0.$$

Therefore, assuming $(b_y \neq d + a_y)$ one finds the positive solution for X

$$X = \frac{a_y}{d + a_y - b_y}\left(a_x - b_x + \sqrt{(a_x - b_x)^2 + (a_y - b_y)^2 - d^2}\right)\,.$$

This shows, that a solution is only possible if $d\,R$ does not exceed the distance between A and B, which has been observed in [5]. It is clear that P_A and P_B have coordinates which are the arithmetic mean between the corresponding coordinates of A and A' and B and B', e.g., $P_{B,x} = R\,\frac{b_x + b'_x}{2}$. Note that $\overline{B, B'} = \frac{d - b_y}{a_y}\,\overline{A, A'}$. Therefore the trapezoid A', A, B' and B becomes a rectangle precisely if $d = a_y + b_y$.

Example 4: Put $R = 1$ length unit and take $d = .4$, $a_x = .8$, $a_y = .3$, $b_x = .1$, $b_y = .2$, then $X = a_x - a'_x \approx 0.77$, $b'_x \approx 0.61$.

Application 1: The case of the heptagon equation

The minimal polynomial of the algebraic number $\rho(7) := 2\cos\left(\frac{\pi}{7}\right) \approx 1.801937736$ (the length ratio of the larger diagonal and the side of a regular 7-gon) is $C(7,x) = x^3 - x^2 - 2x + 1$ (see *e.g.*, [7]), *Table 2* and *section 3*). The three real zeros are known to be $x(7;k) = 2\cos\left(k\frac{\pi}{7}\right)$, for $k = 1$, 3, and 5. They are $x(7;1) = \rho(7)$, $x(7;3) \approx .4450418670$ and $x(7;5) = -2\cos\left(2\frac{\pi}{7}\right) \approx -1.246979604$.

Here we show how these zeros are obtained by three different origamis. We also treat the standard geometric solution of this cubic equation using two right angular rulers, as explained in [6] based on [1] (see also *von Sanden* [14], ch. III, sect. 2, pp. 55-61, with Fig. 17 on p. 55). The corresponding *Figures* are *13*, *14*, and *15*. The slope of the y' axis is chosen as $\alpha = 90^o$, thus $y' = y$. The monic cubic equation $C(7,x) = 0$ has sign pattern $+,-,-,+$. This leads, in the standard geometrical construction, to the right angle pattern l, r, l, with l and r for a 90^o left and right turn, respectively. One starts with some (oriented) horizontal line segment $\overline{B,C}$ of length $a_0 = 1$ (for the monic case in some length unit R). A 90^o left turn gives $\overline{C,D}$ of length $a_1 = 1$, then a 90^o right turn leads to $\overline{D,E}$ of length $a_2 = 2$, and finally the 90^o left turn leads to $\overline{E,A}$ of length $a_3 = 1$. (The starting point has been chosen as B in order to comply with the later origami solution). This pattern ('Streckenzug' or line segment zig-zag) is dictated by the cubic equation and will be the same for each of the three solutions.

In the origami version one needs the two perpendicular axes $y' = y$ and x. As explained in [5], p. 198-199 and *Fig. 4* on p. 208, the y-axis is chosen parallel to $\overline{CD}$, at a perpendicular distance $2a_0 = 2$ from point B. The x-axis is parallel to $\overline{DE}$ at a perpendicular distance $2a_3 = 2$ from point A. See the present *Figure 13*. In our case point B has coordinates $[-2,0]$ and $A: [1,2]$ (if $R = 1$) . In the standard geometrical construction of a solution to the cubic equation one has to find a point F on the axis with line segment $\overline{C,D}$, here $F: [-1,x]$, such that a line perpendicular to $\overline{B,F}$ through F hits point G on the straight line with segment $\overline{D,E}$, and a perpendicular line to $\overline{F,G}$ through G hits point A. In *Figure 13* the solution has $F = P_B \equiv PB$ and $G = P_A \equiv PA$. For a general cubic equation there will always be at least one real solution, and depending on its discriminant one will find either one, two or three real solutions. In general the discriminant is $Disc = p^3 + q^2$, with $q := \frac{1}{2}\left(2\frac{a_1^3}{27} - \frac{a_1 a_2}{3} + a_3\right)$ and $p := \frac{1}{9}(3a_2 - a_1^2)$. In our case $Disc = -\frac{7^2}{2^2 3^3} < 0$, telling that there are three (different) real solutions, in accordance with the explicitly known ones. Therefore, one expects three different constructions for the given right angle zig-zag B, C, D, E, A. In the origami version we expect to find three (different) creases g_1, g_2 and g_3 each for folding simultaneously A onto some A' on the x-axis and B onto some B' on the y-axis. The three *Figures 13, 14* and *15* show these solutions.

For all three figures the zeroes of $C(7,x)$ are $x = \frac{\overline{C,D}}{R} = \frac{P_{B_y}}{R} > 0$. $F = P_B$ and $G = P_A$. The folding $A \to A'$ works like earlier described in connection with *Figure 5*. For $B \to B'$ with $B: [-2R, 0]$ (we use the length unit R here) and $B': [0, \tilde{y}]$ one has for the mid-point $P_B: \left[-R, \frac{\tilde{y}}{2}\right]$.

With $\gamma = \angle(D, P_B, P_A)$, $\tan\gamma = \frac{R + \overline{O, R_B}}{P_{B_y}} = \frac{P_{B_y}}{R}$. Hence $\overline{O, R_B} = \frac{P_{B_y}^2}{R} - R = \frac{\tilde{y}^2}{4R} - R$. $\tan\delta = \tan\left(\frac{\pi}{2} + \gamma\right) = -\frac{1}{\tan\gamma}$. The equation for the crease is $g_B: y = -\frac{1}{\tan\gamma}(x - \overline{O, R_B})$. (Here x is a cartesian varable.) Putting then $\tan\beta = \tan\gamma$ (with $\beta = \angle(A, A', O)$) yields $\tilde{y} = 2\frac{a_y}{\hat{X}} = \frac{4R}{\hat{X}}$ with $\hat{X} := \frac{X}{R}$, where $X = R - a'_x$. Together with $\overline{0, R_A} = \overline{0, R_B} =: rR$ one finds the cubic equation for $\hat{X}$:

$$\hat{X}^3 - 4\hat{X}^2 - 4\hat{X} + 8 = 0$$

for each of the three figures.

Figure 13:

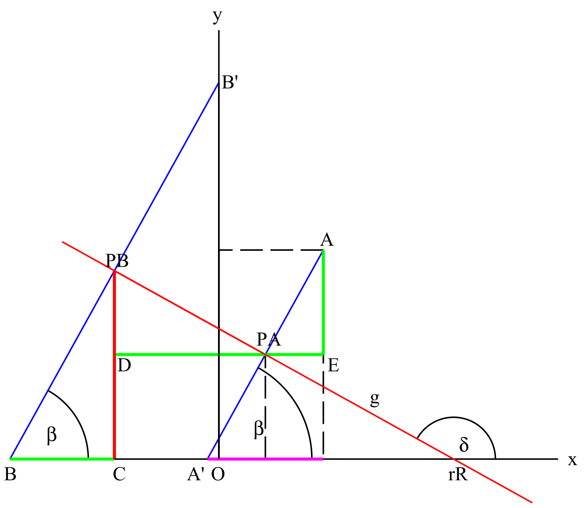
Here $\tilde{x} = a'_x/R < 0$, *i.e.*, $X = R + |a'_x| > 0$. $x = \dfrac{\overline{C, P_B}}{R} = \dfrac{\tilde{y}}{2\,R} = \dfrac{2}{\hat{X}}$. The cubic heptagon equation for x, given above, is compatible with the cubic equation for $\hat{X}$. Because the three solutions for x are known from the heptagon (see above), and since here $x > 1$ one has $x = \rho(7) = 2\,\cos\left(\dfrac{\pi}{7}\right) \approx 1.801937736$, corresponding to $\hat{X} = \dfrac{2}{x} = \dfrac{1}{\cos\left(\frac{\pi}{7}\right)} = \sqrt{1 + \tan\left(\dfrac{\pi}{7}\right)^2} \approx 1.109916264$.

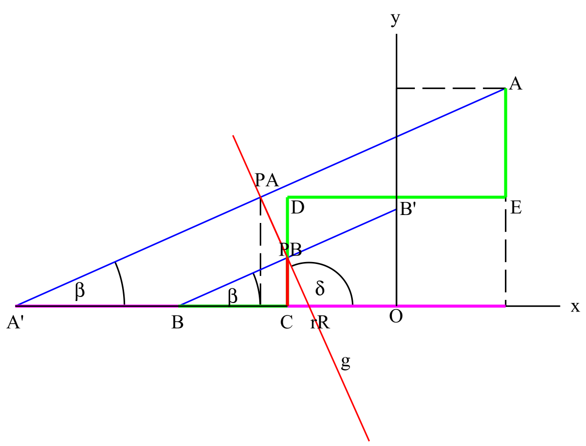
Figure 14:

$\tilde{x} = a'_x/R < 0$, *i.e.*, $X = R + |a'_x| > 0$. $x = \dfrac{\overline{C, P_B}}{R} = \dfrac{\tilde{y}}{2\,R} = \dfrac{2}{\hat{X}}$. Because $0 < x < 1$, one has $x = x(7;3) = 2\,\cos\left(\dfrac{3\,\pi}{7}\right) \approx .4450418670$, corresponding to $\hat{X} = \dfrac{2}{x} = \dfrac{1}{\cos\left(\frac{3\pi}{7}\right)} = \sqrt{1 + \tan\left(\dfrac{3\,\pi}{7}\right)^2} \approx 4.493959217$.

Figure 15:

$\tilde{x} = a'_x/R > 0$, *i.e.*, $X = R - a'_x < 0$. $\tilde{y} = b'_y < 0$. $0 > x = -\dfrac{\overline{C, P_B}}{R} = \dfrac{\tilde{y}}{2\,R} = \dfrac{2}{\hat{X}}$. Hence $x = x(7;5) = -2\,\cos\left(\frac{\pi\,2}{7}\right) \approx -1.246979604$, corresponding to $0 > \hat{X} = \dfrac{2}{x} = -\dfrac{1}{\cos\left(\frac{2\,\pi}{7}\right)} =$- $\sqrt{1 + \tan\left(\dfrac{2\,\pi}{7}\right)^2} \approx -1.603875472$.





Case $\alpha = \dfrac{\pi}{2}$

Figure 13: Heptagon, first origami

Figure 14: Heptagon, second origami

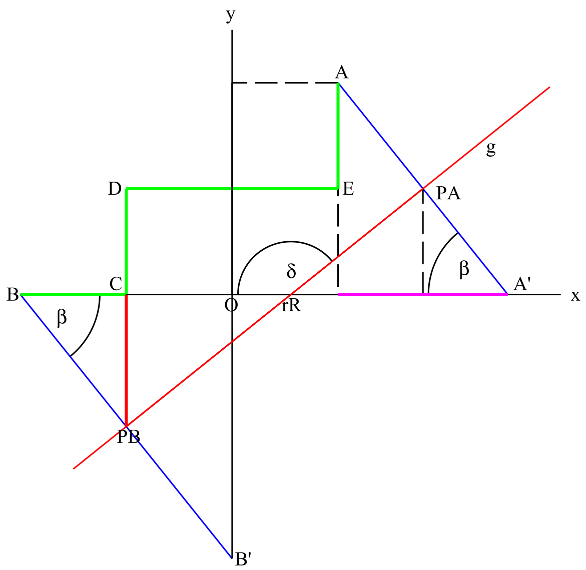


Figure 15: $\alpha = \frac{\pi}{2}$, heptagon, third origami

Application 2: Doubling the cube

This classical problem cannot be solved by ruler and compass, but with Origami this can be accomplished because one has to solve the third order equation $x^3 - 2 = 0$. See [8] and *Figure 16*. (i) First one has to find a third of $\overline{A,B}$. This is a standard origami problem solved by finding the intersection point X of the two creases $\overline{A,C}$ and $\overline{A,E}$ where E is found by halving the square, bringing $B \to A$ and $C \to D$. If the length of the side of the square A, B, C, D is taken as 1 (in some length unit) then $\overline{B,H} = \frac{1}{3} = \overline{C,J}$.

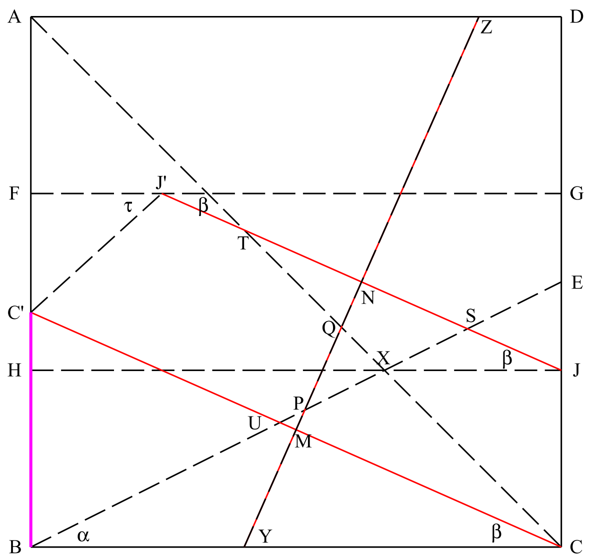


Figure 16: Doubling the cube; finding $\overline{C',B}$

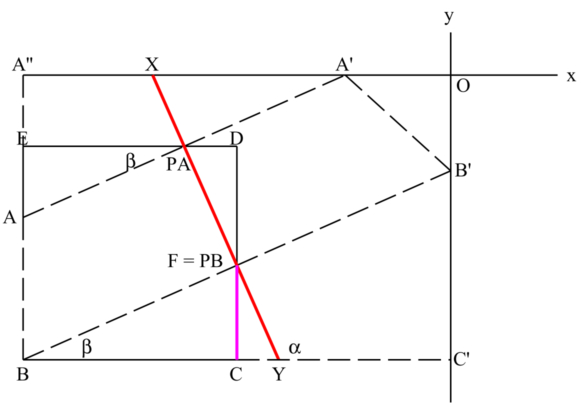


Figure 17: Doubling the cube; standard version

To see this one just has to find the intersection of the two straight lines $y = \frac{1}{2}x$ and $y = -x + 1$ finding $X : \left[\frac{2}{3}, \frac{1}{3}\right]$. (ii) Folding $\overline{A,D}$ onto $\overline{H,J}$ will then generate the crease $\overline{F,G}$ which completes the task to divide the square into three equal parts. $F:\ \left[0, \frac{2}{3}\right]$, $G:\ \left[1, \frac{2}{3}\right]$. (iii) The crucial origami is then to fold at the same time $C \to C'$ with C' on the line $\overline{A,B}$ and $J \to J'$ with J' on the line $\overline{F,G}$. The claim is that $x := \frac{\overline{A,C'}}{\overline{C',B}} = 2^{1/3}$, or with $s := \overline{C',B}$, *i.e.*, $x = \frac{1-s}{s}$, $\mathbf{s^3 - s^2 + s - \frac{1}{3} = 0}$. The discriminant is $Disc = +\frac{1}{3^4}$, showing that there is only one real solution, which is $s = \frac{1}{3}(2^{2/3} - 2^{1/3} + 1) \approx 0.4424933339$. See [9] A246644 for the decimal expansion of s.

The analytic **proof** is obtained from looking at the right triangle $\triangle(C', F, J')$ with angle $\tau := \angle(F\,J', C')$. This angle can be computed by identifying the trapezoid angle $\angle(C', J', J)$ with the one $\angle(J', J\,C) = \beta + \frac{\pi}{2}$. Then $\beta + \tau + (\beta + \frac{\pi}{2}) = \pi$, *i.e.*, $\tau = \frac{\pi}{2} - 2\beta$. Now $\sin\tau = ((1-s) - \frac{1}{3})/(\frac{1}{3}) = 2 - 3s$, which is also $\sin\left(\frac{\pi}{2} - 2\beta\right) = 2\cos^2\beta - 1$. But $\cos\beta = \frac{1}{\sqrt{1+s^2}}$ from $\tan\beta = s$, and thus $\sin\tau = \frac{1-s^2}{1+s^2}$. Equating this with $2 - 3s$ leads to the claimed equation for s, hence the one for the ratio x. ☐

We list more analytic data for *Figure 16* with $s := \overline{C',B}$, $b := \overline{C',M} = \overline{M,C}$ and $a := \overline{J',N} = \overline{N,J}$: $\tan\alpha = \frac{1}{2}$, $\tan\beta = s/1 = s$, $\sin\beta = \frac{s}{2b} = \frac{1}{3}/(2a) = \frac{1}{6a}$. $\tan\tau = \frac{1-s^2}{2s}$ (from the second formula for $\sin\tau$ given above). $J':\ \left[\frac{3s-1}{3s}, \frac{2}{3}\right]$ from $\overline{F,J'} = 1 - \overline{J',G} = 1 - \frac{1}{3\tan\beta} = 1 - \frac{1}{3s} \approx 0.2466929834$. $M:\ \left[\frac{1}{2}, \frac{s}{2}\right]$, $N:\ \left[\frac{6s-1}{6s}, \frac{1}{2}\right] = \left[\frac{1+s-s^2}{2}, \frac{1}{2}\right]$. $P:\ \left[\frac{s^2-1}{s-2}, \frac{1}{2}\frac{s^2-1}{s-2}\right]$, $Q:\ \left[\frac{1+2s-s^2}{2(1+s)}, \frac{1+s^2}{2(1+s)}\right]$. $S:\ \left[\frac{2}{3}\frac{1+3s}{1+2s}, \frac{1}{3}\frac{1+3s}{1+2s}\right]$, $T:\ \left[\frac{1}{3}\frac{2-3s}{1-s}, \frac{1}{3(1-s)}\right]$, $U:\ \left[\frac{2s}{1+2s}, \frac{s}{1+2s}\right]$, $Y:\ \left[\frac{1-s^2}{2}, 0\right]$, $Z:\ \left[\frac{1-s^2}{2} + s, 1\right]$. The equation for the crease $\overline{Y,Z}$ is $y = \frac{1}{s}\left(x - \frac{1-s^2}{2}\right)$.

Standard version to find s with $\mathbf{s^3 - s^2 + s - \frac{1}{3} = 0}$

The cubic equation for $s = \overline{C',B}$ can also be solved geometrically in the standard fashion, similar to finding x in the heptagon case treated above. Here the sign pattern is $+,-,+,-$, which means that the 90^o chain pattern is l, l, l. This leads to the line chain B, C, D, E, A shown in *Figure 17* which is identical with B, C, D, A, F in *Figure 16* (the scale of both figures is different). In this case the y-axis is parallel to $\overline{D,C}$ such that the x-coordinates of B becomes -2 (for $R = 1$ length unit), and the x-axis is parallel to $\overline{E,D}$ (in *Figure 16* this is $\overline{A,D}$) such that the y-coordinate of B becomes $-\frac{1}{3} - 1 = -\frac{4}{3}$, using $\overline{E,A} = \frac{1}{3} = \overline{E,A''}$. Here the F in the standard geometrical construction (not to be confused with F in *Figure 16*) with $\overline{C,F} = s$ is shown in *Figure 17* . That is the $[x, y]$ coordinates of this $F = PB$ from the origami construction are $\left[-1, -\left(\frac{4}{3} - s\right)\right]$. $B \to B'$ on the y-axis with coordinates $B':\ [0, -\frac{4}{3} + 2s]$ (from the continuation of the line element $\overline{B,PB}$ to the $y-$axis). $A \to A'$ with coordinates A': $\left[-2\left(1 - \frac{1}{3s}\right), 0\right]$ because $\overline{A,A'}$ is parallel to B, B' with slope $\tan\beta = s$. The two midpoints defining the crease are $F = P_B$ and $P_A:\ \left[-2(1 - \frac{1}{6s}), -\frac{1}{3}\right]$. The slope of the crease $\overline{X,Y}$, shown in *Figure 17*, is $\tan\alpha = -\frac{1}{\tan\beta} = -\frac{1}{s}$. The equation for the crease is

$y = -\frac{1}{s}\left(x + \frac{s^2 + 6s - 1}{3s}\right)$. The data for the trapezoid A, B, B', A' is: $a := \overline{A, PA} = \overline{PA, A'} = \frac{\sqrt{1+s^2}}{3s} \approx 0.8237612353$, and $b := \overline{B, PB} = \overline{PB, B'} = \sqrt{1+s^2} \approx 1.093526566$. $\overline{A, B} = \overline{A, A''} = \frac{2}{3}$.

Completing the task of doubling a given cube

Up to now we have only found the doubling of the cube with side length s. In *Figure 16* we had $2^{1/3}\,\overline{C', B} = \overline{C', A}$, *i.e.*, $2\,s^3 = (1 - s)^3$. In *Figure 17* we had $2^{1/3}\,\overline{PB, C} = \overline{D, PB}$. For the decimal expansion of $2^{1/3}$ see [9] A002580. But the task is to double a cube with given side length L. If one takes as length of the side of the square $R = \frac{L}{s}$, then $L = \overline{B, C'}$ and $M = \overline{C', A} = 2^{1/3}\,L$ is the side length for the doubled cube. However, we first have to find *via* origami $1/s \approx 2.259921051$. But this can be achieved by considering the parallel to $\overline{C, C'}$ through A. This parallel will hit the continuation of $\overline{B, C}$ on some point C'' with coordinates $\left[\frac{1}{s}, 0\right]$ (origin at B, x-axis along $\overline{B, C}$ and y-axis along $\overline{B, A}$). See *Figure 18*. This means that if we take the length scale $R = L$ the searched length $M = 2^{1/3}\,L$ for the doubled cube is given by $\overline{C, C''}$ which is $\left(1 - \frac{1}{s}\right) L$. It is easy to find the parallel g_2 in *Figure 18* by origami. First find $g1$, the crease perpendicular to crease g through the point A (this can be done, as explained in the introductory remarks; axiom 4 or IV). Then find $g2$ as the crease perpendicular to $g1$ through point A. Finally the square C, C'', D'', A'' can be completed.

The coordinates of some points are: $C:\ [L, 0]$, $C':\ [0, s\,L]$, $C'':\ \left[\frac{L}{s} = (1 + 2^{1/3})\,L),\, 0\right]$, $V:\ [-s\,L, 0]$, $W:\ \left[-\frac{s\,(1-s)}{1 + s^2},\, L\,\frac{s\,(1+s)}{1 + s^2}\,L\right]$, $D'':\ \left[\frac{L}{s},\, 2^{1/3}\,L\right]$.

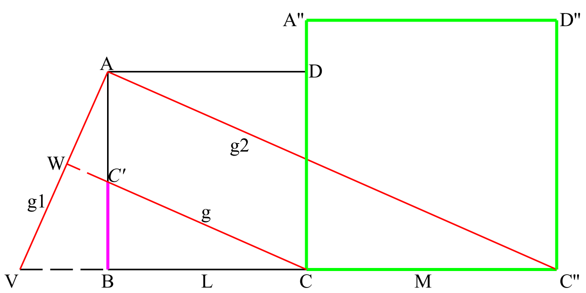


Figure 18: Doubling the cube: finding $M = 2^{1/3}\,L$

Application 3: Trisection of an angle

This is another classical problem not solvable with ruler and compass but with origami. See [3], [5], [8]. We first discuss the origami shown in [5] *Fig. 1*, pp. 204-5, and also in [8]. See the present *Figure 19*, where the angle is $\alpha = \angle(P, B, C)$. Because the origami solution will be based on a cubic equation with three real roots, the question of the meaning of the other two roots arises. The answer can be found in [5]: the origami prescription for trisecting a given angle is not unique and the other two solutions correspond to trisecting the angle $\pi - \alpha$ and $\pi + \alpha$. This will be treated at the end. of this section.

In the square (A, B, C, D) the point P on $\overline{A, D}$ defines the angle $\alpha = \angle PBC$ to be trisected and the crease $g1$. Then an arbitrary horizontal crease $g2$ defining points E and F with distance $2h < 1$ from the base line $\overline{B, C}$ is folded. The dashed crease $g3$ bringing B onto E and C onto F has then a distance h from the base line. The crucial folding g is then to bring point E onto E' on crease $g1$ and simultaneously point B onto B' on crease $g3$. This will also bring point G to G'. The continuation of $\overline{B, B'}$ will intersect the line $\overline{D, C}$ at a point V, defining crease $g4$ .The intersection point of crease g with crease $g3$ is called Y. This defines the blue crease $g5$ with line segment $\overline{B, Y}$ crossing the continuation of the top line $\overline{A, D}$ at a point Q (depending on the choice of P and h this point Q could also lie on $\overline{A, D}$, *e.g.*, for $\alpha = 70^o$ and $h = 0.2$). The claim is now that the blue crease $g5$ and the crease $g4$ trisect the angle α with $\sigma = \frac{\alpha}{3}$. One also shows that G' lies on the blue crease $g5$.

The following analytic data is given for a coordinate system with origin at B, the $x-$axis along $\overline{B, C}$ and the y-axis along $\overline{B, A}$. The input quantities are R, the length of the square in some length unit, $\overline{A, P} = x = \dfrac{1}{tan\,\alpha}$, with input α in radians, and $\overline{B, G} = h\,R$:

$\beta = \frac{\pi}{2} - \sigma$, $\overline{B, C} = R$, $E: [0, 2\,h\,R]$, $G: [0, h\,R]$, $\quad F: [R, 2\,h\,R]$, $H: [R, h\,R]$, $B': \left[\dfrac{h}{\tan\sigma} R,\, h\,R\right]$,
$E': \left[h\,\dfrac{\cos\alpha}{\sin\sigma} R,\, h\,\dfrac{\sin\alpha}{\sin\sigma} R\right]$, $G': \left[\dfrac{1 + \cos(2\sigma)}{\tan(2\sigma)} h\,R,\, (1 + \cos(2\sigma))\,h\,R\right]$, $Y: \left[\dfrac{h}{\tan(2\sigma)} R,\, h\,R\right]$,
$X: \left[\dfrac{h}{2\tan\sigma} R,\, \dfrac{h}{2} R\right]$, $L: \left[0,\, \dfrac{h}{\sin(2\sigma)\,tan\,\sigma} R\right]$, $J: \left[\dfrac{h}{\sin(2\sigma)} R,\, 0\right]$,
$K: \left[\sin(2\sigma)\,\dfrac{(L_y/R - h)}{2} R,\, (h + \sin^2\sigma\,(L_y/R - h)\,R\right]$,
$Z: \left[\sin(2\sigma)\,\dfrac{(L_y/R - 2h)}{2} R,\, (2h + \sin^2\sigma\,(L_y/R - 2h)\,R\right]$, $Q: \left[\dfrac{1}{\tan(2\sigma)} R,\, R\right]$, $V: [R,\, R\tan\sigma]$.

The equations for the creases are (x is here the abscissa): $g1: y = \tan(\alpha)\,x$, $g2: y = 2h$, $g3: y = h$, $g: y = \tan\left(\dfrac{\pi}{2} + \sigma\right)\left(x - \dfrac{h}{\sin(2\sigma)}\right)$, $g4: y = \tan(\sigma)\,x$, $g5: y = \tan(2\sigma)\,x$.

Some lengths in the trapezoid (B, B', E', E) are: $\overline{E', G'} = \overline{E, G} = h\,R = \overline{G', B'} = \overline{G, B}$, $b := \overline{B, X} = \overline{X, B'} = \dfrac{h}{2\sin\sigma} R$, $e := \overline{E, Z} = \overline{Z, E'} = \dfrac{\sin\alpha - 2\,sin\,\sigma}{2\sin^2\sigma} h\,R$, $\overline{B, Y} = \dfrac{h}{\sin(2\sigma)} R$.

Now to the **proof** of the trisection of $\alpha = \angle(P, B, C)$. Name the three angles, called σ in *Figures 19* and *20*, as follows. $\tau := \angle(X, B, J)$, $\sigma := \angle(Y, B, X,)$ and $\eta := \angle(E', B, G') = \alpha - (\tau + \sigma)$. We want to show that $\tau = \sigma$ and $\eta = \sigma$ which implies $\alpha = 3\sigma$.

Consider the angle $\varepsilon := \angle(G', Y, B')$. (Note that at this stage it is not yet clear that $\varepsilon = \angle(Y, B, J)$ which would immediately show that $\sigma = \tau$. This is because it is not yet clear that the line $\overline{Y, G'}$ (obtained from folding along g where Y is the intersection of g with $\overline{G, H}$) really continues to point B. For this one has to prove $\sigma = \tau$.) Because $\angle(G, Y, L) = \angle(B, J, L) =: \beta = \frac{\pi}{2} - \tau$, and also $\angle(K, Y, G') = \angle(K, Y, G) = \beta$ from the folding along g, we have $\pi = 2\beta + \varepsilon$, or $\varepsilon = \pi - 2\beta = 2\tau$. The proof that $\sigma = \tau$ is done by starting with $\sigma = \angle(X, B', Y)$ from the folding along g. Also $\angle(Y, B', G') = \dfrac{\pi}{2} - \varepsilon$ because of the folding along g the right angle $\angle(Y, G, E)$ appears also as $\angle(Y, G', E')$ and E, G' and B' are on a straight line, like E, G and B. Because $\angle(X, B', Y) = \dfrac{\pi}{2} - \tau$ we have from the right angle $\angle(J, B', E')$ also $\angle(J, B', G') = \frac{\pi}{2} = \tau + \sigma + (\frac{\pi}{2} - \varepsilon)$ *i.e.*, $0 = \tau + \sigma - 2\tau$ or $\sigma = \tau$. This proves that the point G' lies on the straight line connecting B and Y, defining the crease $g5$. Finally, $\eta = \sigma$ because the blue line $\overline{B, G'}$ is the height, let its length be k, in $\triangle(B, B', E')$ and $tan\,\sigma = \frac{h}{k} = tan\,\eta$ (because $\overline{G', B'} = \overline{G, B} = \overline{G', E'} = h$ from folding along g). This implies that this triangle is isosceles, *i.e.*, $\overline{B, E'} = \overline{B, B'}$. □

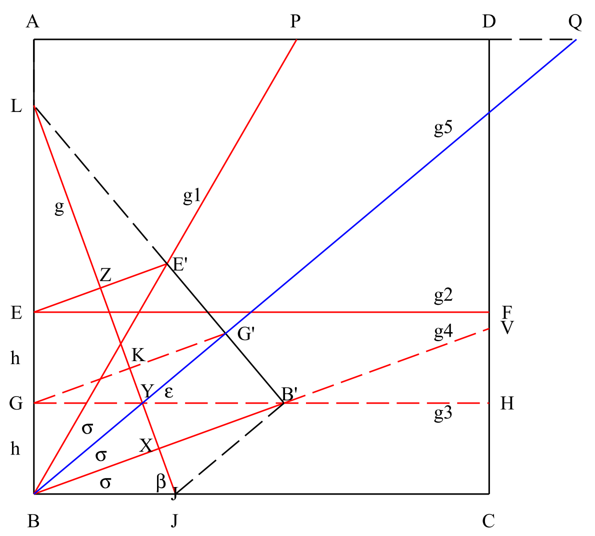


Figure 19: Trisecting an angle $\alpha = 60^o$ and $h = 0.2$

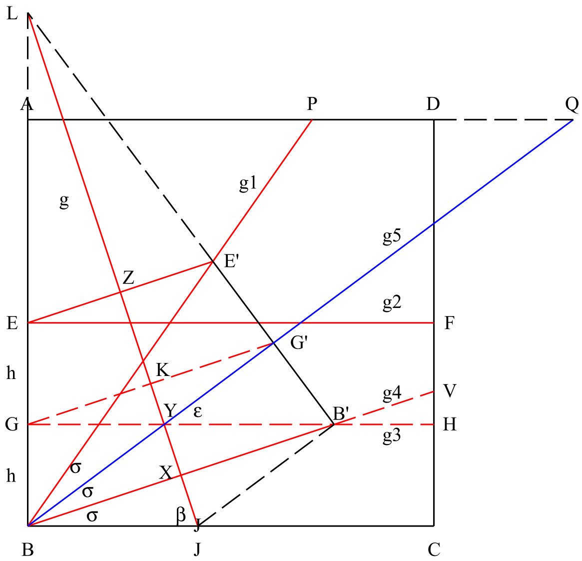


Figure 20: Trisecting an angle $\alpha = 55^o$ and $h = 0.25$

As mentioned above, in [5] the cubic equation relevant for this origami trisection of an angle α is identified and the question of the other two roots is answered. The crucial prescription to fold $B \to B'$ and, at the same time $E \to E'$, has three solutions, corresponding to the three real solutions of the cubic equation governing this folding. In *Figure 21* the origami for the trisection of the angle α is repeated but now $X1 := \overline{G, B'_x}$ has been shown as a fat line segment (in blue). If one uses the results from above for this case one will find that the dimensionless $\hat{X}1 := \dfrac{X1}{R} = \dfrac{\overline{G, B'}}{R} \dfrac{h}{tan\, \frac{\alpha}{3}}$ satisfies the following cubic equation.

$$\hat{X}^3 - \frac{3\, h}{\tan\, \alpha} \hat{X}^2 - 3\, h^2\, \hat{X} + \frac{h^3}{\tan\, \alpha} = 0\ .$$

The discriminant of this cubic is $Disc = -\dfrac{h^6\,(1 + \tan(\alpha)^2)^2}{\tan(\alpha)^4}$, hence negative, therefore there are three different real solutions. The other two solutions $X2$ and $X3$ are shown in the *Figures 22* and *23*, respectively.

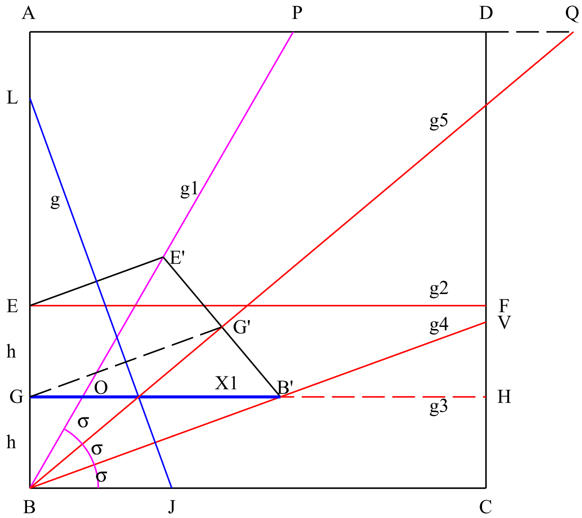


Figure 21: Solution $X1$: Trisecting an angle α
$\alpha\ =\ 60^o$ and $h\ =\ .2$

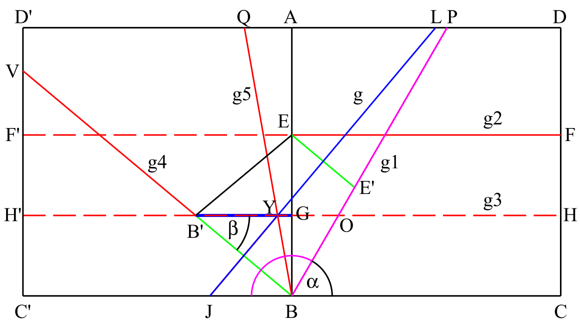
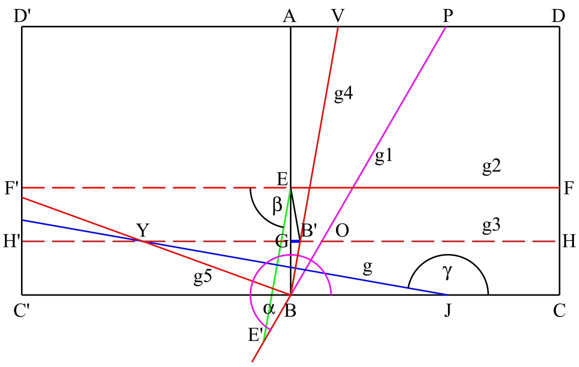


Figure 22: $X2$, Trisecting an angle $\pi\ -\ \alpha$
$180^0\ -\ 60^o\ =\ 120^o,\ h\ =\ .3$



Trisecting an angle $\pi\ +\ \alpha$
$180^0\ +\ 60^o\ =\ 240^o,\ h\ =\ .2$

Finally, we list some analytic data for the three trisection *Figures*. The data for *Figure 21* has been given already. The origin is taken at B with the $x-$axis along $\overline{B,C}$ and the y-axis along $\overline{B,A}$. Note that in the origami for the general cubic equation treated above the y' axis was along the crease $g1$ and the origin was there at $O:\ \left[\dfrac{h\,R}{tan\,\alpha},\ h\,R\right]$. $\sigma\ =\ \dfrac{\alpha}{3}$, as used also above .

Figure 21: $\mathbf{\hat{X}1\ =\ \dfrac{X1}{R}\ =\ \dfrac{\overline{G,B'}}{R}}$ ($\approx$ $\mathbf{0.5495\ for\ R\ =\ 1,\boldsymbol{\alpha}\ =\ 60^o,h\ =\ .2}$)

$$0\ <\ \frac{h}{\hat{X}1}\ =\ tan\,\sigma\ =\ \frac{e'_y\ -\ 2\,h}{e'_x}\ =\ \frac{\sin\,\alpha\ -\ 2\,\sin\,\sigma}{\cos\,\alpha}\ .$$

In the first equation the mapping $B \to B'$ and in the second one $E \to E'$ has been considered. Thus $\hat{X}1 = \frac{h}{\tan\sigma}$, which is in the Figure 21 shown for $R = 1$, $\alpha = 60^o$ and $h = .2$ which has the above given value.

Figure 22: $\mathbf{\hat{X}2 = \frac{X2}{R} = -\frac{\overline{G,B'}}{R}}$ $(\approx -\mathbf{.23835}$ **for** $\mathbf{R = 1, \alpha = 60^o, h = .3})$

$\beta = \frac{\pi - \alpha}{3}$, $\overline{B,G} = h\,R = \overline{G,E}$, $\tan\beta = \tan\left(\frac{\pi-\alpha}{3}\right) = \frac{h}{|\hat{X}2|} = \frac{2\,h\,R - e'_y}{e'_x}$,

$\overline{B,B'} =: 2\,b\,R = \frac{h\,R}{\sin\beta}$, $\overline{E,E'} =: 2\,e\,R = \frac{e'_x}{\cos\beta}$, $B' : \left[-\frac{h\,R}{\tan\beta}, h\,R\right]$,

$E' : \left[2\,h\,R\,\frac{\cos\alpha\,\cos\beta}{\sin(\alpha+\beta)}, 2\,h\,R\,\frac{\sin\alpha\,\cos\beta}{\sin(\alpha+\beta)}\right]$, $J : \left[-\frac{h\,R}{\sin(2\,\beta)}, 0\right]$.

The equations for the creases are: $g:\ y = \frac{1}{\tan\beta}, (x + \frac{h\,R}{\sin(2\,\beta)})$, $g1:\ y = (\tan\alpha)\,x$,
$g4:\ y = (\tan\beta)\,x$, $g5:\ y = -\tan(2\,\beta)\,x$.

Figure 23: $\mathbf{\hat{X}3 = \frac{X3}{R} = \frac{\overline{G,B'}}{R}}$ $(\approx +\mathbf{0.0353}$ **for** $\mathbf{R = 1, \alpha = 60^o, h = .2})$

$\beta = \frac{\pi + \alpha}{3}$, $\gamma = \frac{\pi}{2} + \beta$, $\overline{B,G} = h\,R = \overline{G,E}$, $\tan\beta = \tan\left(\frac{\pi+\alpha}{3}\right) = \frac{h}{\hat{X}3} = \frac{2\,h\,R + |e'_y|}{|e'_x|}$,

$\overline{B,B'} =: 2\,b\,R = \frac{h\,R}{\sin\beta}$, $\overline{E,E'} =: 2\,e\,R = \frac{|e'_x|}{\cos\beta}$, $B' : \left[\frac{h\,R}{\tan\beta}, h\,R\right]$,

$E' : \left[2\,h\,R\,\frac{\cos\alpha\,\cos\beta}{\sin(\beta-\alpha)}, 2\,h\,R\,\frac{\sin\alpha\,\cos\beta}{\sin(\beta+\alpha)}\right]$, $J : \left[\frac{h\,R}{\sin(2\,\beta)}, 0\right]$.

The equations for the creases are: $g:\ y = -\frac{1}{\tan\beta}\left(x + \frac{h\,R}{\sin(2\,\beta)}\right)$, $g1:\ y = (\tan\alpha)\,x$,
$g4:\ y = -(\tan(\beta)\,x$, $g5:\ y = -\tan(\beta - \alpha)\,x$.

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