Karlsruhe
February 9 2011
corr. February 23 2011

Divisor Product Representation for Natural Numbers

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Abstract

The unique representation of positive integers in terms of divisor products is presented. The computation of the minimal polynomials of $\cos\left(\frac{2\pi}{n}\right)$*, given elsewhere, is based on this representation.*

The minimal polynomial $\Psi(n,x)$ of the algebraic number $\cos\left(\frac{2\pi}{n}\right)$, for $n \in \mathbb{N}$, are listed as coefficient array in [4] A181875/A181876, where also details, references [1], [3] and [5], as well as a W. Lang link in A181875 are given. The computation of these minimal polynomials $\Psi(n,x)$ is based on a formula of the following type which is due to the recurrence given in [5],

$$\Psi(n,x) \;=\; \frac{t(n,x)\,\prod_{i=1}^{k}\,t(n_i,x)}{\prod_{j=1}^{l}\,t(m_j,x)}\;, \tag{1}$$

with certain sets of numbers $\{n_i\}_{i=1}^{k}$ and $\{m_j\}_{j=1}^{l}$. The structure numbers k and l will be determined later on, and

$$t(n,x) \;:=\; \begin{cases} \frac{1}{2^{\frac{n}{2}}}\,\left(T(\frac{n}{2}+1,x)\,-\,T(\frac{n}{2}-1,x)\right) & \text{if } n \text{ is even}\;, \\[2ex] \frac{1}{2^{\frac{n-1}{2}}}\,\left(T(\frac{n+1}{2},x)\,-\,T(\frac{n-1}{2},x)\right) & \text{if } n \text{ is odd}\,, \end{cases} \tag{2}$$

with *Chebyshev*'s T-polynomials (see A053120 for their coefficient triangle). The sets of numbers appearing in eq. (1) are determined from the representation of n in terms of the divisor products $a(k) =$ A007955(k) (see $\tau(n) =$ A000005(n) for the number of divisors of n). This representation will be called $dpr(n)$, and it has the form

$$dpr(n) \;=\; \frac{a(n)\,\prod_{i=1}^{k}\,a(n_i)}{\prod_{j=1}^{l}\,a(m_j)}\;. \tag{3}$$

Here one should keep the $a(.)$ and not replace them with their values A007955, and it is essential that $a(1)$, although it evaluates to 1, is not omitted. This is because in eq. (1) the polynomial $t(1,x) = x-1$, and not 1. We also use 'lowest terms', meaning that common a-factors in the numerator and the denominator are canceled. See, *e.g.*, the case $n = 6$ given later. This representation will become unique if we also require decreasing arguments of the a-factors in the numerator as well as in the denominator. This divisor product representation of n is immediately found from the recurrence

$$dpr(1) \;=\; a(1)\;, \tag{4}$$

$$dpr(n) \;=\; a(n)\,\frac{1}{\prod_{\substack{d|n\\ d\neq n}}\,dpr(d)}\;,\; n\geq 2\;. \tag{5}$$

This follows from the triviality $n = \dfrac{a(n)}{\prod_{\substack{d|n\\ d\neq n}}\,d}$. Later we shall use $D(n)$ for the set of all divisors of n, and $D'(n)$ for the one without n (see Definition 2).

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It is clear that for any prime number p one has $dpr(p) = \frac{a(p)}{a(1)}$. For pure powers of any number, *i.e.*, $n = m^q$, $q \in \mathbb{N}_0$, $m \in \mathbb{N}$, one has $dpr(n) = \frac{a(n)}{a(m^{q-1})}$, *e.g.*, $dpr(27) = \frac{a(27)}{a(9)}$. See *Table 1* for a list of the representations $dpr(n)$, $n = 1..80$, where also the pair (k, l) from eq. (3) is given.
An example with numerator-denominator cancellations appears already for $n = 6$:

$$dpr(6) = \frac{a(6)}{dpr(3)\, dpr(2)\, dpr(1)} = \frac{a(6)}{(a(3)/a(1))\,(a(2)/a(1))\,a(1)} = \frac{a(6)\,a(1)}{a(3)\,a(2)}\,, \tag{6}$$

where $(k, l) = (1, 2)$.
Next, we state *Proposition 1* with the solution of the recurrence eqs. (4) and (5) in the case when n is the product of N pairwise different prime numbers. We use the notation p_k for some prime number which should not be confused with the k-th prime number, which will be denoted by $p(k)$ later on.
In the inductive proof (over N) of this proposition we shall employ a *Lemma* which will be proven with the help of the induction assumption.
Proposition 1:

$$dpr\left(\prod_{j=1}^{N} p_j\right) = \frac{\mathcal{O}\left(a(n)\,\Pi\, a(.(N-2).)\,\Pi\, a(.(N-4).)\,\cdots\right)}{\mathcal{O}\left(\Pi\, a(.(N-1).)\,\Pi\, a(.(N-3).)\,\cdots\right)}\,, \tag{7}$$

with pairwise different prime numbers $p_1, ..., p_N$, and $a(.k.)$ stands for $a(\underbrace{...}_{k\ times})$. A product $\Pi\, a(.k.)$ is taken over the $\binom{N}{k}$ factors with k distinct primes from $\{p_1, ..., p_N\}$ inserted as arguments of the divisor function a. $a(.0.) := a(1)$. The last factor in the numerator is $\Pi\, a(.(N - 2\left\lfloor \frac{N}{2} \right\rfloor).)$, and the last one of the denominator is $\Pi\, a(.(N - (2\left\lfloor \frac{N-1}{2} \right\rfloor + 1)).)$. The symbol $\mathcal{O}$ in the numerator and denominator demands to order the evaluated arguments of the a-functions in a (strictly) decreasing way in order to achieve uniqueness. Note that it is not sufficient to order only within each of the a- products, as the following example shows: $N = 4$, $n = 71 \cdot 7 \cdot 5 \cdot 2$ which has in the denominator the products for $a(...)$ and $a(.)$, and the smallest $a(...)$ is $a(7 \cdot 5 \cdot 2) = a(70)$ but the biggest $a(.)$ is $a(71)$. This also holds for the primorials $n = prl(N) :=$ A002110(N), for which p_j is replaced by $p(j)$. In this case the smallest $a(...)$ is always $a(30)$ and the largest $a(.)$ could be $a(31)$ if $N = 31$. A product $\Pi\, a(.(N-k).)$, $k \in \{1, 2, ..., N\}$, has $\binom{N}{N-k}$ a-factors.
Lemma 1:

$$\Pi\, dpr(.(N-j).) = \frac{\mathcal{O}\left(\prod_{k=0}^{\left\lfloor \frac{N-j}{2} \right\rfloor} \left[\Pi\, a(.(N-j-2\,k).)\right]^{e(j,2\,k)}\right)}{\mathcal{O}\left(\prod_{k=0}^{\left\lfloor \frac{N-j-1}{2} \right\rfloor} \left[\Pi\, a(.(N-j-(2\,k+1)).)\right]^{e(j,2\,k+1)}\right)}\,. \tag{8}$$

We use for *dpr* arguments the same convention for the number of factors as above for the a-arguments. Here a product $\Pi\, a(.(N-j-l).)$ has $\binom{N}{N-j-l}$ factors, and the exponents are $e(j, l) := \frac{(j+l)!}{j!\,l!} = \binom{j+l}{l}$.
Before giving the proof, some examples may be helpful. In the proposition take $N = 4$ and $n = 11 \cdot 7 \cdot 5 \cdot 3$. The numerator of $dpr(1155)$ has, besides $a(n) = a(1155)$, the products $\Pi\, a(.2.)$ and $\Pi\, a(.0.)$ which are $a(11 \cdot 7)\, a(11 \cdot 5)\, a(11 \cdot 3)\, a(7 \cdot 5)\, a(7 \cdot 3)\, a(5 \cdot 3)$ (6 factors) and $a(1)$ (1 factor). The ordered version of the numerator is thus $a(1155)\, a(77)\, a(55)\, a(35)\, a(33)\, a(21)\, a(15)\, a(1)$. Similarly the numerator in ordered form is $a(385)\, a(231)\, a(165)\, a(105)\, a(11)\, a(7)\, a(5)\, a(3)$. For this example $(k, l) = (7, 8)$, with the notation from eq. (3).
In the lemma the structure of $\Pi\, dpr(...)$, *e.g.*, $N = 5$, $j = 2$, $n = \prod_{k=1}^{5} p_k$, is in the numerator $\mathcal{O}\left([\Pi\, a(...)]^1\, [\Pi\, a(.)]^6\right)$, and in the denominator $\mathcal{O}\left([\Pi\, a(..)]^3\, [\Pi\, a(1)]^{10}\right)$. Here $\Pi\, dpr(...)$ has 10 factors, and the products in the numerator have factors 10 and 5, and in the denominator 10 and 1. To proceed one

would have to distribute the five primes over the open places, determine the value of each $a-$argument, and order in the numerator as well as denominator. We leave the computation of the final answer as an exercise to the reader.

Proof of Proposition 1:

This is done by induction over N with the help of the recurrence formula (5), together with (4). For $N = 1$ the proposition is true due to $dpr(p) = a(p)/a(1)$. We assume that the proposition is true for all N-values $1, 2, ..., N-1$. After the first step one finds for $n = \prod_{k=1}^{N} p_k$, if we leave out the order sign $\mathcal{O}$ for the moment, reinstalling it at the end of the proof,

$$dpr(n) \,=\, a(n) \prod_{l=1}^{N} \frac{1}{\Pi\, dpr(.(N-l).)}\,, \tag{9}$$

with the last factor being $\Pi\, dpr(.0.) = dpr(.0.) = a(1)$. Here it becomes clear why we formulated *Lemma 1*. It can be applied, due to the induction assumption, for each of the $l = 1, 2, ..., N-1$ factors of the product. The last factor, the one for $l = N$, is, as just seen, $a(1)$. The proof of *Lemma 1* will be given separately but it should to be placed here because it uses the induction hypothesis. If the $N-1$ factors are replaced with the help of *Lemma 1* one will find for each l a numerator/denominator block. Note that a product $\Pi\, a(.(N - 2\,k).)$, with $k \in \{1, 2, .. \lfloor \frac{N}{2} \rfloor\}$, appears in the numerator for the blocks $l = 1, 3, ..., (2\,k-1)$, and in the denominator for $l = 2, 4, ..., 2\,k$. Note also that in the l-block the first exponent in the denominator is always $e(l, 0)$, and the first exponent in the numerator is $e(l, 1)$. A specific a-factor will appear with a certain multiplicity (exponent) in the numerator as well as denominator, and we now show that there are cancellations such that every $a(.(N - 2\,k).)$ or $a(.(N - (2\,k+1)).)$ appears exactly once, in the numerator or the denominator, respectively. This is because for $a(.(N - 2\,k).)$ one finds for the exponents the following alternating sum with $+$ signs for the numerator, and $-$ sign for the denominator:

$$\sum_{l=1}^{2k} (-1)^{k+1}\, e(l, 2\,k - l) \;=\; \sum_{l=1}^{2k} (-1)^{k+1} \binom{2k}{l} \;=\; -0 + 1 \;=\; +1\;. \tag{10}$$

This is due to the fact that the alternating sum of every even numbered row in *Pascal*'s triangle A007318 vanishes. For example, if $2k = 4$ the product $\Pi\, a(.(N - 4).)$, with $\binom{N}{N-4}$ factors, appears in the numerator for the block $l = 1$ as the second factor, for $l = 3$ as first factor, and in the denominator for $l = 2$ as second factor.

Similarly, the product $\Pi\, a(.(N - (2\,k+1)).)$, $k = 0, 1, ... \lfloor \frac{N-1}{2} \rfloor$, appears in denominator for the blocks $l = 1, ..., (2\,k+1)$, and in the numerator for the blocks $l = 2, 4, ..., 2\,k$. Therefore the exponent count is

$$\sum_{l=1}^{2k+1} (-1)^{k+1}\, e(l, 2\,k + 1 - l) \;=\; \sum_{l=1}^{2k+1} (-1)^{k+1} \binom{2k+1}{l} \;=\; -0 + 1 \;=\; +1\;. \tag{11}$$

Here one uses the fact that the odd numbered rows (not the first one, of course) have also vanishing alternating sum. □

Proof of lemma 1:

As already mentioned above, this proof should be placed inside the proof of *Proposition 1* because the induction hypothesis of the latter is used. First the general structure is considered, giving rise to multiplicities (exponents) m for the a-factors in the numerator and denominator. Then it is proved that the exponents of the various a-factors are the ones stated in *Lemma 1*.

The recurrence is used in the first step, and if we omit the order symbols $\mathcal{O}$ for simplicity, we find for $j \in \{1, 2, ..., N-1\}$,

$$\prod dpr(.(N-j).) \;=\; \prod a(.(N-j).)\, \frac{1}{\Pi\, dpr(.(N-j-1).)\, \Pi\, dpr(.(N-j-2).)\, \cdots\, \Pi\, dpr(.0.)}, \tag{12}$$

where the product on the *l.h.s.*, like the overall product on the *r.h.s.*, has $\binom{N}{N-j}$ factors, and the products in the denominator have $\binom{N-j}{N-j-k}$ factors if the argument is $.(N-j-k).$. The last product in the denominator is $\Pi\, dpr(.0.) = a(1)$. Consider, *e.g.*, $N=5$, $j=2$, where we give the number of factors as indices on the products:

$$\prod_{10} dpr(...) \;=\; \prod_{10} a(...)\, \frac{1}{\Pi_3\, dpr(..)\, \Pi_3\, dpr(.)\, a(1)}. \tag{13}$$

Each quotient $\dfrac{1}{\Pi\, dpr(.(N-j-l).)}$ is called block B_l, for $l \in \{1, 2, ..., N-j\}$. For each block the induction hypothesis of *Proposition 1* holds. We now count how often, for given k, any $a(.(N-j-2\,k).)$ or $a(.(N-j-(2\,k+1)).)$ appears in the numerator and denominator. For $a(.(N-j-2\,k).)$, which appears in the assertion of *Lemma 1* in the numerator, one counts the exponents in the numerator as positive and the ones in the denominator as negative. For $a(.(N-j-2\,k+1).)$ the exponents are counted in the opposite way.

Case $\mathbf{a(.(N-j-2\,k).)}$: Any such divisor product appears in the numerators of the odd numbered blocks B_l, for $l \in \{1, 3, ..., 2\,k-1\}$, and in the denominators for the even numbered blocks for $l \in \{2, 4, ..., 2\,k\}$. Define the multiplicity for $a(.(N-j-2\,k).)$, *i.e.*, the number of appearances of divisor products with $N-j-2\,k$ factors, as $m(N, j, 2\,k)$. From the above mentioned number of factors indicated by the various product signs one finds immediately

$$\begin{aligned} m(N, j, 2\,k) \;&=\; \binom{N}{N-j} \left\{ \binom{N-j}{N-j-1} \binom{N-j-1}{N-j-2\,k} - \binom{N-j}{N-j-2} \binom{N-j-2}{N-j-2\,k} \pm \; ... \right. \\ &\qquad \left. - \binom{N-j}{N-j-2\,k} \binom{N-j-2\,k}{N-j-2\,k} \right\} . \end{aligned} \tag{14}$$

Therefore,

$$m(N, j, 2\,k) \;=\; \frac{N!}{(N-j-2\,k)!\, j!\, (2\,k)!} \sum_{l=1}^{2\,k} \binom{2\,k}{l} \;=\; \frac{N!}{(N-j-2\,k)!\, j!\, (2\,k)!} . \tag{15}$$

In the last step the fact that the even numbered rows in *Pascal*'s triangle A007318 have vanishing alternating sum. Because a divisor product $a(.(N-j-2\,k).)$ comes only in $\binom{N}{N-j-2\,k}$ different versions one finds its exponent from

$$e(j, 2\,k) \;=\; \frac{m(N, j, 2\,k)}{\binom{N}{N-j-2\,k}} \;=\; \frac{(j+2\,k)!}{j!\, (2\,k)!} \;=\; \binom{j+2\,k}{2\,k} . \tag{16}$$

Case $\mathbf{a(.(N-j-(2\,k+1)).)}$: This type of divisor product appears in the denominators for the blocks $l = 1, 3, ..., 2\,k+1$ and in the numerators for $l = 2, 4, ..., 2\,k$. The multiplicity, counting with the denominator appearances taken positive, is

$$\begin{aligned} m(N, j, 2\,k+1) \;&=\; \binom{N}{N-j} \left\{ \binom{N-j}{N-j-1} \binom{N-j-1}{N-j-(2\,k+1)} - \binom{N-j}{N-j-2} \binom{N-j-2}{N-j-(2\,k+1)} \right. \\ &\qquad \left. \pm ... + \binom{N-j}{N-j-(2\,k+1)} \binom{N-j-(2\,k+1)}{N-j-(2\,k+1)} \right\} . \end{aligned} \tag{17}$$

Therefore,

$$m(N, j, 2\,k+1) \;=\; \frac{N!}{(N-j-(2\,k+1))!\, j!\, (2\,k+1)!} \sum_{l=1}^{2\,k+1} \binom{2\,k+1}{l} \;=\; \frac{N!}{(N-j-(2\,k+1))!\, j!\, (2\,k+1)!}. \tag{18}$$

Here the vanishing of the odd numbered rows of *Pascal*'s triangle for $2\,k+1 \geq 3$ was used. Again, the exponent of $\Pi\, a(.(N-j-(2\,k+1)).)$ is then

$$e(j, 2\,k+1) \;=\; \frac{m(N, j, (2\,k+1))}{\binom{N}{N-j-(2\,k+1)}} \;=\; \frac{(j+2\,k+1)!}{j!\,(2\,k+1)!} \;=\; \binom{j+2\,k+1}{2\,k+1}\,. \tag{19}$$

□

It is clear by now that only the structure of $dpr(n)$ is important and not the special primes used in the decomposition of n. Therefore, it is sufficient to consider in *Proposition 1* the primorials $n := \prod_{j=1}^{N} p(j) \;=\; A002110(N)$. For example, the structure of $dpr(3\cdot 7\cdot 11)$ is like the one for $dpr(2\cdot 3\cdot 5) \;=\; a(2\cdot 3\cdot 5)\,a(5)\,a(3)\,a(2)/(a(3\cdot 5)\,a(2\cdot 5)\,a(2\cdot 3)\,a(1))$ (see *Table 1*), where the primes $2, 3, 5$ have to be replaced by $3, 7, 11$, respectively. In *Table 2* the result for $dpr(A002110(N))$, for $N \;=\; 1,\, 2,\, \dots,\, 7$ is shown in an abbreviated version were only the arguments of the divisor products a for the numerator and denominator are listed.

For the general prime number decomposition $n \;=\; p_1^{e_1}\,p_2^{e_2}\,\cdots p_N^{e_N}$ it is for the same reason sufficient to consider $n \;=\; p(1)^{e(1)}\,p(2)^{e(2)}\,\cdots p(N)^{e(N)}$, with positive integers $e(j)$ which may be collected as components of the N-tuple $\vec{e}_N \;:=\; (e(1), e(2), .., e(N))$.

For the following *Theorem* we need the definition of uniform multiplication of all a-arguments in a divisor product representation.

Definition 1: $*$-multiplication.

If in $dpr(n)$ the argument of all a-factors is multiplied by some number m this is denoted by $m * dpr(n)$.

Example 1:

$$3^2 * dpr(10) \;=\; a(90)\,a(9)/(a(45)\,a(18))\;. \tag{20}$$

It is clear that this $*$-multiplication satisfies: $m_1 * (m_2 * (m_3 * dpr(n))) \;=\; (m_1\cdot m_2) * (m_3 * dpr(n)) \;=\; (m_1\cdot m_2\cdot m_3) * dpr(n)$. Therefore, no brackets are needed in $*$-products, and $m_1 * (m_2 * dpr(n)) \;=\; m_2 * (m_1 * dpr(n))$. The identity operation is $1 * dpr(n) \;=\; dpr(n)$. It is obvious that the $*$-multiplication respects the order $\mathcal{O}$ as well as the structure numbers (k, l).

Theorem: Reduction of divisor product representation

For $N \in \mathbb{N}$, $e(j) \in \mathbb{N}$, $j \;=\; 1, 2, ..., N$, one has

$$dpr\left(\prod_{j=1}^{N} p(j)^{e(j)}\right) \;=\; p(1)^{e(1)-1} * p(2)^{e(2)-1} * \cdots * p(N)^{e(N)-1} * dpr\left(\prod_{j=1}^{N} p(j)\right) \;= \tag{21}$$

$$\left(\prod_{k=1}^{N} p(k)^{e(k)-1}\right) * dpr\left(\prod_{j=1}^{N} p(j)\right)\;. \tag{22}$$

With the help of this *theorem* all divisor product representations are reduced to the explicitly known result from *proposition 1* .

Example 2:

$$dpr(2^3\cdot 3^2\cdot 5^1) \;=\; 2^2 * 3 * 1 * dpr(2\cdot 3\cdot 5) \;=\; 12 * dpr(30) \;=\; \frac{a(360)\,a(60)\,a(36)\,a(24)}{a(180)\,a(120)\,a(72)\,a(12)}\;. \tag{23}$$

For the proof of the *Theorem* we first need the following

Proposition 2:

$$dpr(p_1^m\,p_2\,\cdots p_N) \;=\; \frac{p_1^m * dpr(p_2\,\cdots p_N)}{p_1^{m-1} * dpr(p_2\,\cdots\, p_N)}\;,\; m\in\mathbb{N},\;\; N\in\mathbb{N}\,. \tag{24}$$

The case $N = 1$ uses $dpr(.0.) := a(1)$, and it has already been considered after the recurrence relation eq. (4). The case $m = 1$ is known explicitely from the *Proposition.* For the proof we shall use the following *Definition 2* and *Lemma 2.*

Definition 2: Set of divisors $\mathbf{D(n)}$ and $\mathbf{D'(n)}$

For $n \in \mathbb{N}$, the set of divisors of n is denoted by $D(n)$, and the set of divisors of n, excluding n, is denoted by $D'(n)$.

Lemma 2:

$$D'(p_1^m\, p_2 \cdots p_N) \;=\; p_1^m \bullet D'(p_2 \cdots p_N) \bigcup_{k=1}^{m} p_1^{m-k} \bullet D(p_2 \cdots p_N)\,,\; m \in \mathbb{N}\;,\; N \in \mathbb{N}\,. \tag{25}$$

Here the $\bullet$ symbol indicates that p_1^m has to be multiplied to all of the $\tau(2^{N-1}) - 1$ elements of the set $D'(p_2 \cdots p_N)$, and similarly for p_1^{m-k} and the $\tau(2^{N-1})$ element set $D(p_2 \cdots p_N)$. Remember that $\tau(n) = A000005(n) = |D(n)|$.
E.g., $p_1 \bullet D'(p_1^2\, p_2) = \{p_1^3, p_1^2\, p_2, p_1^2, p_1\, p_2, p_1\}$.

Proof of lemma 2:

For $N = 1$ the first set involving D' is empty, and in the union the set D has to be put to $D(1) = \{1\}$. Then the statement is clearly true. For other N and m the proof is also obvious: just write, for given $N \geq 2$, the elements of the set of the *l.h.s.* into a union of sets which have in turn the elements p_1^m, p_1^{m-1}, ..., $p_1^1 = p_1$, and those without element p_1 which appear formally for $p_1^0 = 1$. In the first set take p_1^m out, being left with $D'(p_2 \cdots p_N)$, and in every of the other sets take out the corresponding p_1^{m-k}, leaving in each case the full set $D(p_2 \cdots p_N)$. □
E.g., $D'(p_1^2\, p_2) = \{p_1^2\} \cup \{p_1\, p_2,\, p_1\} \cup \{p_2, 1\} = p_1^2 \bullet D'(p_2) \cup p_1 \bullet D(p_2) \cup 1 \bullet D(p_2)$, where $D'(p_2) = \{1\}$.

Notation:

In the recurrence, eqs. (5) and (4), the set $D'(n)$ appears. In the following we will write for $\prod_{\substack{d|n \\ d \neq n}} dpr(d)$ also $\Pi\, dpr(D'(n))$. The same notation will be used also for other sets.

Proof of proposition 2:

Double induction on N and m is used for the proof. For given m we show the statement for all N by induction over N, using the induction hypothesis on m, *i.e.*, assuming that the statement is true for all m-values $1, 2, ..., m - 1$. For $N = 1$ and $m = 1$ the statement reduces to the known result $dp(p_1) = \frac{a(p_1)}{a(1)}$. In the induction step one uses first the recurrence for any m. After employing *Lemma 2* for the denominator we find, with the new *notation*,

$$dpr(p_1^m\, p_2 \cdots p_N) \;=\; \frac{a(p_1^m\, p_2 \cdots p_N)}{\Pi\, dpr(p_1^m \bullet D'(p_2 \cdots p_N)) \prod_{k=1}^{m} \Pi\, dpr(p_1^{m-k} \bullet D(p_2 \cdots p_N))}\;. \tag{26}$$

In all the products in the denominator only up to $N-1$ elements appear and by the induction hypothesis on N we can use the statement of the *proposition* for any $m \to m' \in \{1, 2, .., m\}$. The last product, for $k = m$, is not replaced. This leads to

$$dpr(p_1^m\, p_2 \cdots p_N) \;=\; a(p_1^m\, p_2 \cdots p_N)\;\cdot$$
$$\cdot\frac{\Pi\, p_1^{m-1} * dpr(D'(p_2 \cdots p_N))}{\Pi\, p_1^m * dpr(D'(p_2 \cdots p_N))}\,\frac{\Pi\, p_1^{m-2} * dpr(D(p_2 \cdots p_N))}{\Pi\, p_1^{m-1} * dpr(D(p_2 \cdots p_N))} \cdots \frac{1}{\Pi\, dpr(1 \bullet D(p_2 \cdots p_N))}\;. \tag{27}$$

The last denominator is $\Pi\, dpr(D(p_2 \cdots p_N))$. Now this reduces as a telescopic product to

$$dpr(p_1^m\, p_2 \cdots p_N) \;=\; a(p_1^m\, p_2 \cdots p_N)\, \frac{\Pi\, p_1^{m-1} * dpr(D'(p_2 \cdots p_N))}{\Pi\, p_1^m * dpr(D'(p_2 \cdots p_N))}\, \frac{1}{\Pi\, p_1^{m-1} * dpr(D(p_2 \cdots p_N))}\;. \tag{28}$$

Next, in order to cancel also the first numerator one replaces $D(p_2 \cdots p_N)$ in the last denominator by $D'(p_2 \cdots p_N) \cup \{p_2 \, \cdots \, p_N\}$, to obtain, after writing $a(p_1^m\, p_2 \, \cdots p_N) \,=\, p_1^m \,*\, a(p_2 \, \cdots p_N)$,

$$dpr(p_1^m\, p_2 \, \cdots p_N) \;=\; \left(p_1^{m-1} \;*\; \frac{a(p_2 \, \cdots p_N)}{\Pi\, dpr(D'(p_2 \cdots p_N))}\right) \frac{1}{p_1^{m-1} \;*\; dpr(p_2 \, \cdots \, p_N)} \;, \tag{29}$$

which becomes finally, after employing the recurrence,

$$dpr(p_1^m\, p_2 \, \cdots p_N) \;=\; (p_1^m \;*\; dpr(p_2 \, \cdots \, p_N)) \;\frac{1}{p_1^{m-1} \;*\; dpr(p_2 \, \cdots \, p_N)} \;. \tag{30}$$

□

Corollary to proposition 2:

$$dpr(p_1^m\, p_2 \, \cdots p_N) \;=\; p_1^{m-1} \;*\; dpr(p_1\, p_2 \, \cdots p_N),\;\; m \in \mathbb{N},\;\; N \in \mathbb{N}\;. \tag{31}$$

This is clear from $dpr(p_1^m\, p_2 \, \cdots p_N) \;=\; p_1 \;*\; dpr(p_1^{m-1}\, p_2 \, \cdots p_N)$, used repeatedly.

Example 3:

$$dpr(120) \;=\; dpr(2^3\, 3\, 5) \;=\; 2^2 \;*\; dpr(30) \;=\; \frac{a(4\cdot 30)\, a(4\cdot 5)\, a(4\cdot 3)\, a(4\cdot 2)}{a(4\cdot 15)\, a(4\cdot 10)\, a(4\cdot 6)\, a(4\cdot 1)} \;=\; \frac{a(120)\, a(20)\, a(12)\, a(8)}{a(60)\, a(40)\, a(24)\, a(4)}\;. \tag{32}$$

Before coming to the proof of the *theorem* we state two more *propositions*.

Proposition 3:

$$dpr(p_1^n\, p_2^m\, p_3 \, \cdots p_N) \;=\; \frac{p_1^n \;*\; dpr(p_2^m\, p_2 \, \cdots p_N)}{p_1^{n-1} \;*\; dpr(p_2^m\, p_2 \, \cdots p_N)},\;\; n, m \in \mathbb{N},\;\; N \;=\; 2, 3, \ldots \tag{33}$$

Proof:

This proof runs along the same lines like the one of *proposition 2*. After using the recurrence, eqs. (5) with (4), an analogon of lemma 2 is formulated by just replacing there $p_1^m \to p_1^n$ and $p_2 \to p_2^m$. Its proof is implied by *lemma 2*. Then induction over N for given n and m is used. As mentioned above, one could replace p_j by $p(j)$, for $j \;=\; 1, \ldots N$.

□

In order to prepare further for the general case we note that a general product $p_1^{e_j} \cdots p_N^{e_N}$ with the $e_j \;\geq\; 1$ can, without loss of information, be rewritten using the following multiset convention as $p(1)^e(1)\, p(2)^e(2) \,\cdots\, p(N)^{e(N)}$ with (not strictly) decreasing exponents $e(j) \;\geq\; 1$. An example will make this clear: $p_1^1\, p_2^3\, p_3^5\, p_4^3$ is transformed into $p(1)^5\, p(2)^3\, p(3)^3\, p(4)^1$ with the obvious substitutions of the p_js by the $p(k)$s. In this way it is sufficient to assume in the *theorem* non-increasing sequences $\vec{e}_N \;=\; \{e(1), e(2), \ldots, e(N)\}$ without zero entries.

If we write $n \equiv n(N, M) \;:=\; p(1)^{e(1)} \,\cdots\, p(M)^{e(M)}\, p(M+1) \,\cdots\, p(N)$ with positive non-increasing exponents, and similarly $\hat{n} \equiv \hat{n}(N-1, M-1) \;:=\; p(2)^{e(2)} \,\cdots\, p(M)^{e(M)}\, p(M+1) \,\cdots\, p(N)$, one can prove in the same vein the more general result of

Proposition 4:

$$dpr(n(N, M)) \;=\; \frac{p(1)^{e(1)} \;*\; dpr(\hat{n}(N-1, M-1))}{p(1)^{e(1)-1} \;*\; dpr(\hat{n}(N-1, M-1))},\;\; N \in \mathbb{N},\;\; M \;\in\; \{1, \ldots N\}\,. \tag{34}$$

Proof:

Here *lemma 2* has to be used in the generalized form

$$D'(n(N, M)) \;=\; p(1)^{e(1)} \bullet D'(\hat{n}(N-1, M-1)) \bigcup_{k=1}^{e(1)} p(1)^{e(1)-k} \bullet D(\hat{n}(N-1, M-1))\,,\; N \;\in\; \mathbb{N}\,,\; M \;\in\; \{1, \ldots N\}\,. \tag{35}$$

Then the proof runs like the one for *proposition 2* with double induction over N and M.

Corollary to proposition 4:

$$dpr(n(N,M)) = p(1)^{e(1)-1} * p(2)^{e(2)-1} * \cdots * p(M)^{e(M)-1} * dpr(p(1)\cdots p(N)),\ N \in \mathbb{N},\ M \in \{1, \ldots N\}\,. \tag{36}$$

See the remark to the *corollary to proposition 2*.

Proof of the theorem:

As mentioned above, one may assume also in the *theorem*, modulo substitutions of the primes involved, positive, non-increasing exponents $\vec{e}_N$. Then, with the definition of $n(N,M)$ from above the *theorem* is nothing but the *corollary to proposition 4* with $M = N$. □

The *theorem* and the property of the $*$-multiplication show that the structure numbers (k,l) of a divisor product representation of any number $n = p_1^{a_1} \cdots p_N^{a_N}$ are the same like the ones for $n = p_1 \cdots p_N$ which, in turn, are the same like the ones for the primorials $n = p(1)\cdots p(N) =$ A002110(N). Because *proposition 2* for $m = 1$ allows a further reduction of $dpr(p(1)\cdots p(N))$ one can prove the

Proposition 5:

The structure numbers (k,l) from eq. (3) for $dpr(p_1^{e_1} \cdots p_N^{e_N})$ are the same as those for $dpr(p(1)\cdots p(N))$, and they are $(2^{N-1}-1, 2^{N-1})$. Thus, every $pdr(n)$ formula for $n \geq 2$ is balanced, *i.e.*, the number of $a-$factors in the numerator coincides with the one of the denominator.

Proof:

The only eq. left to show is $l = 2^{N-1}$, because the balance property $k+1 = l$ is already clear. This is done inductively with the help of the $m = 1$ case of the formula of *proposition 2* with appropriate renaming

$$dpr(p(1)\cdots p(N)) = \frac{p(N) * dpr(p(1)\cdots p(N-1))}{dp((p(1)\cdots p(N-1))}\ , \text{ for } N \in \mathbb{N}\,. \tag{37}$$

Start with $dpr(2) = \frac{a(2)}{a(1)}$ with $l = 1 = 2^0$. Assume the validity of $l = 2^{N'}$ for all $dpr(p(1)\cdots p(N'))$, for $N' = 1, 2, \ldots, N-1$. Then because of this formula one has a quotient of two balanced fractions, each with structure number $l = 2^{N-2}$. Because there are no cancellations on obtains one quotient with the doubled number of factors in the numerator as well as in the denominator, hence $l = 2\cdot 2^{N-2} = 2^{N-1}$. □

Note that we have just shown that $l \equiv l(n) = 2^{N(n)-1}$, where $N(n)$ is the number of distinct primes in the prime number decomposition of n. $N(n) =$ A001221(n), with $N(1) := 0$, and $l(n) =$ A007875$(n), n \geq 2$.

In *Table 2* we list the primorial $dpr(prl(N))$ results for $N = 1, \ldots 7$ in an abbreviated form, where only the arguments of the divisor products a are given. This table leads us to formulate the following

Proposition 6: Parity of a-arguments for divisor product representation of primorials

1) In the numerator as well as in the denominator of $dpr(prl(N))$, for $N > 2$, there are as many even as odd a-arguments, *viz* 2^{N-2}.

2) For each of the 2^{N-2} even numerator (resp. denominator) a-arguments q of $dpr(prl(N))$, for $N \geq 2$, there is precisely one odd denominator (resp. numerator) a-argument $q/2$.

Proof:

1) From *proposition 1* with the substitutions $p_j \to p(j)$, in order to obtain primorials, the parity of the numerator a-arguments is the following. Primorials are even, therfore the first (the largest) a-argument is even. In each of the following $\Pi\, a(.(N-2\,k).)$ products one distinguishes between those which do not involve the only even prime $p(1) = 2$ and the others. Thus the number of odd a-arguments in the numerator is $\sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} \binom{N-1}{N-2\,k} = 2^{N-2}$ (due to summing every other entry in row $N-1$, $N \geq 2$, of

Pascal's triangle A007318). The remaining $2^{N-1} - 2^{N-2} = 2^{N-2}$ a-arguments are odd. Similarly, for the denominator the number of odd a-arguments is $\sum_{k=1}^{\lfloor\frac{N-1}{2}\rfloor} \binom{N-1}{N-(2\,k-1)} = 2^{N-2}$. Hence the number of the remaining even ones is also 2^{N-2}.

2) The first numerator a-argument $n = prl(N)$ is even, and the first one in the denominator is $\frac{n}{2}$, for $N \in \mathbb{N}$. If there is an even numerator a-argument q in the product $\Pi\, a(.(N-2\,k).)$, for $k \in \{1, \ldots \lfloor\frac{N}{2}\rfloor\}$, then the corresponding $\frac{q}{2}$ argument will occur once within the $\Pi\, a(.(N-2\,k-1).)$ product where no $p(1) = 2$ is used.
Similarly, if an even denominator a-argument q appears within $\Pi\, a(.(N-(2\,k-1)).)$, for $k \in \{1, \ldots \lfloor\frac{N+1}{2}\rfloor\}$, then $\frac{q}{2}$ will occur once within $\Pi\, a(.(N-2\,k).)$ in the numerator where $p(1)$ is not used. $\square$

Table 2 shows that already for $N = 5$ one cannot find the even-odd partner $a-$arguments at the same position of the ordered numerator and denominator lists. *E.g.*, the 5th argument 154 in the numerator list for the $N = 5$ case has partner argument 77 but this appears at position 6 of the denominator list, whereas on position 5 one has 210. This is related to the different number of these $\Pi\, a(.(N-k).)$-products in the numerator and in the denominator. *E.g.*, for $N = 5$ in the numerator these product numbers are $1, 10$ and 5 but in the denominator they are $5, 10$ and 1. The numerator a-argument 154 appears for the fourth member of the product $\Pi a(\ldots)$, hence the partner denominator argument 77 is found within the product $\Pi\, a(..)$, but this product starts only at position 6. This mismatch has nothing to do with the reordering between factors of different $\Pi\ a(.(N-k).)$ products which is sometimes necessary because of the order prescription $\mathcal{O}$. See the remark in connection to with *proposition 1*. Here, for $N = 5$ one could omit the $\mathcal{O}$ prescription in the numerator as well as in the denominator.

Definition 3: Bijection of even-odd partner a-arguments for primorials

The even-odd partnership in the numerator and denominator a-arguments can be encoded as permutations of labeled elements, where we use an underlined number $\underline{j}$ if the $a-$argument at position j is even. If it is odd we take j without underlining.

The examples for $N = 1, \ldots, 4$ will illustrate this permutation notation of the divisor product representation for primorials.

$$\begin{pmatrix}\underline{1}\\1\end{pmatrix},\quad \begin{pmatrix}\underline{1}&2\\1&\underline{2}\end{pmatrix},\quad \begin{pmatrix}\underline{1}&2&3&\underline{4}\\1&\underline{2}&\underline{3}&4\end{pmatrix},\quad \begin{pmatrix}\underline{1}&2&3&4&\underline{5}&\underline{6}&\underline{7}&8\\1&\underline{2}&\underline{3}&\underline{4}&5&6&7&\underline{8}\end{pmatrix},$$
$$\begin{pmatrix}\underline{1}&2&3&4&\underline{5}&\underline{6}&7&\underline{8}&\underline{9}&\underline{10}&\underline{11}&12&13&14&15&\underline{16}\\1&\underline{2}&\underline{3}&\underline{4}&6&7&\underline{5}&8&9&11&12&\underline{10}&\underline{13}&\underline{14}&\underline{15}&16\end{pmatrix}. \tag{38}$$

For $N = 5$ a nontrivial permutation appears for the first time. In cycle notation, forgetting the underlinings for the moment, ons has $(5, 6, 7)\,(10, 11, 12)$. From *Table 2* one finds in the $N = 6$ case the permutation from the symmetric group S_{32} $(5, 7)\,(6, 8)\,(12, 13, 14, 15, 17)\,(16, 21, 20, 19, 18)\,(25, 27)\,(26, 28)$. To find from *Table 2* the permutation from S_{64} for the case $N = 7$ is left to the reader.

This leads us to conjecture that the cycle pattern of the permutation is always symmetric with respect to its middle position, *i.e.*, the dividing line between the positions $j = 2^{N-3}$ and $j + 1$.

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AMS MSC numbers: 97F30, 11B83, 11A67.

Keywords: Prime number decomposition, sets of divisors, divisor products, primorials, Chebyshev T-polynomials.

Concerned with OEIS sequences A00005, A001221, A002110, A007318, A007875, A007955, A053120,A070826, A181875, A181876 .

Table 1: **Divisor product representation of $\mathbf{n} = \mathbf{1}, \mathbf{2}, ..., \mathbf{80}$.**

n	$(\mathbf{k},\mathbf{l})$	dpr(n)	**n**	$(\mathbf{k},\mathbf{l})$	dpr(n)
1	$(0,0)$	$a(1)$	**41**	$(0,1)$	$a(41)/a(1)$
2	$(0,1)$	$a(2)/a(1)$	**42**	$(3,4)$	$a(42)\,a(7)\,a(3)\,a(2)/$ $(a(21)\,a(14)\,a(6)\,a(1))$
3	$(0,1)$	$a(3)/a(1)$	**43**	$(0,1)$	$a(43)/a(1)$
4	$(0,1)$	$a(4)/a(2)$	**44**	$(1,2)$	$a(44)\,a(2)/(a(22)\,a(4))$
5	$(0,1)$	$a(5)/a(1)$	**45**	$(1,2)$	$a(45)\,a(3)/(a(15)\,a(9))$
6	$(1,2)$	$a(6)\,a(1)/(a(3)\,a(2))$	**46**	$(1,2)$	$a(46)\,a(1)/(a(23)\,a(2))$
7	$(0,1)$	$a(7)/a(1)$	**47**	$(0,1)$	$a(47)/a(1)$
8	$(0,1)$	$a(8)/a(4)$	**48**	$(1,2)$	$a(48)\,a(8)/(a(24)\,a(16))$
9	$(0,1)$	$a(9)/a(3)$	**49**	$(0,1)$	$a(49)/a(7)$
10	$(1,2)$	$a(10)\,a(1)/(a(5)\,a(2))$	**50**	$(1,2)$	$a(50)\,a(5)/(a(24)\,a(10))$
11	$(0,1)$	$a(11)/a(1)$	**51**	$(1,2)$	$a(51)\,a(1)/(a(17)\,a(3))$
12	$(1,2)$	$a(12)\,a(2)/(a(6)\,a(4))$	**52**	$(1,2)$	$a(52)\,a(2)/(a(26)\,a(4))$
13	$(0,1)$	$a(13)/a(1)$	**53**	$(0,1)$	$a(53)/a(1)$
14	$(1,2)$	$a(14)\,a(1)/(a(7)\,a(2))$	**54**	$(1,2)$	$a(54)\,a(9)/(a(27)\,a(18))$
15	$(1,2)$	$a(15)\,a(1)/(a(5)\,a(3))$	**55**	$(1,2)$	$a(55)\,a(1)/(a(11)\,a(5))$
16	$(0,1)$	$a(16)/a(8)$	**56**	$(1,2)$	$a(56)\,a(4)/(a(28)\,a(8))$
17	$(0,1)$	$a(17)/a(1)$	**57**	$(1,2)$	$a(57)\,a(1)/(a(19)\,a(3))$
18	$(1,2)$	$a(18)\,a(3)/(a(9)\,a(6))$	**58**	$(1,2)$	$a(58)\,a(1)/(a(29)\,a(2))$
19	$(0,1)$	$a(19)/a(1)$	**59**	$(0,1)$	$a(59)/a(1)$
20	$(1,2)$	$a(20)\,a(2)/(a(10)\,a(4))$	**60**	$(3,4)$	$a(60)\,a(10)\,a(6)\,a(4)/$ $(a(30)\,a(20)\,a(12)\,a(2))$
21	$(1,2)$	$a(21)\,a(1)/(a(7)\,a(3))$	**61**	$(0,1)$	$a(61)/a(1)$
22	$(1,2)$	$a(22)\,a(1)/(a(11)\,a(2))$	**62**	$(1,2)$	$a(62)\,a(1)/(a(31)\,a(2))$
23	$(0,1)$	$a(23)/a(1)$	**63**	$(1,2)$	$a(63)\,a(3)/(a(21)\,a(9))$
24	$(1,2)$	$a(24)\,a(4)/(a(12)\,a(8))$	**64**	$(0,1)$	$a(64)/a(32)$
25	$(0,1)$	$a(25)/a(5)$	**65**	$(1,2)$	$a(65)\,a(1)/(a(13)\,a(5))$
26	$(1,2)$	$a(26)\,a(1)/(a(13)\,a(2))$	**66**	$(3,4)$	$a(66)\,a(11)\,a(3)\,a(2)/$ $(a(33)\,a(22)\,a(6)\,a(1))$
27	$(0,1)$	$a(27)/a(9)$	**67**	$(0,1)$	$a(67)/a(1)$
28	$(1,2)$	$a(28)\,a(2)/(a(14)\,a(4))$	**68**	$(1,2)$	$a(68)\,a(2)/(a(34)\,a(4))$
29	$(0,1)$	$a(29)/a(1)$	**69**	$(1,2)$	$a(69)\,a(1)/(a(23)\,a(3))$
30	$(3,4)$	$a(30)\,a(5)\,a(3)\,a(2)/$ $(a(15)\,a(10)\,a(6)\,a(1))$	**70**	$(3,4)$	$a(70)\,a(7)\,a(5)\,a(2)/$ $(a(35)\,a(14)\,a(10)\,a(1))$
31	$(0,1)$	$a(31)/a(1)$	**71**	$(0,1)$	$a(71)/a(1)$
32	$(0,1)$	$a(32)/a(16)$	**72**	$(1,2)$	$a(72)\,a(12)/(a(36)\,a(24))$
33	$(1,2)$	$a(33)\,a(1)/(a(11)\,a(3))$	**73**	$(0,1)$	$a(73)/a(1)$
34	$(1,2)$	$a(34)\,a(1)/(a(17)\,a(2))$	**74**	$(1,2)$	$a(74)\,a(1)/(a(37)\,a(2))$
35	$(1,2)$	$a(35)\,a(1)/(a(7)\,a(5))$	**75**	$(1,2)$	$a(75)\,a(5)/(a(25)\,a(15))$
36	$(1,2)$	$a(36)\,a(6)/(a(18)\,a(12))$	**76**	$(1,2)$	$a(76)\,a(2)/(a(38)\,a(4))$
37	$(0,1)$	$a(37)/a(1)$	**77**	$(1,2)$	$a(77)\,a(1)/(a(11)\,a(7))$
38	$(1,2)$	$a(38)\,a(1)/(a(19)\,a(2))$	**78**	$(3,4)$	$a(78)\,a(13)\,a(3)\,a(2)/$ $(a(39)\,a(26)\,a(6)\,a(1))$
39	$(1,2)$	$a(39)\,a(1)/(a(13)\,a(3))$	**79**	$(0,1)$	$a(79)/a(1)$
40	$(1,2)$	$a(40)\,a(4)/(a(20)\,a(8))$	**80**	$(1,2)$	$a(80)\,a(8)/(a(40)\,a(16))$

Table 2: Divisor product representation for primorials prl(N) := A002110(N)

N	**prl(N)**	**prl(N)/2**	**dpr(prl(N)) structure**
1	2	1	[2]/[1]
2	6	3	[6, 1]/[3, 2]
3	30	15	[30, 5, 3, 2]/[15, 10, 6, 1]
4	210	105	[210, 35, 21, 15, 14, 10, 6, 1]/[105, 70, 42, 30, 7, 5, 3, 2]
5	2310	1155	[2310, 385, 231, 165, 154, 110, 105, 70, 66, 42, 30, 11, 7, 5, 3, 2]/
			[1155, 770, 462, 330, 210, 77, 55, 35, 33, 22, 21, 15, 14, 10, 6, 1]

The next two instances are:

6, 30030, 15015,

[30030, 5005, 3003, 2145, 2002, 1430, 1365, 1155, 910, 858, 770, 546, 462, 390, 330, 210, 143, 91, 77, 65, 55, 39, 35, 33, 26, 22, 21, 15, 14, 10, 6, 1]/

[15015, 10010, 6006, 4290, 2730, 2310, 1001, 715, 455, 429, 385, 286, 273, 231, 195, 182, 165, 154, 130, 110, 105, 78, 70, 66, 42, 30, 13, 11, 7, 5, 3, 2]

7, 510510, 255255,

[510510, 85085, 51051, 36465, 34034, 24310, 23205, 19635, 15470, 15015, 14586, 13090, 10010, 9282, 7854, 6630, 6006, 5610, 4290, 3570, 2730, 2431, 2310, 1547, 1309, 1105, 1001, 935, 715, 663, 595, 561, 455, 442, 429, 385, 374, 357, 286, 273, 255, 238, 231, 195, 182, 170, 165, 154, 130, 110, 105, 102, 78, 70, 66, 42, 30, 17, 13, 11, 7, 5, 3, 2]/

[255255, 170170, 102102, 72930, 46410, 39270, 30030, 17017, 12155, 7735, 7293, 6545, 5005, 4862, 4641, 3927, 3315, 3094, 3003, 2805, 2618, 2210, 2145, 2002, 1870, 1785, 1430, 1365, 1326, 1190, 1155, 1122, 910, 858, 770, 714, 546, 510, 462, 390, 330, 221, 210, 187, 143, 119, 91, 85, 77, 65, 55, 51, 39, 35, 34, 33, 26, 22, 21, 15, 14, 10, 6, 1]