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Proof of a Conjecture Involving Chebyshev Polynomials

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Abstract

In the formula section under OEIS A057059 *L. Edson Jeffery* stated a conjecture on a sum of Chebyshev S polynomials evaluated at certain arguments. We give the proof of this conjecture based on two main known identities.

In OEIS [9] A057059 *L. Edson Jeffery* stated the following conjecture (here written in terms of *Chebyshev* S-polynomials (A049310) for even and odd k separately)

$$IdA(n,K): \quad \sum_{j=0}^{n-1} S_{2K-1}\left(2\cos\left(\pi\frac{2j+1}{2n+1}\right)\right) = K, \quad \text{for } K = 1,2,...,n,\ n\in\mathbb{N}, \tag{1}$$

$$IdB(n,K): \quad \sum_{j=0}^{n-1} S_{2K}\left(2\cos\left(\pi\frac{2j+1}{2n+1}\right)\right) = n-K, \quad \text{for } K = 1,2,...,n,\ n\in\mathbb{N}. \tag{2}$$

In the context of the number triangle $Tri(n,k)$ A057059 these identities are only needed for $K = \frac{k}{2} \in \left\{1,\,2,\,...,\,\left\lfloor\frac{n}{2}\right\rfloor\right\}$ and $K = \frac{k-1}{2} \in \left\{1,\,2,\,...,\,\left\lfloor\frac{n-1}{2}\right\rfloor\right\}$, respectively. But we will prove these two identities IdA and IdB for the given extended range of K.
Due to the identity $\cos(\pi - x) = -\cos(x)$ one can rewrite the two conjectures as

$$IdA'(n,K): \quad \sum_{l=1}^{n} (-1)^{l+1} S_{2K-1}\left(2\cos\left(\pi\frac{l}{2n+1}\right)\right) = K, \quad \text{for } K = 1,2,...,n,\ n\in\mathbb{N}, \tag{3}$$

$$IdB'(n,K): \quad \sum_{l=1}^{n} S_{2K}\left(2\cos\left(\pi\frac{l}{2n+1}\right)\right) = n-K, \quad \text{for } K = 1,2,...,n,\ n\in\mathbb{N}. \tag{4}$$

In IdA' the signs appear due to the odd polynomials S_{2K-1}, and in IdB' the signs do not appear because S_{2K} is an even function. Now the arguments of the S-polynomials have become the known positive zeros $xS_l^{(2n)}$, $l = 1, ..., n$, of the polynomials $S_{2n}(x)$. (See. *e.g.,* the Jul 12 2011 comment in A049310.) This makes these identities interesting. If *Chebyshev* T-polynomials (also known as *Chebyshev* polynomials of the first kind, see A053120) are also used the identities become

$$IdA''(n,K): \quad \sum_{l=1}^{n} (-1)^{l+1} S_{2K-1}\left(2T_l\left(\frac{\rho(2n+1)}{2}\right)\right) = K, \quad \text{for } K = 1,2,...,n,\ n\in\mathbb{N}, \tag{5}$$

$$IdB''(n,K): \quad \sum_{l=1}^{n} S_{2K}\left(2T_l\left(\frac{\rho(2n+1)}{2}\right)\right) = n-K, \quad \text{for } K = 1,2,...,n,\ n\in\mathbb{N}, \tag{6}$$

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where we used for the largest positive zero of $S_{2\,n}$ the notation $\rho(2\,n+1) \,:=\, xS_1^{(2\,n)} \,=\, 2\,\cos\left(\pi\,\frac{1}{2\,n+1}\right)$, which is the length ration of the largest diagonal and the side in the regular $(2\,n+1)$-gon, (see, e.g., [6]). Note that $2\,T_l\left(\frac{x}{2}\right)$ are the monic *Chebyshev* T polynomials, called $R(l,\,x)$ in A127672, and $\hat{t}_l(x)$ in [6]. For the proof we shall use the version IdA' and IdB', and the explicit version of the S polynomials will be employed

$$S_k(x) \,=\, = \sum_{q=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^q \binom{k-q}{q}\, x^{k\,-\,2\,q}, \quad \text{for} \quad k\,\in\,\mathbb{N}_0\,. \tag{7}$$

This is known, and it can be proved with the help of *Morse* code polynomials using dots and bars. See, e.g. [4], p. 302. Note that the trigonometric *Binet-de Moivre* version of the S polynomials will not help here immediately to prove the identities.

Proof of IdA$'$:

$$S_{2\,K-1}\left(xS_l^{(2\,n)}\right) \,=\, \sum_{q=0}^{K-1} (-1)^q \binom{2\,K-q-1}{q}\, 2^{2\,(K-q)-1} \left(\frac{xS_l^{(2\,n)}}{2}\right)^{2\,(K-q)-1}\,. \tag{8}$$

Now the known formula for odd powers of cos in terms of a sum over cos functions with multiple odd arguments comes into play.

$$IdA1(Q,\theta): \qquad (\cos(\theta))^{2\,Q-1} \,=\, \frac{1}{2^{2\,(Q-1)}} \sum_{k=0}^{Q-1} \binom{2\,Q-1}{k} \cos\left((2\,(Q-1-k)+1)\,\theta\right)\,, \quad \text{for} \quad Q\,\in\,\mathbb{N}\,. \tag{9}$$

See [3], p. 53, eq. 1.320 7., and the number triangle A122366 for the binomials.

Using this for $Q \,=\, K-q$ and $\theta \,=\, \pi\,\dfrac{l}{2n+1}$ in eq. (8) and interchanging the q and k summations with the l sum in $IdA'(n,K)$ leads to the l-sum

$$\Sigma_A(K,q,k,n) \,:=\, \sum_{l=1}^{n} (-1)^{l+1} 2\,\cos\left(\frac{\pi}{2\,n+1}\,(2(K-q-1-k)\,+\,1)\,l\right)\,. \tag{10}$$

Now the second important identity enters the stage, found, *e.g.*, in *Jolley* [5], p. 80, Nr. (429).

$$IdA2(n,\theta): \qquad \sum_{l=1}^{n} (-1)^{l+1}\, 2\,\cos(\theta\,l) \,=\, 1 \,+\, (-1)^{n+1}\, \frac{\cos\left((2\,n+1)\,\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)}, \quad \text{for} \quad n\,\in\,\mathbb{N}. \tag{11}$$

The short proof of this identity can be given in term of the exponential function, but requires a careful setup like that used in *Courant* [2], pp. 383-4. A proof without using the exponential function can also be given by adapting the *Gauß* trick for summing integers, as done in Maor [7], pp. 112-114. Here is the proof based on exponential functions.

Proof of eq. (11):

We want to prove

$$-\frac{1}{2} \,+\, \sum_{l=1}^{n} (-1)^{l+1}\, \cos(\theta\,l) \,=\, (-1)^{n+1}\, \frac{\cos\left((2\,n+1)\,\frac{\theta}{2}\right)}{2\,\cos\left(\frac{\theta}{2}\right)}\,, \quad \text{for} \quad n\,\in\,\mathbb{N}\,. \tag{12}$$

The *l.h.s.* is, with $-1 = exp(\pm i\,\pi)$, and $x := \pi + \theta$

$$\begin{aligned} -\frac{1}{2}\sum_{l=-n}^{n} exp(i\,x\,l) &= -\frac{1}{2}\exp(-i\,x\,n)\sum_{l=0}^{2\,n} exp(i\,x\,l) = -\frac{1}{2}\exp(-i\,x\,n)\,\frac{exp(i\,x\,(2\,n+1)) - 1}{exp(i\,x) - 1} &(13)\\ &= -\frac{1}{2}(-1)^{n}\exp(-i\,\theta\,n)\,\frac{-exp(i\,\theta\,(2\,n+1)) - 1}{-exp(i\,\theta) - 1} &(14)\\ &= \frac{1}{2}(-1)^{n+1} exp(-i\,\theta\,n)\,\frac{exp(i\,\theta\,(2\,n+1)) + 1}{exp(i\,\theta) + 1}\,\frac{exp\left(-i\frac{\theta}{2}\right)}{exp\left(-i\frac{\theta}{2}\right)} &(15)\\ &= (-1)^{n+1}\frac{\exp(i\left(n+\frac{1}{2}\right)\theta) + \exp(-i\left(n+\frac{1}{2}\right)\theta)}{2\left(\exp(i\frac{\theta}{2}) + \exp(-i\frac{\theta}{2})\right)}\,, &(16)\end{aligned}$$

which is the assertion if converted back to cos functions. □

The crucial point is now that with $\theta = \frac{\pi}{2\,n+1}\,(2\,(K-q-1-k) + 1)$ the sum $\Sigma(K,q,k,n)$ from eq.(11) becomes identically 1 because the cos in the numerator of eq. (12) is then $\cos\left(\frac{\pi}{2}\,(\text{odd number})\right)$, with the odd number $(2\,(K-q-1-k) + 1)$, which vanishes whereas the cos in the denominator does not vanish because $2\,(K-q-k)-1 \in \{1, ..., 2\,n-1\}$ due to $K-q-k \in \{1, ..., n\}$.

IdA' has therefore been reduced to

$$\sum_{l=1}^{n}(-1)^{l+1}\,S_{2\,K-1}(xS_l^{(2\,n)}) = \sum_{q=0}^{K-1}(-1)^{q}\binom{2\,K-q-1}{q}\sum_{k=0}^{K-q-1}\binom{2\,(K-q-1)+1}{k}\cdot 1\,. \qquad (17)$$

Because

$$\sum_{k=0}^{N}\binom{2\,N+1}{k} = 2^{2\,N}\,,\quad \text{for}\ \ N \in \mathbb{N}, \qquad (18)$$

which is trivial, one is left with the q-sum with $N = K-q-1$ and $L = K-1$

$$\sum_{q=0}^{L}(-1)^{q}\binom{(2\,L+1)-q}{q}\,2^{(2\,L+1)-2\,q)}\,\frac{1}{2} = \frac{1}{2}\,S_{2\,L+1}(2) = \frac{1}{2}\,((2\,L+1)+1) = L+1 = K. \qquad (19)$$

which proves IdA' from eq. (3). Here the explicit form of the *Chebyshev* S polynomial from eq. (7) has been used, together with the well known fact that $S_m(2) = m+1$ (proved with the help of the S recurrence relation). □

Proof of IdB′:

The proof runs along the same line as above but at the end some more interesting binomial sums show up.

$$S_{2\,K}\left(xS_l^{(2\,n)}\right) = \sum_{q=0}^{K}(-1)^{q}\binom{2\,K-q}{q}\,2^{2\,(K-q)}\left(\frac{xS_l^{(2\,n)}}{2}\right)^{2\,(K-q)}\,. \qquad (20)$$

Now the known formula for even powers of cos in terms of a sum over cos functions with multiple arguments is

$$IdB1(Q,\theta):\qquad (\cos(\theta))^{2\,Q} = \frac{1}{2^{2\,Q}}\left(\sum_{k=0}^{Q-1}\binom{2\,Q}{k}\cos\left(2\,(Q-k)\,\theta\right) + \binom{2\,Q}{Q}\right)\,,\ \ \text{for}\ \ Q \in \mathbb{N}\,. \qquad (21)$$

See [3], p. 53, eq. 1.320 6.

In eq. (20) this is used with $Q = K - q$ and $\theta = \pi \frac{l}{2\,n+1}$. The q and k summations are interchanged with the l-summation in the first k-dependent term of $IdB'(n, K)$ leading to the following l-sum.

$$\Sigma_B(K, q, k, n) := \sum_{l=1}^{n} 2\, \cos\left(\frac{\pi}{2\,n+1}\, 2\,(K-q-k)\,l\right) . \tag{22}$$

The important identity found, *e.g.*, in *Jolley* [5], p. 78, Nr. (418) rewritten, is used here.

$$IdB2(n, \theta): \qquad \sum_{l=1}^{n} 2\, \cos(\theta\, l) = -1 + \frac{\sin\left((2\,n+1)\,\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}, \quad \text{for} \quad n \in \mathbb{N}. \tag{23}$$

The proof, based on exponential functions, is given in *Courant* [2], p. 383-4.
The crucial point is the vanishing of the sin numerator after insertion of $\theta = \frac{\pi}{2\,n+1}\,(2\,(K-q-k))$, because $\sin(\pi\, m) = 0$ for any integer m. The denominator does not vanish because $K - q - k \in \{1, ..., n\}$. Therefore, $\Sigma_B(K, q, k, n) = -1$. We are led to compute

$$\sum_{l=1}^{n} S_{2\,K}(x S_l^{(2\,n)}) = \sum_{q=0}^{K} (-1)^q \binom{2\,K-q}{q} \left[\sum_{k=0}^{K-q-1} \binom{2\,(K-q)}{k} \cdot (-1) + n \binom{2\,(K-q)}{K-q}\right] . \tag{24}$$

In the second term the l-sum produced the factor n in front of the binomial. In the first term the trivial binomial sum (regarding the row sum of the even numbered rows of *Pascal*'s triangle A007318) employing the symmetry around the central binomial coefficient A000984) is

$$\sum_{k=0}^{N-1} \binom{2\,N}{k} = 2^{2\,N-1} - \binom{2\,N-1}{N}, \quad \textit{for} \quad N \in \mathbb{N}\,. \tag{25}$$

See also A000346$(n-1)$. Note that this identity is not valid for $N = 0$ where the *l.h.s.* vanishes due to the undefined sum but the *r.h.s.* becomes $\frac{1}{2}$. Therefore one has to split off the term $q = K$ in the q-sum. The formula is now

$$\sum_{q=0}^{K-1} (-1)^q \binom{2\,K-q}{q} \left[-2^{2\,(K-q)-1} + \binom{2\,(K-q)-1}{K-q} + n \binom{2\,(K-q)}{K-q}\right] + (-1)^K\, n\,. \tag{26}$$

There are now three binomial sums, but only two of them are independent. For the first identity the summation index q is taken up to K, subtracting again this $q = K$ term, in order to use eq. (7) for $k = 2\,K$.

$$-\frac{1}{2}\sum_{q=0}^{K-1} (-1)^q \binom{2\,K-q}{q}\, 2^{2\,(K-q)} = -\frac{1}{2}\, S_{2\,K}(2) + \frac{1}{2}\,(-1)^K = -\frac{1}{2}\,(2\,K+1) + \frac{1}{2}\,(-1)^K\,, \tag{27}$$

where $S_m(2) = m+1$ has been used again.
Because the first binomial in the solid brackets of eq. (26) can be obtained from the second one, viz $2\binom{2\,(K-q)-1}{K-q} = \binom{2\,K-q}{K-q}$, we prove for the sum involving the latter one the following identity.

$$\sum_{q=0}^{K-1} (-1)^q \binom{2\,K-q}{q} \binom{2\,(K-q)}{K-q} = 1 - (-1)^K\,, \quad \text{for} \quad K \in \mathbb{N}\,. \tag{28}$$

If one uses $\binom{2\,K-q}{q}\binom{2\,(K-q)}{K-q} = \binom{K}{q}\binom{2\,K-q}{K}$ this becomes

$$\sum_{q=0}^{K} (-1)^q \binom{K}{q} \binom{2\,K-q}{K} = 1\,, \quad \text{for} \quad K \in \mathbb{N}\,. \tag{29}$$

The product of the two binomials is triangle $Tri(K, q) =$ A104684(K, q). The identity means that the alternating row sums of A104684 are identically 1, and this is stated there with the reference to a problem posed by *Michel Bataille* [1]. The proof is done for the row reversed signed triangle (with q replaced by $K - q$), observing the convolution property.

$$\sum_{q=0}^{K} (-1)^{K-q} \binom{K}{K-q} \binom{K+q}{K} = 1\,, \quad \text{for} \quad K \in \mathbb{N}\,. \tag{30}$$

Take $a^{(K)}(k) = (-1)^k \binom{K}{k}$ with ordinary generating function (*o.g.f.*) $A(x) = (1 + (-x))^K$ and $b_k^{(K)} = \binom{K+k}{K}$ with *o.g.f.* $B^{(K)}(x) = \frac{1}{(1-x)^{K-1}}$. Then $C^K(x) = A^{(K)}(x)\, B^{(K)}(x) = \frac{1}{(1-x)} = \sum_{k=0}^{\infty} x^k$. This ends the proof adapted from [1].

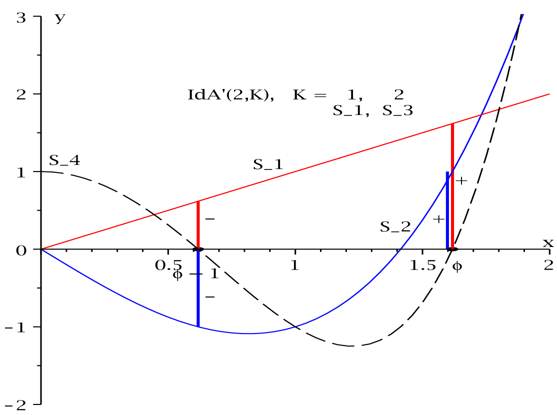
Summing up all four terms of eq. (26) leads finally to

$$S_{2K}\left(xS_l^{(2n)}\right) = \left(-\frac{1}{2}(2K+1) + \frac{1}{2}(-1)^K\right) + \frac{1}{2}(1-(-1)^K) + (1-(-1)^K)\,n + (-1)^K\,n = n - K\,. \tag{31}$$

This ends the proof of $IdB'(n, K)$ from eq. 4. □

We close this note with a graphical representation of some instances of these identities. For $IdA'(n, K)$ we look in *Figure 1* at $n = 2$ and consider only the positive zeros of $S_4(x)$ (in dashed black) which are $xS_1^{(4)} = \phi$ and $xS_2^{(4)} = \phi - 1$ with the golden section ϕ satisfying $\phi^2 - \phi - 1 = 0$ and $\phi > 0$. Then $K = 1, 2$ for $S_1(x) = x$ (in red) and $S_3(x) = x^3 - 2x$ (in blue). The vertical bars have to be added (+ mark) or subtracted (− mark). This means that if a bar in the negative y region has a − mark the length of that bar has to be added. Thus the two blue ($K = 2$) bar lengths add up to 2. The two red ($K = 1$) bar lengths have to be subtracted to yield the length 1.

Similarly for the identity $IdB'(n, K)$ we consider in *Figure 2* the instance $n = 3$, *i.e.*, the three positive zeros of $S_6(x) = x^6 - 5x^4 + 6x^2 - 1$ which are $xS_l^{(6)} = 2\cos\left(\frac{\pi l}{7}\right)$, for $l = 1, 2, 3$. Only $K = 1$, *i.e.*, $S_2(x) = x^2 - 1$, and $K = 2$, *i.e.*, $S_4(x)$, is interesting because for $K = 3$ the identity is trivially true because S_6 vanishes of course for each of its positive zeros. The three bars (red) for $S_2(x)$ add up to $2 = 3 - 1$, one bar has to be subtracted. The three bars (blue) for $S_4(x)$ add up to $1 = 3 - 2$.



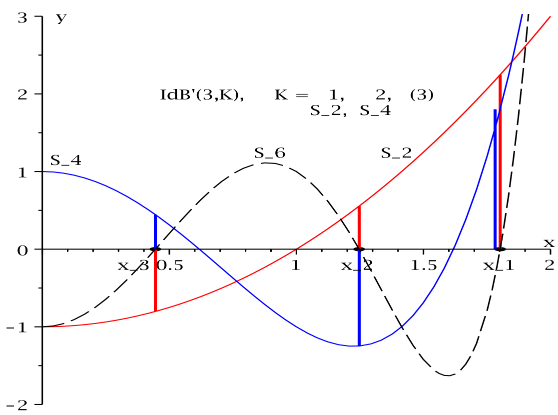


Figure 1 **Figure 2**

See the text for details. Plots with Maple [8].

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