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A176740: E.g.f. Lagrange inversion partition array

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Abstract

The coefficients of the special Lagrange series which provides the compositional inverse of a formal power series, considered as exponential generating function (*e.g.f.*), are obtained from a certain partition array involving the multinomial M_3 numbers A036040.

Given any formal power series (*f.p.s.*) $g(x) := \sum_{k=1}^{\infty} g_k \frac{x^k}{k!} =: x\,h(x)$ with $g_0 = 0$, $g_1 \neq 0$, the compositional inverse $f = g^{[-1]}$ defined by $f(g(x)) \equiv x$ or $g(f(y)) \equiv y$ is defined as *f.p.s.* $f(y) := \sum_{n=1}^{\infty} f_n \frac{y^n}{n!}$. The well known expression for the coefficients f_n of the Lagrange series, which involves higher derivatives of the inverse of the *f.p.s.* h, is *e.g.*, [5], p. 524, [3], p. 205, [6], ch. 4.5, p.146 ff.

$$f_n = \left(\frac{1}{h(x)^n}\right)^{(n-1)}\Bigg|_{x=0} , \; n \geq 2; \; f_1 = \frac{1}{h_0} = \frac{1}{g_1}, \tag{1}$$

i.e.

$$f_n = (n-1)!\,[x^{n-1}] \left(\frac{1}{h(x)}\right)^n , \; n \geq 2, \text{ and } f_1 = \frac{1}{g_1}. \tag{2}$$

Therefore, instead of giving an explicit formula, it demands to find the coefficients of inverse powers of a *f.p.s.* considered as *e.g.f.*. This is achieved with the help of *Bell* polynomials [6] and the proof can be found *e.g.*, in [3], p. 437, eq. (11.43) with p. 428. eq. (11.29) or [2], p. 175, eq. (13.84).

$$f_n = \frac{1}{g_n^n} \sum_{k=1}^{n-1} (-1)^k\, n^{\overline{k}}\, B_{n-1,k}(\hat{g}_1, \hat{g}_2, \ldots, \hat{g}_{n-k}) \; , \; n \geq 2 \; , \tag{3}$$

with $\hat{g}_k := \frac{g_{k+1}}{(k+1)\,g_1}$, and $f_1 = \frac{1}{g_1}$. $n^{\overline{k}}$ is the rising factorial (*Pochhammer* symbol) $n\,(n+1)\cdots(n-(k-1))$.

This formula is the solution of the recurrence formula for the coefficients of the Lagrange series which is (see [2], p. 173, eq.(13.81) where the solution f_n is also given as a determinant due to *Cramer*'s rule in eq.(13.82)).

$$f_n = -\frac{1}{g_1^n} \sum_{k=1}^{n-1} B_{n,k}(g_1, \ldots, g_{n+1-k})\, f_k \; , \; n \geq 2, \; f_1 = \frac{1}{g_1}. \tag{4}$$

Note: This solution of the recurrence for the coefficients of the this special application of a Lagrange series appears in the context of the *Faà di Bruno Hopf* algebra for the formal diffeomorphism group in

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the complex plane as the antipode (*aka* counit). See [8], ch. 3, "*The Power of Combinatorics*", sects. 3.4.4 , 3.4.5, pp. 136 - 140, especially eqs. (3.34), (3.36), and (3.38).
The above solution of the recurrence relation for the $\{f_n\}$ coefficients can be considered as an analogon of *Zimmermann*'s forest formula in the context of renormalizable quantum field theory à la *Bogoliubov-Parasuik-Hepp-Zimmermann* (BPHZ), where the analog of the recurrence formula is the iterative R-operation of *Bogoliubov*.

This "sophisticated Zimmermann type cancellation formula" ([8], p. 139), eq. (3), can be reinterpreted such that its partition structure becomes obvious (W. Lang, Feb. 2010). This just uses the fact that *Bell* polynomials are partition polynomials and some later given proposition. The final formula for the coefficients of this special application of the Lagrange series becomes then

$$f_n \,=\, \frac{1}{g_1^{2n-1}}\,\hat f_n, \quad i.e.,\; f(y) \,=\, g_1 \sum_{n=1}^{\infty} \hat f_n \left(\frac{y}{g_1^2}\right)^n \frac{1}{n!}\,, \tag{5}$$

$\hat f_1 \,=\, 1$, and for $n \,=\, 2, 3, \ldots$ we have

$$\hat f_n \,=\, \sum_{m=1}^{n-1} (-1)^m \, P3(n,m)\,, \tag{6}$$

with

$$P3(n,m) \,:=\, \sum_{p:=(1^{e_1},2^{e_2},\ldots,(n-1)^{e_{n-1}})\text{ from }Pa(n-1,m)} P3(p)\,, \tag{7}$$

where, finally,

$$P3(p) \,:=\, M3(n+m-1,\vec{\hat e}\,) \prod_{j=1}^{n-1} \left(g_1^{j-1}\, g_{j+1}\right)^{e_j} \,. \tag{8}$$

Some more definitions have been used in the above formulae. The set of partitions of n with m parts (m part partitions of n) is here denoted by $Pa(n,m)$. This set will be considered as a list of $p(n,m) \,=\,$ A008284(n,m) partitions ordered lexicographically. Note that we take the partitions of n in the *Abramowitz-Stegun* (*A-St*) order [1], *i.e.* the partitions are ordered with increasing parts number m, and partitions with the same number of parts m are or ordered lexicographically.
The multinomial M_3 partition numbers, called here $M3(n,\vec e\,)$, are given as partition array in A036040 (see also the W. Lang link there). $\vec e \,:=\, (e_1, e_2, \ldots, e_n)$, stands for the non-negative integer exponents of a given m part partition of n: $\sum_{j=1}^{n} j\, e_j \,=\, n$ and $\sum_{j=1}^{n} e_j \,=\, m$, $e_j \,\in\, \mathbb{N}_0$.
In the formula the M_3 numbers for m part partitions of $n+m-1$ with the following exponents, determined from those of the partition p, enter:

$$\vec{\hat e} \,=\, (0, e_1, e_2, \ldots, e_{n-1}, \underbrace{0, \ldots 0}_{m-1})\,. \tag{9}$$

I.e., $\hat e_1 := 0, \hat e_2 \,:=\, e_1, \ldots, \hat e_n \,:=\, e_{n-1}, \hat e_{n+1} \,:=\, 0, \ldots, \hat e_{n+m-1} \,:=\, 0$.
These $\hat e$ exponents belong indeed to an m part partition of $n+m-1$ because $2\,e_1 \,+\, 3\,e_2 \,+\, \ldots \,+\, n\,e_{n-1} \,=\, (1\,e_1 \,+\, 2\,e_2 \,+\, \ldots \,+\, (n-1)\,e_{n-1}) \,+\, \sum_{j=1}^{n-1} e_j \,=\, (n-1) + m$.
The definition of the M_3 numbers is

$$M3(n+m-1,\vec{\hat e}\,) \,:=\, \frac{(n+m-1)!}{\prod_{j=1}^{n+m-1} j!^{\hat e_j}\, \hat e_j!}. \tag{10}$$

In *Table 1* these $M3$ numbers are listed as partition array A176740(n,k) in *OEIS*[7] for partitions of n, $n \geq 1$, in the *A-St* order, as they appear in the above formula for $\hat{f}_{n+1}$. If the numbers corresponding to partitions of n with the same parts number m are summed, the well known triangle A134991 of *OEIS* [7], the so called $2-$associated *Stirling* number triangle of the second kind $S2_2(n,m)$ appears (see [4], p. 222, where these numbers are called $S_2(n,k)$, with $n \rightarrow n-m$ and $k \rightarrow m$). This triangle is shown in *Table 2*. The column sequences without leading zeros and offset 0 are A000012(n) (powers of 1), A000247$(n+3)$, A000478$(n+6)$, A058844$(n+8)$, ... The row sums give A000311$(n+1)$ (*Schroeder*'s fourth problem). The signed triangle has row sums A133942$(n) = (-1)^n\, n!$, $n \geq 1$.

It should be mentioned that the same problem is treated in A134685 in *OEIS*[7], however the order of partitions is different and fewer terms are given.

It turns out that in the $\hat{f}_n$, $n \geq 2$, formula, which derives from partitions of $n-1$ in the *A-St* order, one finds exactly the $g-$coefficients fitting to the partitions of $2\,(n-1)$ with $n-1$ parts, also sorted in *A-St* order. This is the main observation of this note. Before proving this we first give an example. The formula for $\hat{f}_5$ is

$$\hat{f}_5 \;=\; -1\, g_1^3\, g_5 \,+\, 15\, g_1^2\, g_2\, g_4 \,+\, 10\, g_1^2\, g_3^2 \,-\, 105\, g_1\, g_2^2\, g_3 \,+\, 105\, g_2^4\;. \tag{11}$$

Compare the array *Tab.1*, line $n = 4$, for the $M3$ numbers $[-1, 15, 10, -105, 105]$, and [1], p. 831, for the order of all the partitions of 4, whose exponents determine these $M3$ numbers according to the above given prescription, and the 4 parts partitions of 8 which produce the g coefficients.

This $1-1$ mapping of partitions of n in *A-St* order to the partitions of $2\,n$ with n parts (we used here $(n-1) \rightarrow n$) is the content of the following *Proposition*. But first a simple *Lemma* is needed.

Lemma:

Every element of $Pa(2\,n, n)$, the set of partitions of $2\,n$ with n parts, has the last $n-1$ exponents vanishing. This means that there is no part exceeding $n+1$.

Proof:

Every exponent in the set $\{e_{n+2},\, e_{n+3},\, \ldots,\, e_{2\,n}\}$ can at most be either 0 or 1, because a partition of $2\,n$ is considered. Moreover, if any of these exponents is 1, say $e_{n+k} = 1$, for $k \in \{2, 3, ..n\}$, then in order to have n parts one would have to use $n-1$ times the part 1, but this already overshoots $2\,n$ because $(n+k) \,+\, (n-1)\,1 \,>\, 2\,n$. Therefore, all of these exponents have to vanish.

Proposition:

There is a bijective mapping between $Pa(n)$, the set of partitions of n and $Pa(2\,n, n)$, the set of partitions of $2\,n$ with n parts, which respects the *A-St* order. This mapping is the following one.

i) $(e_1, e_2,\, \ldots, e_n) \in Pa(n) \mapsto (n-m, e_1, e_2, \ldots, e_n, 0,\, \ldots, 0) \in Pa(2\,n, n)$.

Here the exponent notation for partitions is used. $m := \sum_{j=1}^{n} e_j$. In the image the last $n-1$ exponents, those for the parts $n+2,\, n+3,\, \ldots, 2\,n$, are all vanishing, in accordance with the *Lemma*.

ii) $(e_1, e_2,\, \ldots, e_{n+1}, 0,\, \ldots, 0) \in Pa(2\,n, n) \mapsto (e_2, e_3,\, \ldots, e_{n+1}) \in Pa(n)$.

The last $n-1$ exponents in any $Pa(2\,n, n)$ partition all vanish due to the *Lemma*. The parts number of the image is $m = \sum_{j=2}^{n+1} e_j = n - e_1$.

Example: The list of the seven $n = 5$ partitions in *A-St* order is (written in exponent notation)
$[(0,0,0,0,1),\ (1,0,0,1,0),\ (0,1,1,0,0),\ (2,0,1,0,0),\ (1,2,0,0,0),\ (3,1,0,0,0),\ (5,0,0,0,0)]$.
This list is mapped, entry by entry, to the following list, which turns out to be also *A-St* ordered.
$[(4,0,0,0,0,1,0,0,0,0),\ (3,1,0,0,1,0,0,0,0,0),\ (3,0,1,1,0,0,0,0,0,0),\ (2,2,0,1,0,0,0,0,0,0),$
$(2,1,2,0,0,0,0,0,0,0),\ (1,3,1,0,0,0,0,0,0,0),\ (0,5,0,0,0,0,0,0,0,0)]$.

Proof:

i) The images are partitions of $Pa(2\,n, n)$ because $1\,(n-m) + \sum_{j=1}^{n} (j+1)\, e_j \;=\; n - \sum_{j=1}^{n} e_j + \sum_{j=1}^{n} (j+1)\, e_j \;=\; n \,+\, n \,=\, 2\,n$.

If in the *A-St* ordered list of the partitions of n, partition p_2 follows partition p_1, then the parts number m is either identical or larger for p_2. If m coincides then the *A-St* order is lexicographic, and the images have identical exponent e_1 and the remaining parts of the two images take over the lexicographic order (the parts are just shifted by $+1$). Therefore, in this case the image of p_2 follows also the image of p_1 because *A-St* order in $Pa(2\,n, n)$ is just lexicographic. If p_2 has larger parts number m than p_1, the exponent e_1 of the image of p_2 is smaller than the one for the image of p_1 and the remaining parts are in lexicographic order. Therefore, also in this case lexicographic order ensues.

ii) It is clear that this map leads to partitions of n with parts number $m \,=\, n \,-\, e_1$, because $\sum_{j=1}^{n} j\, e_{j+1} \;=\; \sum_{j=1}^{n+1} j\, e_j \;-\; \sum_{j=1}^{n+1} e_j = 2\,n \,-\, n \,=\, n.$

For $Pa(2\,n, n)$ partitions the *A-St* order is just lexicographic (the parts number is fixed; it is n). If from this set a partition p_2 follows lexicographically p_1, then the parts number $(n \,-\, e_1)$ of the image of p_2 is never smaller than the one of the image of p_1. If it is larger then the image of p_2 follows in *A-St* order the one of p_1. If the parts number of the images coincide, the original lexicographic order from the two $\{e_2,\, e_3,\, \ldots\, e_{n+1}\}$ exponent sets is carried over to the images, hence they are *A-St* ordered.

This proves the bijective mapping between the lists of *A-St* ordered partitions $L(n)$ for partitions of n and $L(2\,n, n)$ for n parts partitions of $2\,n$: $L(n)[k] \longleftrightarrow L(2\,n, n)[k]$, for $k \,=\, 1, \ldots, p(n) = $ A000041(n).

Corollary:

A000041(n) $=:\; p(n) \;=\; p(2\,n, n) \;:=$ A008284(n, m).

Examples: $p(4) \;=\; 5 \;=\; p(8, 4)$, $p(5) \;=\; 7 \;=\; p(10, 5)$.

Now the assertion about the $\hat{f}_n$ structure in terms of *A-St* ordered partitions of $Pa(2\,(n-1), n-1)$ is obvious. Remember that we have to use the *Proposition* with $n \;\rightarrow\; n-1$. Just observe that the exponent of the coefficient g_1 in eq. (8) is $\sum_{j=1}^{n-1} (j-1)\, e_j \;=\; \sum_{j=1}^{n-1} j\, e_j \;-\; \sum_{j=1}^{n-1} e_j \;=\; (n-1) \,-\, m$, like expected for the map **i**) of the *Proposition* with $n \;\rightarrow\; n-1$. Also the exponents of the other g_k, $k \in \{2, 3, \ldots, n\}$, fit this map. Thus we have shown, that the following formulae can be used for $\hat{f}_n$, instead of eqs. (6) to (8) .

$$\hat{f}_n \;=\; \sum_{k=1}^{p(n-1)} a(n-1, k) \prod_{j=1}^{n} g_j^{e_j(k)} \;,\; n \,\in\, \{2, 3, \ldots,\} \text{ and } \hat{f}_1 \;=\; 1 \;, \qquad (12)$$

with the signed partition array $a(n, k) \;:=$ A176740(n, k) given in *Table 1*, and the exponents $\{e_j(k)\}$ of the $k-th$ partition of $Pa(2\,(n-1), n-1)$ in *A-St*, i.e. here in lexicographic order. The number of partitions is $p(n) :=$ A000041(n). This produces for $\hat{f}_n$ the polynomials given in *Table 3*.

TAB. 1: A176740(n,k) partition array

n/k	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
1	-1											
2	-1	3										
3	-1	10	-15									
4	-1	15	10	-105	105							
5	-1	21	35	-210	-280	1260	-945					
6	-1	28	56	35	-378	-1260	-280	3150	6300	-17325	10395	
7	-1	36	84	126	-630	-2520	-1575	-2100	6930	34650	15400	-51975
8	-1	45	120	210	126	-990	-4620	-6930	-4620	-5775	13860	83160
⋮												

n/k	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22 ...**
1										
2										
3										
4										
5										
6										
7	-138600	270270	-135135							
8	51975	138600	15400	-135135	-900900	-600600	945945	3153150	-4729725	2027025
⋮										

TAB. 2: triangle A134991 from the unsigned partition array A176740

n/m	1	2	3	4	5	6	7	8	9	10	...
1	1										
2	1	3									
3	1	10	15								
4	1	25	105	105							
5	1	56	490	1260	945						
6	1	119	1918	9450	17325	10395					
7	1	246	6825	56980	190575	270270	135135				
8	1	501	22935	302995	1636635	4099095	4729725	2027025			
9	1	1012	74316	1487200	12122110	47507460	94594500	91891800	34459425		
10	1	2035	235092	6914908	81431350	466876410	1422280860	2343240900	1964187225	654729075	
⋮											

TAB. 3: E.g.f. inversion: partition polynomials for $\{\hat{\mathbf{f}}_{\mathbf{n}}\}_1^{10}$, $\mathbf{f_n} = \frac{1}{\mathbf{g}_1^{2\,\mathbf{n}-1}}\,\hat{\mathbf{f}}_{\mathbf{n}}$

n	**partition polynomials for n − 1 parts partitions of 2 (n − 1)**
1	1
2	$-g_2$
3	$-g_1\, g_3 + 3\, g_2^2$
4	$-g_1^2\, g_4 + 10\, g_1\, g_2\, g_3 - 15\, g_2^3$
5	$-g_1^3\, g_5 + 15\, g_1^2\, g_2\, g_4 + 10\, g_1^2\, g_3^2 - 105\, g_1\, g_2^2\, g_3 + 105\, g_2^4$
6	$-g_1^4\, g_6 + 21\, g_1^3\, g_2\, g_5 + 35\, g_1^3\, g_3\, g_4 - 210\, g_1^2\, g_2^2\, g_4 - 280\, g_1^2\, g_2\, g_3^2 + 1260\, g_1\, g_2^3\, g_3 - 945\, g_2^5$
7	$-g_1^5\, g_7 + 28\, g_1^4\, g_2\, g_6 + 56\, g_1^4\, g_3\, g_5 + 35\, g_1^4\, g_4^2 - 378\, g_1^3\, g_2^2\, g_5 - 1260\, g_1^3\, g_2\, g_3\, g_4 - 280\, g_1^3\, g_3^3 + 3150\, g_1^2\, g_2^3\, g_4 + 6300\, g_1^2\, g_2^2\, g_4^2$ $- 17325\, g_1\, g_2^4\, g_3 + 10395\, g_2^6$
8	$- g_1^6\, g_8 + 36\, g_1^5\, g_2\, g_7 + 84\, g_1^5\, g_3\, g_6 + 126\, g_1^5\, g_4\, g_5 - 630\, g_1^4\, g_2^2\, g_6 - 2520\, g_1^4\, g_2\, g_3\, g_5 - 1575\, g_1^4\, g_2\, g_4^2 - 2100\, g_1^4\, g_3^2\, g_4$ $+ 6930\, g_1^3\, g_2^3\, g_5 + 34650\, g_1^3\, g_2^2\, g_3\, g_4 + 15400\, g_1^3\, g_2\, g_3^3 - 51975\, g_1^2\, g_2^4\, g_4 - 138600\, g_1^2\, g_2^3\, g_3^2 + 270270\, g_1\, g_2^5\, g_3 - 135135\, g_2^7$
9	$- g_1^7\, g_9 + 45\, g_1^6\, g_2\, g_8 + 120\, g_1^6\, g_3\, g_7 + 210\, g_1^6\, g_4\, g_6 + 126\, g_1^6\, g_5^2 - 990\, g_1^5\, g_2^2\, g_7 - 4620\, g_1^5\, g_2\, g_3\, g_6 - 6930\, g_1^5\, g_2\, g_4\, g_5$ $- 4620\, g_1^5\, g_3^2\, g_5 - 5775\, g_1^5\, g_3\, g_4^2 + 13860\, g_1^4\, g_2^3\, g_6 + 83160\, g_1^4\, g_2^2\, g_3\, g_5 + 51975\, g_1^4\, g_2^2\, g_4^2 + 138600\, g_1^4\, g_2\, g_3^2\, g_4 + 15400\, g_1^4\, g_3^4$ $- 135135\, g_1^3\, g_2^4\, g_5 - 900900\, g_1^3\, g_2^3\, g_3\, g_4 - 600600\, g_1^3\, g_2^2\, g_3^3 + 945945\, g_1^2\, g_2^5\, g_4 + 3153150\, g_1^2\, g_2^4\, g_3^2 - 4729725\, g_1\, g_2^6\, g_3 + 2027025\, g_2^8$
10	$- g_1^8\, g_{10} + 55\, g_1^7\, g_2\, g_9 + 165\, g_1^7\, g_3\, g_8 + 330\, g_1^7\, g_4\, g_7 + 462\, g_1^7\, g_5\, g_6 - 1485\, g_1^6\, g_2^2\, g_8 - 7920\, g_1^6\, g_2\, g_3\, g_7 - 13860\, g_1^6\, g_2\, g_4\, g_6$ $- 8316\, g_1^6\, g_2\, g_5^2 - 9240\, g_1^6\, g_3^2\, g_6 - 27720\, g_1^6\, g_3\, g_4\, g_5 - 5775\, g_1^6\, g_4^3 + 25740\, g_1^5\, g_2^3\, g_7 + 180180\, g_1^5\, g_2^2\, g_3\, g_6 + 270270\, g_1^5\, g_2^2\, g_4\, g_5$ $+ 360360\, g_1^5\, g_2\, g_3^2\, g_5 + 450450\, g_1^5\, g_2\, g_3\, g_4^2 + 200200\, g_1^5\, g_3^3\, g_4 - 315315\, g_1^4\, g_2^4\, g_6 - 2522520\, g_1^4\, g_2^3\, g_3\, g_5 - 1576575\, g_1^4\, g_2^3\, g_4^2$ $- 6306300\, g_1^4\, g_2^2\, g_3^2\, g_4 - 1401400\, g_1^4\, g_2\, g_3^4 + 2837835\, g_1^3\, g_2^5\, g_5 + 23648625\, g_1^3\, g_2^4\, g_3\, g_4 + 21021000\, g_1^3\, g_2^3\, g_3^3 - 18918900\, g_1^2\, g_2^6\, g_4$ $- 75675600\, g_1^2\, g_2^5\, g_3^2 + 91891800\, g_1\, g_2^7\, g[3] - 34459425\, g_2^9$
⋮	

References

[1] Abramowitz and I. A. Stegun, eds., *Handbook of Mathematical Functions*, National Bureau of Standards Applied Math. Series 55, Tenth Printing, reprinted as Dover publication, seventh printing 1968, New York, 1972

[2] R. Aldrovandi, *Special Matrices of Mathematical Physics*, World Scientific, 2001

[3] Ch. A. Charalambides, *Enumerative Combinatorics*, Chapman &Hall/CRC, 2002

[4] L. Comtet, *Advanced Combinatorics*, D. Reidel, Dordrecht, Boston,1974

[5] G. M. Fichtenholz, *Differential- und Integralrechnung II*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1964

[6] J. Riordan, *Combinatorial Identities*, Robert E. Krieger Publ. Comp., Huntington, New York, 1979

[7] N.J.A. Sloane's On-Line Encyclopedia of Integer Sequences,
http//:www.research.att.com/˜njas/sequences/index.html

[8] E. Zeidler, *Quantum Field Theory II, Quantum Electrodynamics*, Springer 2009

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