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A181872/A181873. Minimal Polynomials of $\sin\left(\frac{2\pi}{n}\right)$

Wolfdieter L a n g [1]

The minimal polynomial of an algebraic number α of degree d_α is the monic, minimal degree rational polynomial which has as root, or as one of its roots, α. This minimal degree d_α is 1 iff α is rational, and the minimal polynomial in this case is $p(x) = x - \alpha$. For the notion 'minimal polynomial of an algebraic number' see, *e.g.*, [4], p. 28. They are irreducible by the minimality requirement.

For the algebraic number $\sin\left(\frac{2\pi}{n}\right)$, for $n \in \mathbb{N}$, the degree (called here $\delta(n)$) is $\delta(4) = 1$, and $\delta(n) = \varphi(n)$ if $gcd(n,8) < 4$, $\delta(n) = \frac{\varphi(n)}{4}$ if $gcd(n,8) = 4$, and $\delta(n) = \frac{\varphi(n)}{4}$ if $gcd(n,8) > 4$, with *Euler*'s totient function $\varphi(n)$ = A000010(n). See [4], Theorem 3.9, p. 37. In *Niven*'s book this theorem is attributed to *D. H. Lehmer*, but in [3] Theorem 2 is incorrect as the counterexample for $n = 12, k = 1$ shows: according to *Lehmer* the degree of $2\,sin(2\,\pi/12)$ is $\varphi(12) = 2$, however it has to be 1 because this quantity is rational, *viz* 1, with the minimal polynomial $x - 1$. The degree is correctly given in *Niven*'s book as $\varphi(12)/4 = 1$. In the 'proof' *Lehmer* distinguished three cases, but in the third case (n is a multiple of 4) the dependence on k was not taken into account. Compare with the proof of *Niven*. In [5] one finds this degree sequence as $\delta(n)$ =A093819(n). The Sines-table in [3] on p. 166 is therefore also incorrect. The characteristic sequence for $\sin\left(\frac{2\pi}{n}\right)$ being rational is $[1,1,0,1,0,0,0,0,0,0,0,1,\text{followed by zeros}]$, given as A183919.

The minimal polynomials of $\sin\left(\frac{2\pi}{n}\right)$, which we will call $\Pi(n,x)$, can be found from a certain mapping c, desribed below, from those of $\cos\left(\frac{2\pi}{c(n)}\right)$. The minimal polynomials of $\cos\left(\frac{2\pi}{n}\right)$ have been discussed in [6] where they have been called $\Psi_n(x)$. We have called them $\Psi(n,x)$, and gave a list of the first 30 polynomials in a link found in A181875. They were also given in a comment by *A. Jasinski* in A023022. The trigonometric identity used in [3] and [4] is $\sin\left(\frac{2\pi}{n}\right) = \cos 2\pi\left|\frac{4-n}{4n}\right|$. Because all zeros of the minimal polynomials $\Psi_n(x)$ are known (see the *Lemma* in [6], p. 473) , *viz* $\cos\left(\frac{2\pi k}{n}\right)$ for $k \in \{0,1,...,\lfloor\frac{n}{2}\rfloor\}$ and $gcd(k,n) = 1$, one can give the map $c : \mathbb{N} \to \mathbb{N}$, $n \mapsto c(n)$ such that $\Pi(n,x) = \Psi(c(n),x)$. Note that this map c is neither surjective nor injective because $\sin\left(\frac{2\pi}{n}\right)$ is never $-\frac{1}{2}$ for $n \geq 1$ (this follows from *Corollary 3.12* in [4], p. 41), and $c(1) = 4 = c(2)$, $c(3) = 12 = c(6)$, $c(9) = 36 = c(18)$, $c(11) = 44 = c(22)$, *etc.* For example, $c(7)$ is found from $\sin\left(\frac{2\pi}{7}\right) = \cos 2\pi\left(\frac{3}{4\cdot 7}\right)$. Now $\Psi(28)$ has this zero due to the lemma on all of its zeros mentioned above. Because the degree of this minmal polynomial of $\cos\left(\frac{2\pi}{28}\right)$ is $\varphi(28)/2 = \varphi(7) = 6$ and coincides with $\delta(7)$ we have $c(7) = 28$. It is clear that always $\delta(n) = d(c(n))$. The map c is given by $c(n) =$ denominator$(\left|\frac{n-4}{4n}\right|)$ (in lowest

[1] wolfdieter.lang@kit.edu, http://www-itp.particle.uni-karlsruhe.de/~wl

terms), and is found under A178182. In the notes given as a link to A181875 we have illustrated how to compute $\Psi(n,x)$ due to the recurrence from [6]. It is easy to write a program for $\Psi(n,x)$.

The minimal polynomials $\Pi(n,x)$ are given in *Table 1*. The numerator and denominator arrays of the coefficients of these polynomials are given as A181872(n,m) and A181873(n,m) in *Table 2* and *Table 3*, respectively. The rational coefficients for the monic polynomials $\Pi(n,x)$ will be given in *Table 4*. *Table 5* shows the head of the integer coefficient array of the non-monic $\pi(n,x) := 2^{\delta(n)}\,\Pi(n,x)$ polynomials. This is A181871(n,m).

Note added, Feb 28 2011

In the paper by *Beslin* and *de Angelis* [1] the explicit form for the (integer) minimal polynomial for $\sin\left(\frac{2\,\pi}{n}\right)$, for odd prime p has been given. It is called there $S_p(x)$. The result is, with $p\,=\,2\,k+1$,

$$S_p(x) \;=\; \sum_{l=0}^{k}(-1)^k\binom{p}{2\,l+1}(1-x^2)^{k-l}\,x^{2\,l}\;. \tag{1}$$

The leading term is $(-1)^k\,2^{p-1}\,x^{p-1}$. $S_p(x)$ checks with $(-1)^k\,2^{p-1}\,\Pi(p,x)$.

References

[1] S. Beslin and V. de Angelis, The minimal Polynomials of $\sin(\frac{2\pi}{p})$ and $\cos(\frac{2\pi}{p})$, Mathematics Magazine 77,2 (2004) 146-9. http://www.maa.org/pubs/mag_apr04_toc.html.

[2] Chan-Lye Lee and K. B. Wong, On Chebyshev's polynomials and certain combinatorial identities, Bulletin of the Malaysian Mathematical Sciences Soc., November 17, 2009, http://www.emis.de/journals/BMMSS/accepted_papers.htm.

[3] D. H. Lehmer, A Note on Trigonometric Algebraic Numbers, Am. Math. Monthly 40,8 (1933) 165-6.

[4] I. Niven, Irrational Numbers, The Math. Assoc, of America, second printing, distributed by John Wiley and Sons, 1963.

[5] N.J.A. Sloane's On-Line Encyclopedia of Integer Sequences, http://oeis.org/.

[6] William Watkins and Joel Zeitlin, The Minimal Polynomial of $\cos(2\pi/n)$, Am. Math. Monthly 100,5 (1993) 471-4.

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Table 1: Minimal polynomials of $\sin\left(\frac{2\pi}{n}\right)$ for $n = 1, 2, ..., 30$.

n	**c(n)**	$\mathbf{\Pi(n, x)}$
1	4	x
2	4	x
3	12	$x^2 - 3/4$
4	1	$x - 1$
5	20	$x^4 - (5/4)\,x^2 + 5/16$
6	12	$x^2 - 3/4$
7	28	$x^6 - (7/4)\,x^4 + (7/8)\,x^2 - 7/64$
8	8	$x^2 - 1/2$
9	36	$x^6 - (3/2)\,x^4 + (9/16)\,x^2 - 3/64$
10	20	$x^4 - (5/4)\,x^2 + 5/16$
11	44	$x^{10} - (11/4)\,x^8 + (11/4)\,x^6 - (77/64)\,x^4 + (55/256)\,x^2 - 11/1024$
12	6	$x - 1/2$
13	52	$x^{12} - (13/4)\,x^{10} + (65/16)\,x^8 - (39/16)\,x^6 + (91/128)\,x^4 - (91/1024)\,x^2 + 13/4096$
14	28	$x^6 - (7/4)\,x^4 + (7/8)\,x^2 - 7/64$
15	60	$x^8 - (7/4)\,x^6 + (7/8)\,x^4 - (1/8)\,x^2 + 1/256$
16	16	$x^4 - x^2 + 1/8$
17	68	$x^{16} - (17/4)\,x^{14} + (119/16)\,x^{12} - (221/32)\,x^{10} + (935/256)\,x^8 - (561/512)\,x^6 + (357/2048)\,x^4$ $-(51/4096)\,x^2 + 17/65536$
18	36	$x^6 - (3/2)\,x^4 + (9/16)\,x^2 - 3/64$
19	76	$x^{18} - (19/4)\,x^{16} + (19/2)\,x^{14} - (665/64)\,x^{12} + (1729/256)\,x^{10} - (2717/1024)\,x^8 + (627/1024)\,x^6$ $-(627/8192)\,x^4 + (285/65536)\,x^2 - 19/262144$
20	5	$x^2 + (1/2)\,x - 1/4$
21	84	$x^{12} - (11/4)\,x^{10} + (11/4)\,x^8 - (39/32)\,x^6 + (15/64)\,x^4 - (1/64)\,x^2 + 1/4096$
22	44	$x^{10} - (11/4)\,x^8 + (11/4)\,x^6 - (77/64)\,x^4 + (55/256)\,x^2 - 11/1024$
23	92	$x^{22} - (23/4)\,x^{20} + (115/8)\,x^{18} - (1311/64)\,x^{16} + (1173/64)\,x^{14} - (2737/256)\,x^{12} +$ $+(2093/512)\,x^{10} - (16445/16384)\,x^8 + (9867/65536)\,x^6 - (3289/262144)\,x^4 + (253/524288)\,x^2$ $-23/4194304$
24	24	$x^4 - x^2 + 1/16$
25	100	$x^{20} - 5\,x^{18} + (85/8)\,x^{16} - (25/2)\,x^{14} + (2275/256)\,x^{12} - (4005/1024)\,x^{10} + (1075/1024)\,x^8$ $-(2675/16384)\,x^6 + (875/65536)\,x^4 - (125/262144)\,x^2 + 5/1048576$
26	52	$x^{12} - (13/4)\,x^{10} + (65/16)\,x^8 - (39/16)\,x^6 + (91/128)\,x^4 - (91/1024)\,x^2 + 13/4096$
27	108	$x^{18} - (9/2)\,x^{16} + (135/16)\,x^{14} - (273/32)\,x^{12} + (1287/256)\,x^{10} - (891/512)\,x^8 + (693/2048)\,x^6$ $-(135/4096)\,x^4 + (81/65536)\,x^2 - 3/262144$
28	14	$x^3 - (1/2)\,x^2 - (1/2)\,x + 1/8$
29	116	$x^{28} - (29/4)\,x^{26} + (377/16)\,x^{24} - (725/16)\,x^{22} + (7337/128)\,x^{20} - (51359/1024)\,x^{18} +$ $+(127281/4096)\,x^{16} - (28101/2048)\,x^{14} + (140505/32768)\,x^{12} - (121771/131072)\,x^{10} +$ $(70499/524288)\,x^8 - (6409/524288)\,x^6 + (2639/4194304)\,x^4 - (1015/67108864)\,x^2 +$ $+29/268435456$
30	30	$x^8 - (7/4)\,x^6 + (7/8)\,x^4 - (1/8)\,x^2 + 1/256$
⋮		

Table 2: **A181872(n, m) array for numerators of coefficients of minimal polynomials of $\sin\left(\frac{2\pi}{n}\right)$**

n/m	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	...
1	0	1										
2	0	1										
3	-3	0	1									
4	-1	1										
5	5	0	-5	0	1							
6	-3	0	1									
7	-7	0	7	0	-7	0	1					
8	-1	0	1									
9	-3	0	9	0	-3	0	1					
10	5	0	-5	0	1							
11	-11	0	55	0	-77	0	11	0	-11	0	1	
⋮												

Table 3: A181873(n, m) array for denominators of coefficients of minimal polynomials of $\sin\left(\frac{2\pi}{n}\right)$

n/m	0	1	2	3	4	5	6	7	8	9	10	...
1	1	1										
2	1	1										
3	4	1	1									
4	1	1										
5	16	1	4	1	1							
6	4	1	1									
7	64	1	8	1	4	1	1					
8	2	1	1									
9	64	1	16	1	2	1	1	1				
10	16	1	4	1	1							
11	1024	1	256	1	64	1	4	1	4	1	1	
⋮												

Table 4: A181872$(\mathbf{n},\mathbf{m})$/A181873$(\mathbf{n},\mathbf{m})$ array for coefficients of minimal polynomials of sin $\left(\frac{2\,\pi}{\mathbf{n}}\right)$

n/m	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	...
1	0	1										
2	0	1										
3	-3/4	0	1									
4	-1	1										
5	5/16	0	-5/4	0	1							
6	-3/4	0	1									
7	-7/64	0	7/8	0	-7/4	0	1					
8	-1/2	0	1									
9	-3/ 64	0	9/16	0	-3/2	0	1					
10	5/16	0	-5/4	0	1							
11	-11/1024	0	55/256	0	-77/64	0	11/4	0	-11/4	0	1	
⋮												

Table 5: A181871 array for integer coefficients of minimal polynomials of $\pi(n,x) := 2^{\delta(n)}\,\Pi(n,x)$

n/m	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	...
1	0	2										
2	0	2										
3	-3	0	4									
4	-2	2										
5	5	0	-20	0	16							
6	-3	0	4									
7	-7	0	56	0	-112	0	64					
8	-2	0	4									
9	-3	0	36	0	-96	0	64					
10	5	0	-20	0	16							
11	-11	0	220	0	-1232	0	2816	0	-2816	0	1024	
⋮												