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A181878, Coefficient array for square of Chebyshev S-polynomials

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The ordinary generating function (*o.g.f.*) for the square of the *Chebyshev* S-polynomials A049310 is found from the identity [?], p. 261, second formula with $U_n\left(\frac{x}{2}\right) = S_n(x)$ and $n \to n+1$

$$(S_n(x))^2 = \frac{1}{2\,(1-(\frac{x}{2})^2)}\left(1 - T_{2\,(n+1)}\left(\frac{x}{2}\right)\right)\,, \tag{1}$$

with the *Chebyshev* T-polynomials A053120. This identity is proved by employing the *de Moivre-Binet* formulae for the S- and T- polynomials for $x = q + q^{-1}$, *viz*

$$S_{n-1}(q + q^{-1}) = \frac{q^n - q^{-n}}{q - q^{-1}} \quad , \quad T_n\left(\frac{q + q^{-1}}{2}\right) = \frac{1}{2}\,(q^n + q^{-n})\,, \tag{2}$$

after some simple algebra.
From the known *o.g.f.* (*e.g.* from [?], p. 259 with $T_0 = 1$ or [?], eq. (1.105), p. 41)

$$T(x,z) := \sum_{n=0}^{\infty} T_n\left(\frac{x}{2}\right) z^n = \frac{1 - z\,\frac{x}{2}}{1 - x\,z + z^2}\,, \tag{3}$$

one finds from one part of the bisection (remember that $G_{even}(x) = \frac{1}{2}\,(G(\sqrt{x}\,) + G(-\sqrt{x}\,))$ for the *o.g.f.*s)

$$\sum_{n=0}^{\infty} T_{2(n+1)}\left(\frac{x}{2}\right) z^n = \frac{-1 + \frac{x^2}{2} - z}{(1+z)^2 - x^2\,z}. \tag{4}$$

Therefore,

$$S(x,z) := \sum_{n=0}^{\infty} S_n^2(x) z^n = \frac{1}{2\,(1-(\frac{x}{2})^2)}\left(\frac{1}{1-z} - \frac{-(1+z) + \frac{x^2}{2}}{(1+z)^2 - x^2\,z}\right) \tag{5}$$

$$= \frac{1+z}{1-z}\,\frac{1}{1 + (2 - x^2)\,z + z^2}\,. \tag{6}$$

This produces, for example, the correct *o.g.f.* for the square of the even indexed *Fibonacci* numbers if one puts $x = 3$, using $S_n(3) = F_{2\,(n+1)}$. See A049684. Other *o.g.f.* instances follow by putting $x = 0$: A000035; $x = +1$ or $x = -1$: A011655; $x = 2$: A000290; $x = i$: A007598 (squared Fibonacci); $x = \sqrt{2}$: A007877$(n+1)$ and $x = 6$: A001110.
Because of the denominator factor $1 - z$ the polynomials $S_n^2(x)$ generated by $S(x,z)$ are the partial sums of the polynomials $s_n(x)$ generated by $s(x,z) := \frac{1+z}{1 + (2 - x^2)\,z + z^2}$. The coefficients of these $s_n(x)$ polynomials in x^2 constitute the *Riordan* triangle $\left(\frac{1}{1+x}, \frac{x}{(1+x)^2}\right)$ A129818(n,k). This means that

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the *o.g.f.* for the sequence in the column with number k (with leading zeros) is $\frac{1}{1+x}\left(\frac{x}{(1+x)^2}\right)^k$, $0 \leq k \leq n$.

The present coefficient array A181878(n,m) is obtained for even row numbers $n = 2k$ from $[x^{2m}]\, S_{2k}^2(x)$ (this notation stands for the coefficient of x^{2m} of the $S_{2k}^2(x)$ polynomial of degree $2k$), and for odd row numbers $n = 2k+1$ from $[x^{2m}]\, \frac{S_{2k+1}^2(x)}{x^2}$. This produces the array shown in table 1. The row length sequence is A109613, the repeated odd numbers. If one tabulates $[x^{2m}]\, S_n^2(x)$ one will find the number triangle A158454(n,k), the *Riordan* array $\left(\frac{1}{1-x^2}, \frac{x}{(1+x)^2}\right)$, shown in table 2, which means that the *o.g.f.* for the sequence in the column with number k (with leading zeros) is $\frac{1}{1-x^2}\left(\frac{x}{(1+x)^2}\right)^k$, $0 \leq k \leq n$. The *o.g.f.* $S(x,z)$ for the row polynomials in the variable x^2 coincides indeed with the one obtained for this *Riordan* triangle. From the (ordinary) convolution of the two sequences generated by $\frac{1}{1-x^2}$ and $\left(\frac{x}{(1+x)^2}\right)^k$ one finds the exlicit sum form

$$\text{A158454}(n,k) = (-1)^{n-k} \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n+k-1-2j}{2k-1}, \quad 0 \leq k \leq n. \tag{7}$$

The present array has the odd numbered rows shifted by one entry to the left due to the division by x^2. The *o.g.f.*s for the polynomials of the even and odd numbered rows are found after bisection of the *o.g.f* for $S_n(x^2)$ given above. For the even row numbers one has

$$\sum_{k=0}^{\infty} S_{2k}^2(\sqrt{x}\,)\, z^n = \frac{1}{2}\left(S(\sqrt{x}, \sqrt{z}\,) + S(\sqrt{x}, -\sqrt{z}\,)\right). \tag{8}$$

The result is

$$\sum_{k=0}^{\infty} S_{2k}^2(\sqrt{x}\,)\, z^n = \frac{1 - 2\,(1-x)\,z + z^2}{(1-z)\,(1-(2-4\,x+x^2)\,z+z^2)}\,. \tag{9}$$

Similarly, for the odd numbered rows with

$$\frac{1}{x}\sum_{k=0}^{\infty} S_{2k+1}^2(\sqrt{x}\,)\, z^n = \frac{1}{2\,x\,\sqrt{z}}\left(S(\sqrt{x}, \sqrt{z}\,) - S(\sqrt{x}, -\sqrt{z}\,)\right), \tag{10}$$

one finds

$$\frac{1}{x}\sum_{k=0}^{\infty} S_{2k+1}^2(\sqrt{x}\,)\, z^n = \frac{1+z}{1-z}\,\frac{1}{1-(2-4\,x+x^2)\,z+z^2}\,. \tag{11}$$

From the well known explicit form of the coefficients of *Chebyshev*'s $T-$polynomials A053120, see *e.g.* [?], eq. (1.96), p. 37 and p. 42 (with a misprint in the binomial identity in the middle of the page: on the *r.h.s.* the factor should read $\frac{k}{n-k}$, not the reciprocal of this), *viz*

$$T_n\left(\frac{x}{2}\right) = \frac{1}{2}\sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j\, \frac{n}{n-j}\binom{n-j}{j}\, x^{n-2j}, \tag{12}$$

one can compute from eq. (1) the (ordinary) convolution implied by the *r.h.s.*, to find the coefficients of x^{2m} of $S_{2n}^2(x)$, as well as of $\frac{S_{2n+1}^2(x)}{x^2}$. The coefficients of the first factor in eq. (1) are in both cases

$a_n = \frac{1}{2^{2n+1}}$, and in the first computation the second factor yields the coefficients

$$b_{2n,l} = \begin{cases} 2 & \text{if l=0 ,} \\ -(-1)^{l-1} \frac{2n+1}{2n+1+l} \binom{2n+1+l}{2n+1-l} & \text{if l=1,2,...,2n+1 .} \end{cases} \tag{13}$$

Convolution of these sequences leads to the result for the even row numbers $a(2n,m)$:= A181878$(2n,m)$

$$a(2n,m) = \frac{1}{2^{2m}} + \sum_{k=0}^{m-1} (-1)^{m-k} \frac{1}{2^{2k+1}} \frac{2n+1}{2n+1+m-k} \binom{2n+1+m-k}{2n+1-(m-k)} . \tag{14}$$

From the *Riordan* property of A158454 these coefficients have already been computed with the result

$$a(2n,m) = (-1)^m \sum_{j=0}^{n} \binom{2n+m-1-2j}{2m-1} . \tag{15}$$

Thus these two expressions have to coincide, providing a binomial identity.

In the computation of the odd numbered rows one uses the coefficients of the second factor in eq. (1)

$$b(2n+1,m) = \begin{cases} 0 & \text{if l=0 ,} \\ -(-1)^{l} \frac{2(n+1)}{2(n+1)+l} \binom{2(n+1)+l}{2(n+1)-l} & \text{if l=1,2,...,2(n+1) .} \end{cases} \tag{16}$$

Convolution of the sequences leads to the result for the odd numbered rows $a(2n+1,m)$:= A181878$(2n+1,m)$

$$a(2n+1,m) = \sum_{k=0}^{m} (-1)^{m-k} \frac{1}{2^{2k+1}} \frac{2(n+1)}{2(n+1)+m+1-k} \binom{2(n+1)+m+1-k}{2(n+1)-(m+1-k)} . \tag{17}$$

From the *Riordan* property of A158454 these coefficients have already been computed, and after taking the left shift by one entry into account one has

$$a(2n+1,m) = (-1)^m \sum_{j=0}^{n} \binom{2n+1+m-2j}{2m+1} . \tag{18}$$

Thus the two sums have to coincide, establishing yet another binomial identity.

The row sums of the present array coincide with those of the *Riordan* triangle A158454. Therefore their *o.g.f.* is $\frac{1+x}{1-x} \frac{1}{1-x-x^2} = \frac{1+x}{1-x^3}$, the period length three sequence $\overline{[1,1,0]}$, A011655$(n+1), n \geq 1$ (see the above above $S(\pm 1, z)$ comment). The alternating row sums of A158454 are generated by $\frac{1+x}{1-x} \frac{1}{1+3x-x^2}$, A007598$(n+1)\,(-1)^n$, $n \geq 0$. For the even row numbers of the present array these alternating sums coincide. For the odd numbered row $n = 2k+1$ one has the alternating sums A049684$(k+1), k \geq 0$ (squares of even indexed *Fibonacci* numbers).

For a *Riordan* triangle the concept of the so-called $A-$ and $Z-$ sequence is of interest in order to find recurrence relations (which may sometimes turn out not to be the simplest one, however). See the W. Lang link "Sheffer a- and z- sequences" under A006232, where also references are given. To find these sequences the compositional inverse of F of a *Riordan* triangle $G(x), F(x) = x\,\hat{F}(x)$ is needed. In this case A158454 one finds $F^{[-1]}(y) = -1 + c(x) = y\,c(y)^2$, with the *o.g.f.* $c(y)$ of the *Catalan* numbers A000108. Then the $A-$sequence is generated by $\frac{1}{c(y)^2} = 1 - y\,(1 + c(y))$. This is the sequence A115141 $= [1, -2, -1, -2, -5, -14, -42, \ldots]$. The $Z-$ sequence turns out to be generated by $F^{[-1]}(y) = -1 + c(y)$. This the *Catalan* sequence with a leading 0. The recurrence for the entries in column $m = 0$ of the number triangle A158454 is then given by

$$T(n,0) \;=\; \sum_{j=0}^{n-1} Z(j)\,T(n-1,j)\;,\quad n\,\geq\,1\,. \tag{19}$$

E.g., $1 \,=\, T(6,0) \,=\, 0\cdot 0 \,+\, 1\cdot 9 \,+\, 2\cdot(-24) \,+\, 5\cdot 22 \,+\, 14\cdot(-8) \,+\, 42\cdot 1$.
For the other columns one has

$$T(n,m) \;=\; \sum_{j=0}^{n-m} A(j)\,T(n-1,m-1+j)\;,\quad 1\,\leq\,m\,\leq\,n\,. \tag{20}$$

E.g.,
$-80 \,=\, T(7,2) \,=\, 1\cdot(-12) + (-2)\cdot 46 + (-1)\cdot(-62) + (-2)\cdot 37 + (-14)\cdot(-10) + (-5)\cdot(-10) + (-14)\cdot 1$
.
These recurrences can be reformulated for the present array $a(n,m)$ as

$$a(2p,0) \;=\; \sum_{j=0}^{2(p-1)} Z(j+1)\,a(2p-1,j)\;,\quad p\,\geq\,1\,, \tag{21}$$

$$a(2p+1,0) \;=\; \sum_{j=0}^{2p} A(j)\,a(2p,j)\;,\quad p\,\geq\,0\,. \tag{22}$$

and for $m\,\geq\,1$ (putting $a(n,-1)\,=\,0$)

$$a(2p,m) \;=\; \sum_{j=0}^{2p-m} A(j)\,a(2p-1,m-2+j)\;,\quad p\,\geq\,1\,, \tag{23}$$

$$a(2p+1,m) \;=\; \sum_{j=0}^{2p-m} A(j)\,a(2p,m+j)\;,\quad p\,\geq\,1\,. \tag{24}$$

References

[1] W. Magnus, F. Oberhettinger and R. P. Soni, Formulas and theorems for the special functions of mathematical physics, 3rd edition, Springer, Berlin, 1966.

[2] Th. J. Rivlin, Chebyshev Polynomials, 2nd ed., John Wiley & Sons, New York, 1990.

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Concerned with OEIS sequences A000035, A000290, A000108, A001110, A007598, A007877, A011655, A049310, A049684, A053120, A115141, A129818, A158454, A181878 .

Table 1: Riordan triangle $\left(\frac{1}{1-x^2}, \frac{x}{(1+x)^2}\right)$ A158454(n, m)**, Paul Barry**

n/m	0	1	2	3	4	5	6	7	8	9	...
0	1										
1	0	1									
2	1	-2	1								
3	0	4	-4	1							
4	1	-6	11	-6	1						
5	0	9	-24	22	-8	1					
6	1	-12	46	-62	37	-10	1				
7	0	16	-80	148	-128	56	-12	1			
8	1	-20	130	-314	367	-230	79	-14	1		
9	0	25	-200	610	-920	771	-376	106	-16	1	
⋮											

Table 2: A181878(n,m) array

n/m	0	1	2	3	4	5	6	7	8	9	10...
0	1										
1	1										
2	1	-2	1								
3	4	-4	1								
4	1	-6	11	-6	1						
5	9	-24	22	-8	1						
6	1	-12	46	-62	37	-10	1				
7	16	-80	148	-128	56	-12	1				
8	1	-20	130	-314	367	-230	79	-14	1		
9	25	-200	610	-920	771	-376	106	-16	1		
10	1	-30	295	-1106	2083	-2232	1444	-574	137	-18	1
⋮											