Wolfdieter Lang, Jun 25 2012
A212359 partition (tabf) array: coefficients of the representative color multinomial corresponding to the k-th partition of n in Abramowitz-Stegun (A-St) order. See Abramowitz-Stegun, pp. 831-2 (this order has been used by C. F. Hindenburg in 1779; see a comment on A036036, where also the reference is given.)

n\k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	29	21	22
1	1																					
2	1	1																				
3	1	1	2																			
4	1	1	2	3	6																	
5	1	1	2	4	6	12	24															
6	1	1	3	4	5	10	16	20	30	60	120											
7	1	1	3	5	6	15	20	30	30	60	90	120	180	360	720							
8	1	1	4	7	10	7	21	35	54	70	42	105	140	210	318	210	420	630	840	1260	2520	5040

...

The rows from n=9 to n=15 are:
n=9: 1, 1, 4, 10, 14, 8, 28, 56, 70, 84, 140, 188, 56, 168, 280, 420, 560, 840, 336, 840, 1120, 1680, 2520, 1680, 3360, 5040, 6720, 10080, 20160, 40320.

n=10: 1, 1, 5, 12, 22, 26, 9, 36, 84, 126, 128, 252, 318, 420, 72, 252, 504, 630, 756, 1260, 1680, 1896, 2520, 504, 1512, 2520, 3780, 5040, 7560, 11352, 3024, 7560, 10080, 15120, 22680, 15120, 30240, 45360, 60480, 90720, 181440, 362880.

n=11: 1, 1, 5, 15, 30, 42, 10, 45, 120, 210, 252, 180, 420, 630, 840, 1050, 90, 360, 840, 1260, 1260, 2520, 3150, 4200, 3780, 6300, 8400, 720, 2520, 5040, 6300, 7560, 12600, 16800, 18900, 25200, 37800, 5040, 15120, 25200, 37800, 50400, 75600, 113400, 30240, 75600, 100800, 151200, 226800, 151200, 302400, 453600, 604800, 907200, 1814400, 3628800.

n=12: 1, 1, 6, 19, 43, 66, 80, 11, 55, 165, 330, 462, 250, 660, 1160, 1386, 1542, 2310, 2896, 110, 495, 1320, 2310, 2772, 1980, 4620, 6930, 9240, 11550, 6940, 13860, 17340, 23100, 30804, 990, 3960, 9240, 13860, 13860, 27720, 34650, 46200, 41580, 69300, 92400, 103980, 138600, 7920, 27720, 55440, 69300, 83160, 138600, 184800, 207900, 277200, 415800, 623760, 55440, 166320, 277200, 415800, 554400, 831600, 1247400, 332640, 831600, 1108800, 1663200, 2494800, 1663200, 3326400, 4989600, 6652800, 9979200, 19958400, 39916800.

n=13: 1, 1, 6, 22, 55, 99, 132, 12, 66, 220, 495, 792, 924, 330, 990, 1980, 2772, 2640, 4620, 5544, 6930, 132, 660, 1980, 3960, 5544, 2970, 7920, 13860, 16632, 18480, 27720, 34650, 11880, 27720, 41580, 55440, 69300, 92400, 1320, 5940, 15840, 27720, 33264, 23760, 55440, 83160, 110880, 138600, 83160, 166320, 207900, 277200, 369600, 249480, 415800, 554400, 11880, 47520, 110880, 166320, 166320, 332640, 415800, 554400, 498960, 831600, 1108800, 1247400, 1663200, 2494800, 95040, 332640, 665280, 831600, 997920, 1663200, 2217600, 2494800, 3326400, 4989600, 7484400, 665280, 1995840, 3326400, 4989600, 6652800, 9979200, 14968800, 3991680, 9979200, 13305600, 19958400, 29937600, 19958400, 39916800, 59875200, 79833600, 119750400, 239500800, 479001600.

n=14: 1, 1, 7, 26, 73, 143, 217, 246, 13, 78, 286, 715, 1287, 1716, 432, 1430, 3225, 5148, 6016, 4290, 8580, 12012, 15030, 18018, 156, 858, 2860, 6435, 10296, 12012, 4290, 12870, 25740, 36036, 34320, 60060, 72072, 90090, 19320, 51480, 90120, 108108, 120120, 180180, 225270, 240240, 300300, 1716, 8580, 25740, 51480, 72072, 38610, 102960, 180180, 216216, 240240, 360360, 450450, 154440, 360360, 540540, 720720, 900900, 1201200, 540600, 1081080, 1351440, 1801800, 2402400, 17160, 77220, 205920, 360360, 432432, 308880, 720720, 1081080, 1441440, 1801800, 1081080, 2162160, 2702700, 3603600, 4804800, 3243240, 5405400, 7207200, 8108280, 10810800,

154440, 617760, 1441440, 2162160, 2162160, 4324320, 5405400, 7207200, 6486480, 10810800, 14414400, 16216200, 21621600, 32432400, 48648960, 1235520, 4324320, 8648640, 10810800, 12972960, 21621600, 28828800, 32432400, 43243200, 64864800, 97297200, 8648640, 25945920, 43243200, 64864800, 86486400, 129729600, 194594400, 51891840, 129729600, 172972800, 259459200, 389188800, 259459200, 518918400, 778377600, 1037836800, 1556755200, 3113510400, 6227020800.

n=15: 1, 1, 7, 31, 91, 201, 335, 429, 14, 91, 364, 1001, 2002, 3003, 3432, 546, 2002, 5005, 9009, 12012, 6676, 15015, 24024, 28032, 30030, 42042, 50452, 182, 1092, 4004, 10010, 18018, 24024, 6006, 20020, 45045, 72072, 84084, 60060, 120120, 168168, 210210, 252252, 30030, 90090, 180180, 252252, 240240, 420420, 504504, 630630, 560568, 840840, 1051050, 2184, 12012, 40040, 90090, 144144, 168168, 60060, 180180, 360360, 504504, 480480, 840840, 1009008, 1261260, 270270, 720720, 1261260, 1513512, 1681680, 2522520, 3153150, 3363360, 4204200, 1081080, 2522520, 3783780, 5045040, 6306300, 8408400, 11211216, 24024, 120120, 360360, 720720, 1009008, 540540, 1441440, 2522520, 3027024, 3363360, 5045040, 6306300, 2162160, 5045040, 7567560, 10090080, 12612600, 16816800, 7567560, 15135120, 18918900, 25225200, 33633600, 22702680, 37837800, 50450400, 240240, 1081080, 2882880, 5045040, 6054048, 4324320, 10090080, 15135120, 20180160, 25225200, 15135120, 30270240, 37837800, 50450400, 67267200, 45405360, 75675600, 100900800, 113513400, 151351200, 227026800, 2162160, 8648640, 20180160, 30270240, 30270240, 60540480, 75675600, 100900800, 90810720, 151351200, 201801600, 227026800, 302702400, 454053600, 681080400, 17297280, 60540480, 121080960, 151351200, 181621440, 302702400, 403603200, 454053600, 605404800, 908107200, 1362160800, 121080960, 363242880, 605404800, 908107200, 1210809600, 1816214400, 2724321600, 726485760, 1816214400, 2421619200, 3632428800, 5448643200, 3632428800, 7264857600, 10897286400, 14529715200, 21794572800, 43589145600, 87178291200.

The sequences of row lengths is given by A000041 = [1, 2, 3, 5, 7, 11, 15, 22,...].
The sequence of the row sums is A072605(n)= [1, 2, 4, 13, 50, 270, 1641, 11945, 96784, 887982, 8939051, 99298354, 1195617443 ...].
The sequence of the alternating row sums is [1, 0, 2, 5, 16, 80, 459, -3205, -26046, -234192, -2328495, 25572182, 308215979, 4001133267, -55988511503, 837876658026, 13405435888388, ...]

A note on the 1-1 (bijective) mapping of partitions of n (n-partitions) to n-multiset representatives, used here to find the color assignment representative for an n-necklace. See also the bijective map between the positive integers, given by n → A212361(n) and its inverse A212362.
The j-th partition of n of length m (parts number) is pa(n,m,j), where j is from {1,2,...,p(n,m)} (see the array p(n,m)=A008284(n,m)). This uses a lexicographical ordering like in A-St but with reversed order of the parts. Therefore, pa(6,3,2)= 3,2,1 not 1,2,3 like in A-St. This reversed order is taken in order to use a partition of n as exponents over the set {1,2,..,n} to produce a certain n-multiset. This map of the set Pa of all partitions of n, n=1,2,...,infinity, to the set Ms of all multiset representatives ms(n), n=1,2,...,infinity, is bijective. E.g. the partitions of n=4 (in A-St order), 4; 3,1; 2^2; 2,1^2; 1^4 are mapped to the five 4-multiset representatives denoted by 1^4; 1^3,2^1; 1^2,2^2; 1^2,2^1,3^1; 1^1,2^1,3^1,4^1, respectively. They stand for the 4-multiset representatives {1,1,1,1}, {1,1,1,2}, {1,1,2,2}, {1,1,2,3}, {1,2,3,4}, respectively. These 4-multisets will appear in the list of all multiset representatives Ms, arranged in A-St order (note that here the partitions are written in non-decreasing form like in A-St, not in reversed form) at certain places. E.g., The multiset {1,1,2,3}, denoted by 1^2,2^1,3^1, is a partition of N=7, and it appears as the first of 4 such multiset representatives which are partitions of N=7 (the other three are, in order, 1^3,2^2; 1^5,2^1 and 1^7), namely for m=n=4 (number of parts or length) as second partition, i.e., 1^2,2,3 = pa(7,4,2). This exponentiation of partitions defines a bijective map e: Pa → Ms, pa(k) → ms(e(k)) , with the k-th partition in A-St order pa(k), k=1,2,...,infinity and the l-th multiset representative in A-St

order ms(l), l=1,2,..,infinity. This map induces a permutation of the positive integers: e: N → N, k-> e(k), given in A212361. The inverse map e^{-1} is given in A212362.
The formula for a(n,k), the necklace partition numbers, is based on Z(C_n,c_n), with the cycle index polynomial of the cyclic group C_n and the figure counting polynomial c_n:=sum(c[i],i=1..n). The formula for Z(C_n) is given, e.g., in the Harary-Palmer reference on p.36, Theorem, eq. (2.2.10). We use here variables x[j] instead of their s_j. See also p.42, Theorem (PET= P\'olya's enumeration theorem) for the abbreviation Z(C_n,c), which stands for the replacement of the Z(C_n) variables x[j], j=1,2,...,n, by the j-th power sum sum(c[i]^j,i=1..n). See also a link under A212357 for the cycle indices Z(C_n), n=1..15. One has to compute in Z(C_n,c_n) the coefficient of the multinomial corresponding to the multiset representative belonging to a partition of n, specifically from pa(n,m,j) for m=1,2,...,n and j=1,...,p(n,m) (see the preceding note). Call the corresponding multiset representative ms(n,m,j)=e(pa(n,m,j) (in an abuse of notation for the map e). E.g., the multinomial in Z(C_n,c_n) corresponding to the 4-multiset representative denoted by 1^2,2,3, obtained, after exponentiation, from the 4-partition 2,1^2 = 2,1,1, is c[1]^2*c[2]*c[3]. This example should suffice to explain the general prescription. For the computation of sum(c[i]^d,i=1..n)^(n/d), for a given divisor of n, one uses the multinomial numbers called M_1 (called here M1) in A-St, pp. 823 and 831-2. For this partition array see A036038(n,k), k=1,...,p(n)=A000041(n). For each d|n (see the array A027750 for all of the A000005(n) divisors of n) all partitions of nhat:=n/d are scanned in order to find out whether pa(n,m,j) = d*pa(nhat,mhat,jhat), where the multiplication with d indicates that all parts are multiplied by d (e.g. 4*(1^2,2,3) = 4^2,8,12. Here the order of a partition is not relevant, one could also take the reversed A-St form mentioned above). If this eq. holds, the numbers mhat and jhat are read off. The contribution to the multinomial coefficient in Z(C_n,c_n) arising from pa(n,m,j), for given d|n, is then N(n,m,j,d):=phi(d)*M1(nhat,mhat,jhat)/n with Euler's totient function phi(d)=A000010(d). This is then summed over all the divisors d of n: N(n,m,j):=sum(N(n,m,j,d),d|n), and finally, for each n>=1, a coefficient list Nlist(n,m):= [seq(N(n,m,j),j=1..p(n,m)=A008284(n,m))] is obtained.
The list of lists Nlist(n):=[seq(Nlist(n,m),m=1..n)] can be flattened to obtain row nr. n of the present array a(n,k), k=1,...,p(n)=A000041(n), because the partitions have been taken in A-St order. Each multiset ms(n,m,j) representative, obtained from a partition pa(n,m,j) via the above described exponentiation, which determines a specific multinomial in Z(C_n,c_n), stands for a whole equivalence class of n-necklaces with different color choices but the same color type or signature. The order of these equivalence classes are shown in the partition array A035206(n,k), k=1,2,...,p(n)=A000041(n) (in A-st order). E.g., the multiset representative {1,1,2,3}, obtained from the partition 2,1,1 of n=4, encodes the color monomial c[1]^2*c[2]*c[3] which represents an equivalence class of 12 members, namely (in short notation c[j]-> j) 112, 113, 114, 221, 223, 224, 331, 332, 334, 441, 442 and 443. This order 12 = A035206(4,k=4), because the signature (the set of exponents) is provided by the partition 2,1,1, the 4-th in the A-St ordered list of the n=4 partitions.
Therefore, the weighted row sum sum(A03506(n,k)*a(n,k),k=1..p(n)=A000041(n)) is the total number of necklaces with n beads, the colors taken from a repertoire of n distinct colors. These numbers are [1, 3, 11, 70, 629, 7826, 117655, 2097684, 43046889, 1000010044,...]. This is the sequence A056665.
The partition array with the numbers A03506(n,k)*a(n,k) can be found as A212360(n,k). Its row sums yield A056665(n), n>=1.

The color representative polynomials for n=1..10 are (note that these are not the partition polynomials: the partitions in A-St order appear here as exponents):

n=1: c[1].

n=2: c[1]^2 + c[1]*c[2].

n=3: c[1]^3 + c[1]^2*c[2] + 2*c[1]*c[2]*c[3].

n=4: c[1]^4 + c[1]^3*c[2] + 2*c[1]^2*c[2]^2 + 3*c[1]^2*c[2]*c[3] +
6*c[1]*c[2]*c[3]*c[4].

n=5: c[1]^5 + c[1]^4*c[2] + 2*c[1]^3*c[2]^2 + 4*c[1]^3*c[2]*c[3] +
6*c[1]^2*c[2]^2*c[3] + 12*c[1]^2*c[2]*c[3]*c[4] + 24*c[1]*c[2]*c[3]*c[4]*c[5].

n=6: c[1]^6 + c[1]^5*c[2] + 3*c[1]^4*c[2]^2 + 4*c[1]^3*c[2]^3 + 5*c[1]^4*c[2]*c[3]
+ 10*c[1]^3*c[2]^2*c[3] + 16*c[1]^2*c[2]^2*c[3]^2 + 20*c[1]^3*c[2]*c[3]*c[4] +
30*c[1]^2*c[2]^2*c[3]*c[4] + 60*c[1]^2*c[2]*c[3]*c[4]*c[5] +
120*c[1]*c[2]^2*c[3]*c[4]*c[5]*c[6].

n=7: c[1]^7 + c[1]^6*c[2] + 3*c[1]^5*c[2]^2 + 5*c[1]^4*c[2]^3 + 6*c[1]^5*c[2]*c[3]
+ 15*c[1]^4*c[2]^2*c[3] + 20*c[1]^3*c[2]^3*c[3] + 30*c[1]^3*c[2]^2*c[3]^2,
30*c[1]^4*c[2]*c[3]*c[4] + 60*c[1]^3*c[2]^2*c[3]*c[4] +
90*c[1]^2*c[2]^2*c[3]^2*c[4] + 120*c[1]^3*c[2]*c[3]*c[4]*c[5] +
180*c[1]^2*c[2]^2*c[3]*c[4]*c[5] + 360*c[1]^2*c[2]*c[3]*c[4]*c[5]*c[6] +
720*c[1]*c[2]*c[3]*c[4]*c[5]*c[6]*c[7].

n=8: c[1]^8 + c[1]^7*c[2] + 4*c[1]^6*c[2]^2 + 7*c[1]^5*c[2]^3 +10*c[1]^4*c[2]^4 +
7*c[1]^6*c[2]*c[3] + 21*c[1]^5*c[2]^2*c[3] + 35*c[1]^4*c[2]^3*c[3] +
54*c[1]^4*c[2]^2*c[3]^2 + 70*c[1]^3*c[2]^3*c[3]^2 + 42*c[1]^5*c[2]*c[3]*c[4]
+105*c[1]^4*c[2]^2*c[3]*c[4] + 140*c[1]^3*c[2]^3*c[3]*c[4] +
210*c[1]^3*c[2]^2*c[3]^2*c[4] + 318*c[1]^2*c[2]^2*c[3]^2*c[4]^2 +
210*c[1]^4*c[2]*c[3]*c[4]*c[5] + 420*c[1]^3*c[2]^2*c[3]*c[4]*c[5]
+630*c[1]^2*c[2]^2*c[3]^2*c[4]*c[5] + 840*c[1]^3*c[2]*c[3]*c[4]*c[5]*c[6] +
1260*c[1]^2*c[2]^2*c[3]*c[4]*c[5]*c[6] + 2520*c[1]^2*c[2]*c[3]*c[4]*c[5]*c[6]*c[7]
+ 5040*c[1]*c[2]*c[3]*c[4]*c[5]*c[6]*c[7]*c[8].

n=9: c[1]^9 + c[1]^8*c[2] + 4*c[1]^7*c[2]^2 + 10*c[1]^6*c[2]^3 + 14*c[1]^5*c[2]^4 +
8*c[1]^7*c[2]*c[3] + 28*c[1]^6*c[2]^2*c[3] + 56*c[1]^5*c[2]^3*c[3] +
70*c[1]^4*c[2]^4*c[3] + 84*c[1]^5*c[2]^2*c[3]^2 + 140*c[1]^4*c[2]^3*c[3]^2 +
188*c[1]^3*c[2]^3*c[3]^3 + 56*c[1]^6*c[2]*c[3]*c[4] + 168*c[1]^5*c[2]^2*c[3]*c[4] +
280*c[1]^4*c[2]^3*c[3]*c[4] + 420*c[1]^4*c[2]^2*c[3]^2*c[4] +
560*c[1]^3*c[2]^3*c[3]^2*c[4] + 840*c[1]^3*c[2]^2*c[3]^2*c[4]^2 +
336*c[1]^5*c[2]*c[3]*c[4]*c[5] + 840*c[1]^4*c[2]^2*c[3]*c[4]*c[5] +
1120*c[1]^3*c[2]^3*c[3]*c[4]*c[5] + 1680*c[1]^3*c[2]^2*c[3]^2*c[4]*c[5] +
2520*c[1]^2*c[2]^2*c[3]^2*c[4]^2*c[5] + 1680*c[1]^4*c[2]*c[3]*c[4]*c[5]*c[6] +
3360*c[1]^3*c[2]^2*c[3]*c[4]*c[5]*c[6] + 5040*c[1]^2*c[2]^2*c[3]^2*c[4]*c[5]*c[6] +
6720*c[1]^3*c[2]*c[3]*c[4]*c[5]*c[6]*c[7] +
10080*c[1]^2*c[2]^2*c[3]*c[4]*c[5]*c[6]*c[7] +
20160*c[1]^2*c[2]*c[3]*c[4]*c[5]*c[6]*c[7]*c[8] +
40320*c[1]*c[2]*c[3]*c[4]*c[5]*c[6]*c[7]*c[8]*c[9].

n=10: c[1]^10 + c[1]^9*c[2] + 5*c[1]^8*c[2]^2 + 12*c[1]^7*c[2]^3 + 22*c[1]^6*c[2]^4
+ 26*c[1]^5*c[2]^5 + 9*c[1]^8*c[2]*c[3] + 36*c[1]^7*c[2]^2*c[3] +
84*c[1]^6*c[2]^3*c[3] + 126*c[1]^5*c[2]^4*c[3] + 128*c[1]^6*c[2]^2*c[3]^2 +
252*c[1]^5*c[2]^3*c[3]^2 + 318*c[1]^4*c[2]^4*c[3]^2 + 420*c[1]^4*c[2]^3*c[3]^3 +
72*c[1]^7*c[2]*c[3]*c[4] + 252*c[1]^6*c[2]^2*c[3]*c[4] +
504*c[1]^5*c[2]^3*c[3]*c[4] + 630*c[1]^4*c[2]^4*c[3]*c[4] +
756*c[1]^5*c[2]^2*c[3]^2*c[4] + 1260*c[1]^4*c[2]^3*c[3]^2*c[4] +
1680*c[1]^3*c[2]^3*c[3]^3*c[4] + 1896*c[1]^4*c[2]^2*c[3]^2*c[4]^2 +
2520*c[1]^3*c[2]^3*c[3]^2*c[4]^2 + 504*c[1]^6*c[2]*c[3]*c[4]*c[5] +
1512*c[1]^5*c[2]^2*c[3]*c[4]*c[5] + 2520*c[1]^4*c[2]^3*c[3]*c[4]*c[5] +
3780*c[1]^4*c[2]^2*c[3]^2*c[4]*c[5] + 5040*c[1]^3*c[2]^3*c[3]^2*c[4]*c[5] +
7560*c[1]^3*c[2]^2*c[3]^2*c[4]^2*c[5] + 11352*c[1]^2*c[2]^2*c[3]^2*c[4]^2*c[5]^2 +
3024*c[1]^5*c[2]*c[3]*c[4]*c[5]*c[6] + 7560*c[1]^4*c[2]^2*c[3]*c[4]*c[5]*c[6] +

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10080*c[1]^3*c[2]^3*c[3]*c[4]*c[5]*c[6] + 15120*c[1]^3*c[2]^2*c[3]^2*c[4]*c[5]*c[6]
+ 22680*c[1]^2*c[2]^2*c[3]^2*c[4]^2*c[5]*c[6] +
15120*c[1]^4*c[2]*c[3]*c[4]*c[5]*c[6]*c[7] +
30240*c[1]^3*c[2]^2*c[3]*c[4]*c[5]*c[6]*c[7] +
45360*c[1]^2*c[2]^2*c[3]^2*c[4]*c[5]*c[6]*c[7] +
60480*c[1]^3*c[2]*c[3]*c[4]*c[5]*c[6]*c[7]*c[8] +
90720*c[1]^2*c[2]^2*c[3]*c[4]*c[5]*c[6]*c[7]*c[8] +
181440*c[1]^2*c[2]*c[3]*c[4]*c[5]*c[6]*c[7]*c[8]*c[9] +
362880*c[1]*c[2]*c[3]*c[4]*c[5]*c[6]*c[7]*c[8]*c[9]*c[10].
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Corrections, Jul 19 2012:
Typos in the text. In the color polynomials for n=9 typos, and for n=10 typos and missing terms added.

--------------------------- e.o.f. --