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A215037: Application of the partial summation formula to some sums over cubes of Fibonacci numbers

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The partial summation formula is, e.g. [1], eq. (1.11), p. 8,

$$\sum_{k=0}^{N} a_k\, b_k \;=\; A_N\, b_N \;-\; \sum_{k=0}^{N-1} A_k\,(b_{k+1} \,-\, b_k), \text{with } A_k \;:=\; \sum_{j=0}^{k} a_j \;, \tag{1}$$

identically in $\{a_k\}_0^N$ and $\{b_k\}_0^N$, for $N \,\in\, \mathbb{N}_0$ (as usual, an undefined sum is put to 0). The proof is simple: just collect terms proportional to b_k, for k=0,...,N, use $A(k) \,-\, A(k-1) \;=\; a_k$, and compare both sides of the assertion.

This formula, which looks similar to the partial integration formula (hence its name, which should not be confused with 'partial sum' even though partial sums are used), can be used to derive relations among finite sums over products of sequences, provided the sum A_k with less factors is known. The lower summation index is here 0, but one could use another offset. As a simple first example we (re)derive the well known formula on the sum over the third powers of *Fibonacci* numbers [4], A000045, given in A005968. See also [3], eq. 38., p. 89 (attributed to Rao, 1953, [5], note a misprint in the last of the set of equations in the middle of p. 682: it should be .. + 2, not ... − 2. Use this equation with $U_n \;=\; F_{n+1}$ and the second of the starting identities to arrive at the *Koshy* eq. 38.).

Example 1: $a_k \;=\; F_k^2$, $b_k \;=\; F_k$. Using the known sum $A_k \;=\; \sum_{j=0}^{k} F_j^2 \;=\; F_{k+1}\, F_k$ (This can be proved directly using the F recurrence and a shift of the summation index. See A001654 and [6], Nr. 45, p. 179) one obtains

$$\sum_{k=0}^{N} F_k^3 \;=\; F_{N+1}\, F_N^2 \;-\; \sum_{k=0}^{N-1} F_{k+1}\, F_k\, F_{k-1} \;, \tag{2}$$

where the F recurrence has been used for $b_{k+1} \,-\, b_k$. For the sum on the *r.h.s.* we use, after adding and subtracting the $k \,=\, N$ term, the well known *Cassini* identity $F_{k+1}\, F_{k-1} \;=\; F_k^2 \,+\, (-1)^k$ (this is obtained by the determinant product formula applied to powers of the Fibonacci matrix with known determinant; see, *e.g.*, [3], eq. (5.4), p. 74 and p. 363). In this way one recovers the negative of the sum on the *l.h.s.* and the remaining terms lead to the result, after also the known alternating sum $\sum_{k=0}^{N} (-1)^k\, F_k \;=\; (-1)^N\, (F_{N-1} \,+\, (-1)^{N-1}) \;=\; (-1)^N$ A008346$(N-1)$ is used.

$$F^{(0,0,0)}(N) \;:=\; \sum_{k=0}^{N} F_k^3 \;=\; \frac{1}{2}\, \left(F_{N+1}^2\, F_N \,+\, (-1)^{N-1}\, (F_{N-1} \,+\, (-1)^{N-1})\right) \;. \tag{3}$$

In this case the partial summation formula first led to eq. (2), a relation between two sums, namely, after a shift in the summation index,

$$\sum_{k=0}^{N} F_{k+2}\, F_{k+1}\, F_k \;=\; -\sum_{k=0}^{N} F_k^3 \;+\; F_{N+3}\, F_{N+1}\, F_N \;. \tag{4}$$

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With the above result this sum becomes (first for $N-1$, then rewritten for N)

$$\frac{1}{2}\,F^{(2,1,0)}(N) \;:=\; \sum_{k=0}^{N} \frac{1}{2}\,F_{k+2}\,F_{k+1}\,F_k \;=\; \frac{1}{4}\,\left(F_{N+2}^2\,F_{N+1} \,-\, (-1)^N\,(F_N \,+\, (-1)^N)\right)\,. \tag{5}$$

This is the sum over the fibonomial coefficients $fibonomial(k+2,3)$ (see A001655$(k-1)$, with A001655$(-1) := 0$). For this sum see A215037$(N-1)$, with A215037$(-1) := 0$.

Using the F-recurrence in twice this sum one obtains an expression for the sum of the two sums $\sum_{k=0}^{N} F_{k+1}^2\,F_k + \sum_{k=0}^{N} F_{k+1}\,F_k^2$. On the other hand, if the F-recurrence is used in the second sum one obtains $\sum_{k=0}^{N} F_{k+1}\,F_k^2 = \sum_{k=0}^{N} F_k^3 + \sum_{k=0}^{N-1} F_{k+1}^2\,F_k$, after a shift of the summation index. This leads to the difference of the two sums $\sum_{k=0}^{N} F_{k+1}^2\,F_k - \sum_{k=0}^{N} F_{k+1}\,F_k^2 = -\sum_{k=0}^{N} F_k^3 + F_{N+1}^2\,F_N$. If the sum over the F-cubes from eq. (3) is inserted, and the result for the sum of the two sums is also used, one finds the following expressions for both sums separately, using the $F-$recurrence, implying also $F_{N+1}^2 - F_N^2 = F_{N+2}\,F_{N-1}$ ([6], eq. (12), p. 176, or [3], p. 90, eq. 56. with $k=1$), and the *Cassini* identity from above.

$$F^{(1,0,0)}(N) \;:=\; \sum_{k=0}^{N} F_{k+1}\,F_k^2 \quad=\quad \frac{1}{2}\,(F_{N+1}^3 \,-\, (-1)^N\,F_{N+1}) \;=\; \frac{1}{2}\,F_{N+2}\,F_{N+1}\,F_N\,, \tag{6}$$

$$F^{(1,1,0)}(N) \;:=\; \sum_{k=0}^{N} F_{k+1}^2\,F_k \quad=\quad \frac{1}{2}\,\left(F_{N+2}\,F_{N+1}^2 \,-\, (-1)^N\,(F_N \,+\, (-1)^N)\right). \tag{7}$$

The sum $F^{(1,0,0)}(N) = fibonomial(N+2,3)$ is found as A001655$(N-1)$, $N \geq 1$, with A001655(-1):=0. The sum $F^{(1,1,0)}(N)$ is found as A215038(N).

Using both results as input one can immediately obtain by induction on $m \geq 0.$, *via* the $F-$recurrence, the expressions for the following $m-$family of sums.

$$F^{(m,1,0)}(N) \;\equiv\; s(m;N) \;:=\; \sum_{k=0}^{N} F_{k+m}\,F_{k+1}\,F_k \;=\; \frac{1}{2}\,\left(F_{N+m}\,F_{N+2}\,F_{N+1} \,-\, (-1)^N\,F_m\,(F_N \,+\, (-1)^N)\right)\;. \tag{8}$$

Similarly, with the expression for the sum over the F-cubes, eq. (3) and eq. (6) as input one can prove by induction on $m \geq 0$

$$F^{(m,0,0)}(N) \;\equiv\; t(m;N) \;:=\; \sum_{k=0}^{N} F_{k+m}\,F_k^2 \;=\; \frac{1}{2}\,\left(F_{N+1}\,F_N\,F_{N+m+1} \,-\, (-1)^N\,F_{m-1}\,(F_{N-1} \,+\, (-1)^{N-1})\right)\;. \tag{9}$$

Here $F_{-1} = 1$ is used (from the F-recurrence).

The corresponding ordinary generating functions ($o.g.f.$s) $G^{(a,b,c)}(x)$ for these partial sum sequences $\{F^{(a,b,c)}(N)\}_{N=0}^{\infty}$ are given immediately by those for the sequences of the summands by multiplication with $\dfrac{1}{1-x}$. Therefore (see for the first four cases A056570, A001655$(k-1)$, A066258, A066259,respectively),

$$
\begin{aligned}
G^{(0,0,0)}(x) &= \frac{x\,(1 - 2\,x - x^2)}{(1 + x - x^2)\,(1 - 4\,x - x^2)\,(1 - x)}, && (10)\\
\frac{1}{2}\,G^{(2,1,0)}(x) &= \frac{x}{(1 + x - x^2)\,(1 - 4\,x - x^2)\,(1 - x)}, && (11)\\
G^{(1,0,0)}(x) &= \frac{x}{(1 + x - x^2)\,(1 - 4\,x - x^2)}, && (12)\\
G^{(1,1,0)}(x) &= \frac{x\,(1 + x)}{(1 + x - x^2)\,(1 - 4\,x - x^2)\,(1 - x)}, && (13)\\
G^{(m,1,0)}(x) &= \frac{x\,(F_{m+1} + e(m)\,x)}{(1 + x - x^2)\,(1 - 4\,x - x^2)\,(1 - x)}, && (14)\\
&\text{with } e(0) = -1, e(1) = +1, e(2) = 0, \text{ and } e(m) = 1 \text{ for } m \geq 3, &&\\
G^{(m,0,0)}(x) &= \frac{x\,(F_{m+1} - L_m\,x - \varepsilon(m)\,x^2)}{(1 + x - x^2)\,(1 - 4\,x - x^2)\,(1 - x)} && (15)\\
&\text{with } \varepsilon(0) = 1, \varepsilon(1) = 0, \text{ and } \varepsilon(m) = 1 \text{ for } m \geq 2. &&\\
& && (16)
\end{aligned}
$$

L are the *Lucas* numbers A000032, and $G^{(0,1,0)}(x) = G^{(1,0,0)}(x)$.

Of course one could go on to fourth and higher powers of *Fibonacci* numbers and apply the partial summation formula.

For alternating sums of powers of *Fibonacci* numbers with various indices the partial summation formula leads also to interesting relations and explicit forms. As an example for the latter we consider the sum $\sum_{k=0}^{N} (-1)^k\, F_k^3$. This appears as A119284, and an explicit formula and the *o.g.f.* is shown there. However, if we apply formula eq. (1) with $a_k = (-1)^k\, F_k$ and $b_k = F_k^2$ it turns out that this sum appears also on the *r.h.s.* (hence drops out) and one obtains an explicit formula for the sum $\sum_{k=0}^{N} (-1)^k\, F_{k+1}\, F_k^2$ which is presently not yet in [4]. This is the content of example 2.

Example 2: $a_k = (-1)^k\, F_k$ and $b_k = F_k^2$

$$
Falt^{(1,0,0)} := \sum_{k=0}^{N} (-1)^k\, F_{k+1}\, F_k^2 = \frac{1}{2}\left(1 - F_{N+1} + (-1)^N\, F_{N+2}\, F_N^2\right). \tag{17}
$$

Proof: Starting with $b_{k+1} - b_k = F_{k+2}\, F_{k-1}$ (a standard formula, *e.g.*, [6], eq. (12), p. 176) , the $F-$recurrence for F_{k+2} and the known sum $\sum_{k=0}^{k} (-1)^j\, F_j = (-1)^k\, F_{k-1} - 1$, one has

$$
\sum_{k=0}^{N} (-1)^k\, F_k^3 = ((-1)^N\, F_{N-1} - 1)\, F_N^2 + \sum_{k=0}^{N-1} (-1)^{k-1}\, F_{k-1}^3 - 2\sum_{k=0}^{N-1} (-1)^k\, F_k\, F_{k-1}^2 + \sum_{k=0}^{N-1} F_{k+2}\, F_{k-1}\,. \tag{18}
$$

The non-alternating sum is known: $\sum_{k=0}^{N-1} F_{k+2}\, F_{k-1} = 1 + \sum_{k=0}^{N-2} F_{k+3}\, F_k = 1 + F_{2\,N-3} - 1 + 2\, F_{N-1}\, F_{N-2}$. See also A080097, and a comment on A080144 for such sums. The other sum reproduces the sum on the *l.h.s.* after separating the $k = 0$ term, shifting the index and adding and subtracting the missing terms for $k = N-1$ and $k = N$. This leads to

$$0 = (-1)^N F_{N-1} F_N^2 \quad - \quad F_N^2 - (-1)^{N-1} F_{N-1}^3 - (-1)^N F_N^3 + 2 \sum_{k=0}^{N-2} (-1)^k F_{k+1} F_k^2 \qquad (19)$$
$$+ F_{2(N-2)+1} - 1 + 2 F_{N-1} F_{N-2} \; .$$

With $F_{2(N-2)+1} = F_{N-1}^2 + F_{N-2}^2$ (see *e.g.*, , [6], eq. (11), p. 176) and replacing $(F_{N-1} + F_{N-2})^2 = F_N^2$ this leads to

$$-2 \sum_{k=0}^{N-2} (-1)^k F_{k+1} F_k^2 = -1 + (-1)^N \left(2 F_N F_{N-1}^2 - 2 F_{N+1} F_N^2 + F_{N-1} F_N^2 + F_{N-1}^3 - F_N^3\right) \; . \qquad (20)$$

One can then show that the terms multiplying $(-1)^N$ boil down to $-(F_N^3 + F_{N+2} F_{N+1} F_{N-2})$. This can be rewritten as $-(2\, fibonomial(N+2) - 2\, fibonomial(N+1) - F_{N+1}^2 F_{N-1})$, and $2\, fibonomial(N+2) - 2\, fibonomial(N+1) = 2 F_{N+1} F_N^2$. Applying once more the *Cassini* identity finally ends the proof. □

The *o.g.f.* of $\{Falt^{(1,0,0)}(N)\}_{N=0}^{\infty}$ is obtained from the one of the summand, given in A066258, after multiplication with $\frac{1}{1-x}$.

$$Galt^{(1,0,0)}(x) = \frac{-x\,(1+x)}{(1+4\,x-x^2)\,(1-x-x^2)\,(1-x)} \; . \qquad (21)$$

Applying the partial summation formula on $Falt^{(1,0,0)}$, which is now known, results in the following identity.

Example 3: $a_k = (-1)^k F_{k+1}$ and $b_k = F_k^2$

$$Falt^{(3,1,0)} := \sum_{k=0}^{N} (-1)^k F_{k+3} F_{k+1} F_k = \frac{1}{2} \left(1 - F_{N+3} + (-1)^N F_{N+2}^2 F_{N+1}\right) \; . \qquad (22)$$

Proof: With $b_{k+1} - b_k = F_{k+2} F_{k+1}$ and $A(k) = \sum_{j=0}^{k} (-1)^j F_{j+1} = (-1)^k F_k - 1$ one finds

$$Falt^{(1,0,0)}(N) \quad = \quad \sum_{k=0}^{N} (-1)^k F_{k+1} F_k^2 = ((-1)^N F_N - 1) F_N^2 + \sum_{k=0}^{N-2} (-1)^k F_{k+3} F_{k+1} F_k + 1$$
$$+ \sum_{k=0}^{N-2} (-1)^k F_{k+3} F_k \; . \qquad (23)$$

The non-alternating sum is known (see the comment on A080144 for such sums): $\sum_{k=0}^{N-2} (-1)^k F_{k+3} F_k = F_{2(N-2)+1} - 1 + F_{N-1} F_{N-2}$. Replacing $F_{2(N-2)+1}$ (see above, [6], p. 176, eq. (11)), inserting the known result for $Falt^{(1,0,0)}(N)$ from eq. (16) leads finally to the desired result. □

The *o.g.f.* of $\{Falt^{(3,1,0)}(N)\}_{N=0}^{\infty}$ is readily obtained from the one of the summand, using the $F-$recurrence for F_{N+3}. See A065563 and A066259.

$$Galt^{(3,1,0)}(x) = \frac{-x\,(3-x)}{(1+4\,x-x^2)\,(1-x-x^2)\,(1-x)} \; . \qquad (24)$$

Another alternating sum of interest is $Falt^{(1,1,0)}(N)$ given by

$$Falt^{(1,1,0)}(N) \;:=\; \sum_{k=0}^{N} (-1)^k\, F_{k+1}^2\, F_k \;=\; (-1)^N\, fibonomial(N+2)\,, \tag{25}$$

with *o.g.f.* (compare this with eq. (20))

$$Galt^{(1,1,0)}(x) \;=\; \frac{-x\,(1-x)}{(1+4\,x-x^2)\,(1-x-x^2)\,(1-x)}\;. \tag{26}$$

For the unsigned summand see A066259. We have not found a way to derive eq. (24) directly from some partial summation formula. Therefore this will be proved here *via* some detour using as input the known alternating sum $Falt^{0,0,0}(N)$ for the cubes.

The application of the formula eq. (1) with $a_k \;=\; (-1)^k\, F_{k+2}$ and $b_k \;=\; F_{k+1}\, F_k$ leads to the analog of a the relation eq. (4) for alternating sums.

Example 4: $a_k \;=\; (-1)^k\, F_{k+2}$ and $b_k \;=\; F_{k+1}\, F_k$, implying $b_{k+1} \,-\, b_k \;=\; F_{k+1}^2$.

$$Falt^{(2,1,0)}(N) \;:=\; \sum_{k=0}^{N} (-1)^k\, F_{k+2}\, F_{k+1}\, F_k \;=\; \sum_{k=0}^{N} (-1)^k\, F_k^3 \,+\, (-1)^N\, F_{N+1}^2\, F_N\,. \tag{27}$$

Using the F-recurrence for F_{k+2}, inserting the result for $Falt^{(1,0,0)}(N)$ from eq. (16), yields the relation

$$Falt^{(1,1,0)}(N) \;=\; \sum_{k=0}^{N} (-1)^k\, F_k^3 \,+\, \frac{1}{2}\,\left(F_{N-1} \,-\, 1 \,+\, (-1)^N\, F_{N+3}\, N_N\, F_{N-1}\right)\;. \tag{28}$$

Up to now no expression for $Falt^{(0,0,0)}(N) \;=\; \sum_{k=0}^{N} (-1)^k\, F_k^3$ has been found this way. Of course one could use the known result derivable from the recurrence of the cubic powers of Fibonacci numbers (*e.g.*, [2], exercise 6.58, p. 315 with the solution on p. 556) $F_{n+1}^3 \,-\, 4\,F_n^3 \,-\, F_{n-1}^3 \;=\; 3\,(-1)^n\, F_n$ (this follows from the *o.g.f.* for the cubes of *Fibonacci* numbers, obtained *via* the *Binet-de Moivre* formula, and the result $F_{3\,n} \;=\; F_{n+1}^3 \,+\, F_n^3 \,-\, F_{n-1}^3$ (see [5], p. 680, eq. III, with $U_n \;=\; F_{n+1}$). This produces after summation

$$Falt^{(0,0,0)}(N) \;=\; \frac{1}{4}\,\left(2 \,-\, 3\,F_{N+2} \,-\, (-1)^N\,(F_{N+1}^3 \,-\, F_N^3)\right)\;. \tag{29}$$

Using $F_N^3 \;=\; \frac{1}{4}\,\left(F_{3\,N} \,-\, 3\,(-1)^N\, F_N\right)$ (see also A056570) this can be rewritten as

$$Falt^{(0,0,0)}(N) \;=\; \frac{1}{10}\,\left(5 \,-\, 6\,F_{N+2} \,+\, (-1)^N\, F_{3\,N+1}\right)\;. \tag{30}$$

See A005968. The *o.g.f.* , obtained from the one of the alternating cubic powers of the *Fibonacci* numbers (see $\{(-1)^n$ A056570$(n)\}$), is

$$Galt^{(0,0,0)}(x) \;=\; \frac{-x\,(1+2\,x-x^2)}{(1+4\,x-x^2)\,(1-x-x^2)\,(1-x)}\,. \tag{31}$$

With this information one finds from eq. (27), using the F-recurrence and the *Cassini* identity

$$Falt^{(1,1,0)}(N) \;:=\; \sum_{k=0}^{N} (-1)^k\, F_{k+1}^2\, F_k \;=\; \frac{1}{10}\,\left(4\,F_N - F_{N+1} + (-1)^N\,(F_{3\,N+1} + 5\,F_{N+2}\,F_N^2)\right)\;. \tag{32}$$

With $F_{3\,N+1} \;=\; \frac{1}{2}\,\left(F_{3\,(N+1)} \,-\, F_{3\,N}\right)$ (see A033887 with the *o.g.f.* from A001076), and $5\,F_{n+2}\,F_n^2 \;=\; \frac{1}{2}\,\left(F_{3\,(N+1)} \,-\, F_{3\,N}\right) + (-1)^N\,F_{N+2} - 5\,F_N^3$, from $F_{3\,n} \;=\; 5\,F_n^3 + 3\,(-1)^n\,F_n$, [3], p. 89, eq. 46. (attributed to *Holton* 1965), and again the *Cassini* identity, this becomes

$$Falt^{(1,1,0)}(N) \;:=\; \sum_{k=0}^{N} (-1)^k\,F_{k+1}^2\,F_k \;=\; \frac{1}{10}\,\left(-2\,F_{N+1} \,+\, (-1)^N\,F_{3\,(N+1)}\right)\;. \tag{33}$$

The bracket is, using again the $F_{3\,n}$ formula from above, with $n \to N+1$, and $F_{N+1}^3 \;=\; F_{N+2}\,F_{N+1}\,F_N + (-1)^N\,F_{N+1}$ (this is eq. 32. of [3], p. 89, but with a sign misprint corrected, see A065563) rewritten as $10\,fibonomial(N+2) + 2\,(-1)^N\,F_{N+1}$. $Falt^{(1,1,0)}(N)$ is now seen to coincide indeed with the mentioned result given in eq. (24). □

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