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Albrecht Dürer's approximation of a regular 5-gon

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In the second volume (Zweites Buch) of Albrecht Dürer's "Underweysung der Messung ..." from 1525 [4] one finds in figure 16 an approximation to a (regular) pentagon (regular 5−gon). To be sure, he also gives the standard exact construction of a regular 5−gon in the preceding Figure 15. This approximate construction can be found also in [11], p. 21, and from there the author became interested to compute the dimensionless area $\hat{F}_D := \dfrac{F_D}{r^2}$, when the radius of each of the five circles is r, which coincides with the length of each side. A further aim was to compute the three angles shown in the second figure as α, $\beta := \dfrac{\pi}{2} + \gamma - \delta$, and $2\,\sigma = \pi - 2\,\gamma$ in order to compared them with the angle $\dfrac{3\pi}{5} \,\hat{=}\, 108^o$ in the regular 5-gon.

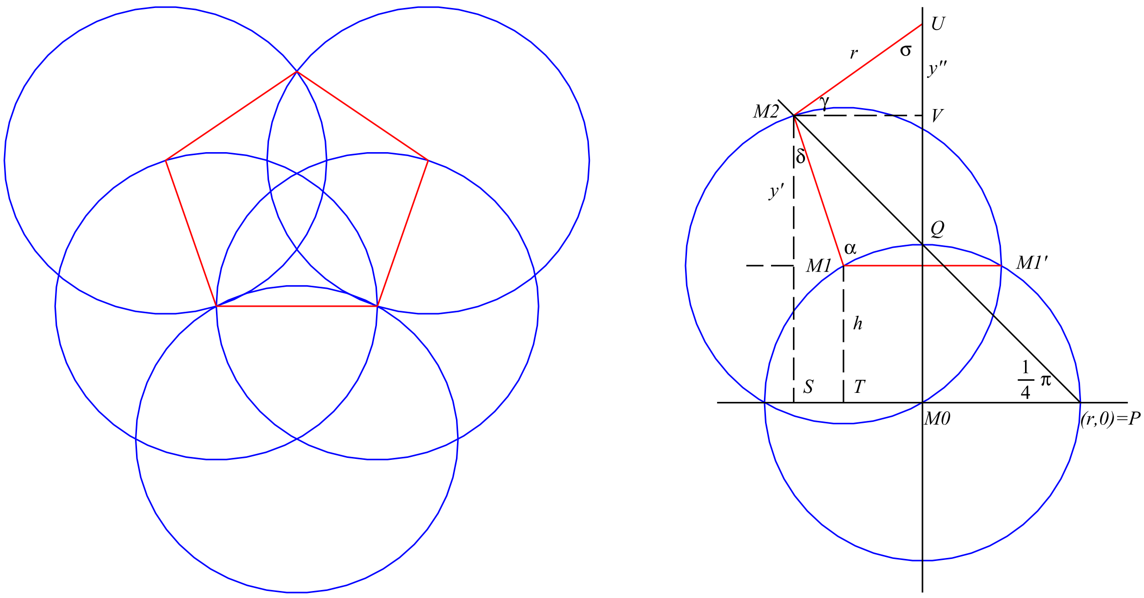

Figure 1 : Dürer's approximation of a regular 5-gon and the construction elements. $\overline{M0,P} = r$, $\overline{M0,T} = r/2$, $\overline{M0,S} = |x2|$, $\overline{T,M1} = h$, $\overline{S,M2} = y2 = h + y'$, $\overline{V,U} = y''$, $\angle(M1', M1, M2) = \alpha$, $\angle(V, M2, U) = \gamma$, $\angle(S, M2, M1) = \delta$, $\angle(M1, M2, U) = \beta = \pi/2 + \gamma - \delta)$, $\angle(M2, U, V) = \sigma = \pi/2 - \gamma$.

The coordinates of the center of the second circle are $M1\left(-\dfrac{r}{2}, h\right)$, with $\hat{h} := \dfrac{h}{r} = \dfrac{\sqrt{3}}{2}$ if the first circle is centered at $M0(0,0)$. The center $M2$ of the third circle is obtained by one of the intersection points of the circle around $M1$ with the straight line $y = -x + r$. Therefore $M2 = (-x2, y2)$ with $x2$ obtained from one of the solutions of the quadratic equation $x^2 + \dfrac{1}{2}\,(2\,h - r)\,x + \dfrac{1}{2}\left((r-h)^2 - \dfrac{3}{4}\,r^2\right) = 0$.

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Or $x^2 + \frac{1}{2}\,a\,r\,x - \frac{1}{2}\,a\,r^2 \,=\, 0$, with $a \,:=\, \sqrt{3} - 1$. The relevant solution is $\hat{x} \,=\, \frac{x2}{r} \,=\, -\frac{1}{4}\,(a + \sqrt{a(a\,+\,8)}\,)$ $= -\frac{1}{4}\left(\sqrt{3}\,-\,1\,+\,\sqrt{(\sqrt{3}-1)\,(\sqrt{3}+7)}\right)$. This is approximately $\hat{x} \,\approx\, -.8150878978$ The dimensionless ordinate $y2$ of $M2$ is $\hat{y} \,:=\, \frac{y2}{r} \,=\, -\hat{x}\,+\,1$. That is $\hat{y}' \,:=\, \hat{y}\,-\,\hat{h} \,=\, \frac{1}{4}\left(3\,-\,\sqrt{3}\,+\,\sqrt{2\,(3\,\sqrt{3}\,-\,2)}\right)$. This is approximately $\hat{y}' \,\approx\, 0.9490624938$. The dimensionless height of the top triangle is $\hat{y}'' \,=\, \sqrt{1\,-\,\hat{x}^2}$ which is approximately 0.5793373101.

This leads, with $\sin(\pi\,-\,\alpha) \,=\, \hat{y}'$, to the angle $\alpha \,\approx\, 108.3661201^\circ$, that is a relative error of around $+0.00339$, or an overshooting of about one third of a percent. The angle γ is obtained from $\cos(\gamma) \,=\, |\hat{x}|$, or $\gamma \,\approx\, 35.40394606^\circ$. The angle δ follows from $\cos(\delta) \,=\, \hat{y}'$, or $\delta \,\approx\, 18.36612013^\circ$. This means $\beta \,=\, \frac{\pi}{2} + \gamma - \delta \,\approx\, 107.0378260^\circ$, a relative error of around -0.0089, or an undershooting of roughly one percent. Finally the third angle $2\,\sigma \,=\, \pi\,-\,2\,\gamma$ is approximately 109.1921078°, a relative error of about $+0.011$, or an overshooting of about one percent. Because in any 5−gon the sum of angles is $3\,\pi$ (from triangulation), one has $\alpha\,+\,\beta\,+\,\sigma \,=\, \frac{3\,\pi}{2} \,\widehat{=}\, 270^\circ$, which checks.

The area of *Duerer*'s 5-gon is obtained from adding the one of the trapezoid with vertices $M1, M2$ together with their mirror points, and the triangle on top of it. Therefore, $\hat{F}_D \,:=\, \frac{F_D}{r^2} \,=\, \frac{1}{2}\,(1 + 2\,|\hat{x}|)\,\hat{y}' + |\hat{x}|\,\hat{y}'' \,\approx$ $1.248100600\,+\,0.472210830 \,=\, 1.720311430$. This should be compared with the dimensionless area of the pentagon with side length r which is, adding again the areas of the trapezoid and the top triangle, $\hat{F}_5 \,:=\, \frac{F_5}{r^2} \,=\, \frac{1}{2}\,(1\,+\,2\,\hat{x}_5)\,\hat{h}_5\,+\,\hat{x}_5\,\hat{y}_5''$ with $\hat{x}_5 \,:=\, \frac{1}{2}\,+\,\cos\left(\frac{2\,\pi}{5}\right) \,=\, \frac{1}{2}\varphi$, with the golden section $\varphi \,:=\, \frac{1}{2}\,(1\,+\,\sqrt{5}\,)$. $\hat{h}_5 \,=\, \sin\left(\frac{2\,\pi}{5}\right) \,=\, \frac{1}{2}\,\sqrt{2\,+\,\varphi}$. $\hat{y}_5'' \,=\, \cos\left(\frac{3\,\pi}{10}\right) \,=\, \frac{1}{2}\,\sqrt{3\,-\,\varphi}$. This produces $\hat{F}_5 \,=\, \frac{1}{4}\,\left((1\,+\,\varphi)\,\sqrt{2\,+\,\varphi} + \varphi\,\sqrt{3\,-\,\varphi}\right) \,=\, \frac{1}{4}\,\varphi\,\sqrt{3\,-\,\varphi}\,(2\,+\,\varphi) \,=\, \frac{1}{4}\,(3\,\varphi\,+\,1)\,\sqrt{3\,-\,\varphi}$ which is approximately 1.7204774005889669228, and more digits are found under A102771. The relative error for the area is therefore about $-9.6\,10^{-5}$, or roughly -10^{-4}.

Remark: These are, of course, trivial calculations, and no claim for originality is made. The purpose of this note is to explain the area digits given in OEIS [9] A220674.

Note added Jan 30 2013:

1. Alonso del Arte pointed out the *Hughes* reference [8]. See p. 7 for the title page, p.9, column 5, for the 5−gon construction, and p. 16 for the description of this construction (also in English). See also the comment and the following remarks on p. 17. This construction is there said to be apparently due to Pappus of Alexandria (without giving the precise reference). *Pietro Cataldi* is mentioned to have ‘derived extensive equations to show that ‘Dürer's pentagon’ is not regular’ in 1620 (no reference is given). For some earlier dates see below.

2. A demonstration is found under [13].

3. In [2] this approximate construction is considered in more detail and the angle α in our figure has been computed as 108.36612...°. It is mentioned there that the angle, which in our text is called β, is ‘a little less than 108°’, and the angle called $2\,\sigma$ in the figure above is ’a little more than 108°’. The area is not computed there. The following remark is made there: ‘Mark Reynold's points out that the construction is found in the book ”Geometrica Deutsch” probably published by Roriczer ca. 1472-1484’. Quoted is an article by R. A. Simon [10], where one also finds these remarks on the angles.

Note added Feb 02 2013:

The two diagonals in *Dürer*'s 5-gon are $d1 \,=\, \overline{M1, U}$ and $d2 \,=\, \overline{M1', M1}$. Scaled with the length of the side they are $\hat{d}1 \,:=\, \frac{d1}{r} \,=\, \sqrt{1/4 + (\hat{y}' + \hat{y}'')^2} \,\approx\, 1.608106327$. Therefore the relative error with respect to the regular 5-gon with the dimensionless diagonal $d_5 = \varphi \approx 1.618033988$ is about -0.0061, or about $-.6$ percent. The other dimensionless diagonal is $\hat{d2} := \frac{d2}{r} = \sqrt{\hat{y}'^2 + (1/2 + |\hat{x}|)^2} \approx 1.621781673$, a relative error of about $+0.0023$, or about $+1/4$ percent.

Note added Feb 14 2013:

1) The relative error of the angle β was wrong. It has been corrected.

2) Some more references are given where this approximate 5-gon construction has been considered. In [1] one finds this as FIG. 36 on p. 67. There it is said (on p. 68) that "this leads to the approximation $\sin(27°) \approx 2\,\sin(60°)\,\sin(15°)$, $0.454 \approx 0.448$" (I changed the notation slightly). This is equivalent to $\frac{1}{2}\sqrt{2 - \sqrt{3 - \varphi}} \approx \frac{\sqrt{3}}{2}\sqrt{2 - \sqrt{3}}$. I cannot see how this comparison arises.
In the very interesting small book with problems [12] one finds the construction in section XI., pp. 27-30 and in section XVII., pp. 44-48. In XI. the author quotes from [7], II., p.5 and §6., pp.7-8, the construction given in *Geometria deutsch*, (neither author, year nor publisher are known, 6 pages, 9 problems, the second page with this construction, found by *Günther* in the Nürnberg town library. He gives a conjecture for the printer). It is assumed that this predates *Dürer* who gave the same construction but with a different text. The angles are given up to arc seconds in [12] as twice $107°2'\,13''$, twice $108°22'$ and $109°11'\,33''$, *i.e.*, about 107.0370, 108.3667, 109.1925 (in [7] they are given up to arc minutes, taken from [5], p. 625, (French p. 530 footnote 1) as twice $107°2'$, twice $108°22'$ and $109°12''$). In [5] the references *J. Bapt. de Benedictis* [3] from 1585 and Clavius [6] from 1606 are given, for having noticed that not all angles are equal in the *Dürer* 5-gon, and in [3], p. 370 and [6] p. 362 the angles are the ones cited by [5]. It is checked in these references that the angles have to sum to $540°$, *viz* $2\cdot 22 - 2\cdot 58 + 1\cdot 72 = 0$.

2) Due to a theorem on isoperimetrical polygons among those with the same number of angles the one with identical angles is the biggest one. Hence the regular $5-$gon has the maximal area and the smaller *Düerer* $5-$gon has to have an equal area companion which is also axially symmetric but has different angles compared with those of *Dürer*'s 5-gon. From *Figure* 2 of a general symmetrical equal side length 5-gon one has, if $h1$ is chosen as variable, $h2 = r\sqrt{1 - \overline{D,B}^2}$ and $\overline{D,B} = r\left(\frac{1}{2} + \sqrt{1 - \hat{h1}^2}\right)$, with $\hat{h1} := \frac{h1}{r}$. The formula for the scaled area $\hat{F} := \frac{F}{r^2}$ as a function of $x = \hat{h1} := \overline{O,D}/r$ (see *Figure* 2) is, with $H(x) := \sqrt{1 - x^2}$ (adding the area of the trapezoid and the triangle on top of it)

$$\hat{F}(x) = (1 + H(x))\,x + \frac{1}{4}(1 + 2\,H(x))\,\sqrt{4\,x^2 - 1 - 4\,H(x)}\,. \tag{1}$$

The range of $x \equiv \hat{h1}$ is given by the two degenerate cases *i)* the 5-gon becoming a triangle with A, B and C on a line, and the height $y_{C,max} = \frac{\sqrt{3\cdot 5}}{2}$, *i.e.*, $\hat{h1} = \frac{\sqrt{3\cdot 5}}{4}$ which coincides with $\hat{h2}$, and *ii)* the 5-gon becoming a symmetrical trapezoid with B, C and its mirror point C' on a line. In this case $\hat{y}_{C,min} = \hat{h1} = \frac{\sqrt{3}}{2}$ and $\hat{h2} = 0$. See the plot of $\hat{F}(x)$ for $x = \hat{h1} \in \left(\frac{\sqrt{3}}{2}, \frac{\sqrt{15}}{4}\right)$ in *Figure* 3*a*). The $x \equiv \frac{h1}{r}$ values for the *Dürer* 5-gon, the regular 5-gon and the companion of *Dürer*'s 5-gon are, with $a := \sqrt{3} - 1$, $xD = \hat{y}' = \frac{1}{4}(2 - a + \sqrt{a\,(a+8)}) \approx 0.949$, $x5 = \frac{1}{2}\sqrt{2 + \varphi} \approx 0.951$ and $xD' \approx 0.953$, respectively. The xD' value comes from the larger of the two solutions of the equation $\hat{F}(x) = \hat{F}_D$ which is, with more digits about 0.9530288824. In *Figure* 3*a*) the small difference between the areas is not visible. In *Figure* 3*b*) a close up is shown. We do not know a geometrical construction for the companion of *Dürer*'s 5-gon. Its angles (see *Figure* 2) are computed from $\sin(\delta) = \widehat{D,B} - \frac{1}{2}$, with $\widehat{D,B} = \frac{\overline{D,B}}{r}$ from above,

$\sin(\gamma) = \hat{h2} = \sqrt{1 - \widehat{D,B}^{\,2}}$, and $\cos(\sigma) = \hat{h2}$. One finds then, with the given value of $\hat{h1} = xD'$, $\alpha = \frac{\pi}{2} + \delta \approx 107.631°$, $\beta = \frac{\pi}{2} + \gamma - \delta \approx 108.963°$ and $\sigma \approx 53.406°$. This checks with $\alpha + \beta + \sigma = 270°$. Now α undershoots and β overshoots, with a relative errors about -0.0034 and $+0.0089$. $2\,\sigma \approx 106.812°$ with relative error about -0.011. Thus over- or undershooting is reversed compared with *Dürer*'s 5-gon.

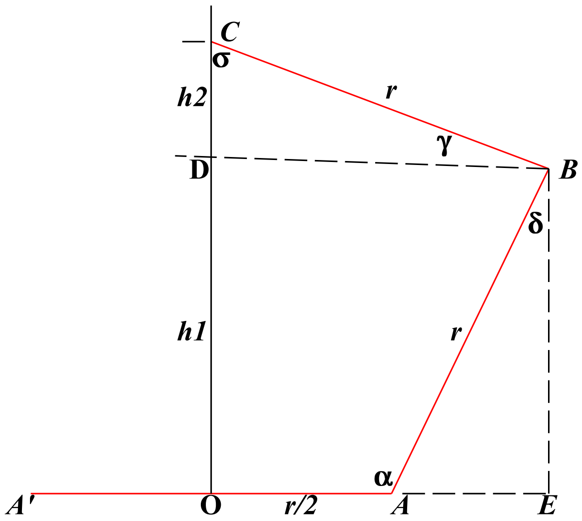
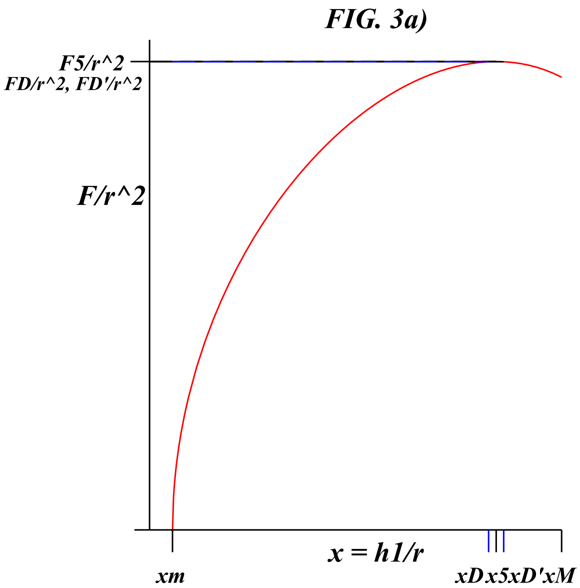


Figure 2: A $y-$axis symmetric 5-gon with sides of the same length r. A and A' are fixed, B (hence D) moves when C is moved on the vertical axis, with $\hat{h1} := h1/r$ from the open interval $\left(\frac{\sqrt{3}}{2}, \frac{\sqrt{15}}{4}\right)$.



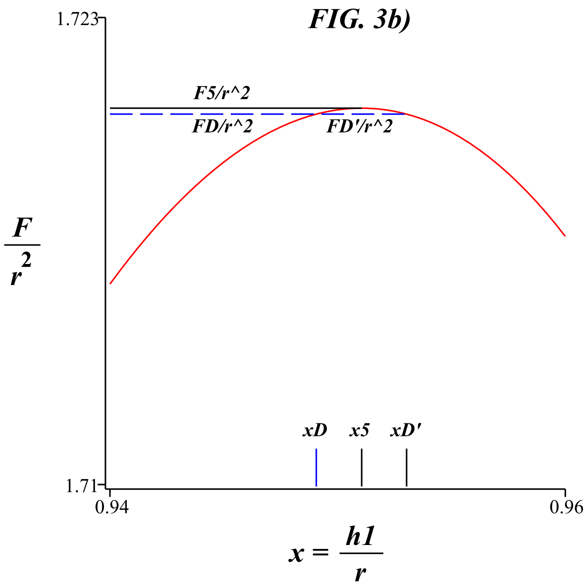


Figure 3a): Scaled area $\hat{F} = \frac{F}{r^2}$ as a function of $x \equiv \hat{h1} = \frac{h1}{r}$ from the lowest value $xm = \frac{\sqrt{3}}{2}$ to the largest one $xM = \frac{\sqrt{15}}{4}$. The maximum of $hatF$ occurs

for $x5 = \frac{1}{2}\sqrt{2+\varphi} \approx 0.951$. The x values for *Dürer*'s 5-gon and its companion are marked as xD and xD'.

Figure 3b): A close up of Figure 3a) showing also the scaled area FD/r^2 of *Dürer*'s 5-gon and its companion slightly below the scaled area for the regular 5-gon $F5/r^2$.

References

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