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A225953: Periods of Indefinite Binary Quadratic Forms, Continued Fractions and the Pell ± 4 Equations

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1 Indefinite Binary Quadratic Forms and Periods

These notes are based on the books by Scholz-Schoeneberg [14] and Buell [1]. The classical texts are *Lagrange* [7] and *Gauss* [3]. For a historical review see *Dickson* [2]. The relation between the periods (or cycles) of indefinite binary quadratic forms and the continued fraction expansion of associated quadratic irrationals is treated in *Buell*'s book. We use a slightly different version, which is the reason for these notes.

An indefinite binary quadratic form $F = a\,x^2 + b\,x\,y + c\,y^2$, with integers a, b, and c, of discriminant $D > 0$ is denoted by $F([a,b,c],\,[x,y])$ or $F(\mathbf{A},\,\vec{x})$ with the matrix $\mathbf{A} = Matrix([[a,\,b/2],[b/2,\,c]])$ and the column vector $\vec{x} = (x,y)^{\top}$. $F = \vec{x}^{\top}\,\mathbf{A}\,\vec{x}$. Sometimes one denotes a form just by $[a,\,b,\,c]$. (Note that [3] uses $2\,b$ instead of b, and the discriminant, called in the English translation determinant, is $\frac{D}{4}$.)
Only *primitive* forms with $gcd(a,b,c) = 1$ are considered. A representation of an integer k by a form F is called *proper* if $gcd(x,y) = 1$, and *improper* otherwise. The discriminant of a form, and of the corresponding characteristic polynomial $F(x) = F([a,b,c],[x,1]) = a\,x^2 + b\,x + c$, is $D = b^2 - 4\,a\,c$ and for indefinite forms it is positive. The possible values for D are $0\,(mod\,4)$ or $1\,(mod\,4)$ and no (pure) squares are allowed. See [12] A079896 (such A-numbers will be used later on without giving the reference to *OEIS*). *Reduced* forms are defined in the two mentioned books in a different but equivalent way.

Definition: Reduced forms (see also [3], art. 183, p. 152)

I) [14], p. 112: An indefinite form of discriminant D is called *reduced* if, with $f(D) := \left\lceil \sqrt{D} \right\rceil$ (ceiling function),

$$0 < b, \qquad f(D) - min\{2\,|a|, 2\,|c|\} \le b < f(D)\,. \tag{1}$$

II) [1], p. 21: An indefinite form of discriminant D is called *reduced* if

$$0 < b < \sqrt{D}, \qquad \sqrt{D} - b < 2\,|a| < \sqrt{D} + b\,. \tag{2}$$

and it follows ([1], Proposition 3.1.) that also $\sqrt{D} - b < 2\,|c| < \sqrt{D} + b$.

Proof of the equivalence:

Before giving the proof we notice that in both versions one has for reduced forms

$$a\,c < 0 \qquad \text{and} \qquad 4\,|a\,c| < D. \tag{3}$$

The first statement can be proven indirectly in version $I)$ because $f(D) - 1 < \sqrt{D} < f(D)$. Here the irrationality of $\sqrt{D}$ and the definition of $f(D)$ have been used. Assume that $a\,c > 0$ (neither a nor c vanishes because otherwise b would also have to vanish). Then $f(D) - 1 < \sqrt{b^2 - 4\,a\,c} < b$ (the positive root is always taken). But together with $b < f(D)$ this leads to a contradiction because $f(D)$

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and b are integers. In version II) this is proven directly by squaring $b < \sqrt{D}$ (monotony of the function x^2), $b^2 < b^2 - 4\,a\,c$. The second inequality in eq. (3) is then $-4\,a\,c < D$ which is trivial because $b^2 > 0$ holds in both definitions.

I) $\Rightarrow$ II): $b \geq f(D) - min\{2\,|a|, 2\,|c|\} \geq f(D) - 2\,|a| > \sqrt{D} - 2\,|a|$, and the same works with $2\,|c|$. $b > 0$ and from above $0 < -4\,a\,c = D - b^2$, hence (monotony of the $\sqrt{x}$ function) $b < \sqrt{D}$. One of the remaining inequalities to prove is $|2\,a| - b < \sqrt{D}$ which becomes, after using the D formula and squaring, $b + |c| > |a|$. Similarly the other one is equivalent to $b + |a| > |c|$. Both inequalities follow from the just derived result $b + 2\,|a| > \sqrt{D}$, and similarly for $2\,|c|$. Squaring this and using the D formula leads to $b + |c| > |a|$ and $b + |a| > |c|$.

II) $\Rightarrow$ I): $0 < b < \sqrt{D} < f(D)$. $(f(D) - 1) - min(2\,|a|, 2\,|c|) < \sqrt{D} - min(2\,|a|, 2\,|c|) < b$ from $f(D) = \sqrt{D}$, and this is true for $2\,|a|$ and $2\,|c|$ separately. But from $f(D) - min(2\,|a|, 2\,|c|) < b+1$ follows that actually $f(D) - min(2\,|a|, 2\,|c|) \leq b$ because the *l.h.s.* is an integer. $\square$

The number of reduced forms for each discriminant D is finite (see [1], Proposition 3.2. on p. 22). See A082174 for these numbers for primitive and A082175 for imprimitive forms.

Two forms $F(\mathbf{A}, \vec{x})$ and $F(\mathbf{A}', \vec{x}')$ are said to be *properly equivalent* if $\mathbf{A}' = \mathbf{M}^{-1,T}\,\mathbf{A}\,\mathbf{M}^{-1}$ with some integer matrix $\mathbf{M}$ with $Det\,\mathbf{M} = +1$. *I.e.*, $\mathbf{M} \in SL(2,\mathbb{Z})$ (a special *Moebius* transformation, see *e.g.*, [1] pp. 9-12, for $SL(2,\mathbb{Z})$). Note that with $\vec{x}' = \mathbf{M}\vec{x}$ one has for the representation of an integer $k = F(\mathbf{A}, \vec{x}) = F(\mathbf{A}', \vec{x}')$. Proper equivalence will mostly be used in these notes, therefore we just write equivalent, using the symbol $F1 \sim F2$. In the other case we write improper equivalence.

With [14], p. 113, we define a *half-reduced* form with the weaker requirement $f(D) - |2\,a| \leq b < f(D)$. A *half-reduced right neighbour form* F^R equivalent to $F(\mathbf{A}, \vec{x})$ is obtained from $\mathbf{A}^R = \mathbf{R}^\top\,\mathbf{A}\,\mathbf{R}$ as $F^R = [c, -b + 2\,c\,t, F(\mathbf{A}, (-1,t)^\top)]$ with the $\mathbf{R} = \mathbf{R}(t) = Matrix([[0,-1],[1,t]])$, and this form is unique because t is determined by $t = \left\lceil \frac{f(D)+b}{2\,c} - 1 \right\rceil$ if $c > 0$ and $t = \left\lfloor 1 - \frac{f(D)+b}{2\,|c|} \right\rfloor$ if $c < 0$. The proof follows immediately from the half-reduced requirement for F^R. We denote such a half-reduced right neighbour relation of $F2$ to $F1$ by $F1 \overset{\mathbf{R}}{\sim} F2$. Starting with any indefinite form of some discriminant D a unique *chain* of half-reduced right neighbours can be built. At some step one will reach a reduced form, and from then on all members of the chain will be reduced. This is the content of [14] proposition (Satz) 79 on p. 113. The proof is lengthy and implies that if F_1 is a reduced form then $F_1 \overset{\mathbf{R}}{\sim} F_2$ implies that also F_2 is reduced. Because of the finite number of reduced forms for D, each chain will become periodic starting with the first reduced form. The period length will be called L. Because each form of a given D is equivalent to a reduced form (via the chain of half-reduced right neighbours) the number of equivalence classes for D, called the *class number*, is finite. This number is denoted by $h(D)$ or $H(D)$ in [14], p. 101, and [1], p. 7, respectively. It coincides with the number of different (periodic) chains of reduced forms for given D. See [1], Appendix 2, pp. 235-243, where Table 2A lists the numbers for odd D and Table 2B for even D, for $D < 10000$. See also A087048.

Lemma 1: $\mathbf{t\,c > 0}$ for half-reduced right neighbour of reduced forms

If a reduced form $F_1 = [a, b, c]$ of discriminant D has the half-reduced right neighbour form $F_2 = [a_2, b_2, c_2]$, *i.e.*, $F_1 \overset{\mathbf{R}}{\sim} F_2$ with some matrix $\mathbf{R}(t)$, then $c\,t > 0$.

Proof: This follows from the formulas for t and $f(D)-1 < \sqrt{D} < f(D)$, because $\sqrt{D}$ is irrational. If $c > 0$ the $2ct \geq f(D) + b - 2c > \sqrt{D} + b$ and from eq. (2) for reduced F this is $> 2c$, hence $ct > 0$. If $c < 0$ then $2|c|t \leq 2|c| - (f(D) + b) < 2|c| - (\sqrt{D} + b)$ and this is < 0 because $\sqrt{D} + b > 2|c|$. Hence $-ct < 0$, or $ct > 0$. $\square$

Note that this *Lemma* is not true if F is only half-reduced, e.g., $F = [8, -8, 1]$ and $F_2 = [1, 4, -4]$, with $t = -2$, hence $ct < 0$,

This result will later (in *Lemma 5*) be employed for a certain period of forms, showing alternating t-signs.

In [1], p. 23, two reduced forms $F = [a, b, c]$ and $F' = [c, b', c']$ of D are called *adjacent* if $b + b' \equiv 0\,(mod\,2\,a)$. We denote this relation by $F \overset{adj}{\sim} F'$. Clearly, if $F \overset{\mathbf{R}}{\sim} F'$ then $F \overset{adj}{\sim} F'$. On the other hand, if $F \overset{adj}{\sim} F'$ then both forms are reduced and F' is the right neighbour form of F because $a' = c$ and $b' = -b + 2\,c\,t$ with some uniquely determined non-vanishing $t \in \mathbb{Z}$. The equivalence matrix is $\mathbf{R}\left(t = \dfrac{b+b'}{2\,c}\right)$. But then F' is certainly half-reduced and $F \overset{\mathbf{R}}{\sim} F'$. Hence, for reduced forms both equivalence relations coincide. In [1] it is shown that the set of reduced forms for D is partitioned into cycles of adjacent forms (Proposition 3.4. on p. 23), *i.e.*, into periods of half-reduced right neighbour forms, each in fact reduced. The period length L is always even, $L = 2\,l$ (Proposition 3.6. on p. 24, and see the remark in the proof of *Lemma 5* later). For a list of the periods of discriminant D, for $D = 5, 8, 12, ..., 232$ see the *W. Lang* link under A225953.

Of interest are also the two forms $F = [a, b, c]$ and $F' = [c, b, a]$ of D (exchange of a and c, or reversed order). They are certainly improperly equivalent due to the transformation matrix $\mathbf{M}^{-1} = Matrix([[0, 1][1, 0]])$. See [14], p. 106, and [1], p. 24 . Such forms are called *associated*. It is clear that if a form F is reduced its associated form F' is also reduced. (There is no self-associated reduced form, having $a = c$, due to the first of eqs. 3.) However, a reduced form F' associated to a reduced form F may not lie in the same period (or cycle). The smallest D where this happens is $D = 136$, $F = [-5, 4, 6]$ and $F' = [6, 4\, -5]$ lie in two different periods, each of length 6. Whenever this happens each member of the period with member F has its associate in the period with member F'. This is *Proposition 3.7* of [1], p. 25. If F_1 is the right neighbour of a reduced form F then $\mathbf{A}_1 = \mathbf{R}(t)^{\top}\,\mathbf{A}\,\mathbf{R}(t)$. With the definition $\mathbf{A}_2 := \mathbf{M}(t)^{\top}\,\mathbf{A}_1\,\mathbf{M}$ (exchange of the diagonal elements in $\mathbf{A}_1$) it follows that $\mathbf{A}_2 = \mathbf{R}^{\top}(t)\,\mathbf{A}'\,\mathbf{R}(t)$ by analogy. F_2, the associate of F_1, is therefore reduced and it is the left neighbour in the period of F', seen by using $\mathbf{R}^{-1}(t)$ to transform F' into F_2. Example: $D = 136$ with F and F' as given above. $F_1 = [-3, 10, 3]$ and $F_2 = [3, 10, -3]$. (In [3], art. 187, p. 158, the example for his determinant 79 is used. This corresponds to the discriminant $D = 4 \cdot 79 = 312$, with six classes of period lengths $[6, 6, 6, 6, 4, 4]$, each of the two length 4 periods is self-associated, and the length 6 periods come in two associated pairs.) In this way a period is either self-associated (each of its forms has its associate in the same period) or a period of some (even) length L has an associated different period of the same length, *i.e.*, each form of the first period has its associate somewhere in the associated period. In the example $D = 136$ the two periods of length 6 are associated, as are the two periods of length 4. In both cases the order of the associate partners are the same (no mixing). For $D = 145$ two of the periods of length 6 are associated, but not member by member as one cycles through the period (mixing). The other period of length 6 is self-associate, like the remaining length 10 period.

With each period of (reduced) forms of length $L = 2\,l$ of discriminant D there comes an L-tuple of integers $\vec{t} = (t_1, t_2, ..., t_L)$ whose entries appear in the matrix $\mathbf{R}(t_j)$ which transforms F_{j-1}, in the period with some fixed starting form F_0, into its right neighbour form F_j. It is clear that this $t-$list is to be considered as cyclic, *i.e.*, $t_{L+1} = t_1$, and the starting form is arbitrary. Later on we are interested in the special period starting with the so called principal form of D, which will be the unique form with $a = 1$.

2 Continued fractions and Pell equations

Each form $[a, b, c]$ of discriminant $D > 0$ is related to a *quadratic irrational* (mixed surd) called $\omega = \omega(a, b, c) := \dfrac{-b + \sqrt{D}}{2\,a}$ [1], p. 31. It is the root of the quadratic polynomial $F([a, b, c], [x, 1]) = a\,x^2 + b\,x + c$ with a positive square root. This number has, shown by *Lagrange* in 1770, a periodic continued fraction approximation [13], ch. III., §20, p. 65. (See also [1], sect. 3.3 for a short account on regular (also called simple) continued fractions.) If two forms $F(\mathbf{A}, \vec{x})$ and $F(\mathbf{A}', \vec{x}')$ are equivalent, *i.e.*, $\mathbf{A}' = \mathbf{M}^{-1,\top}\,\mathbf{A}\,\mathbf{M}^{-1}$ and $\vec{x}' = \mathbf{M}\,\vec{x}$ with $Det\,\mathbf{M} = +1$ and $\mathbf{M} = Matrix([[\alpha, \beta], [\gamma, \delta]]) \in SL(2, \mathbb{Z})$

then this irrational number transforms like (see [1], p. 31 where ω is called *principal root* of the form F)

$$\omega' \,=\, \frac{\alpha\,\omega \,+\, \beta}{\gamma\,\omega \,+\, \delta} \qquad \alpha\,\delta \,-\, \beta\,\gamma \,=\, +1\,. \tag{4}$$

This can be proved by comparing terms proportional to 1 and $\sqrt{D}$ on both sides of the equation with the help of the known transformation laws for a', b', c', which are $a' \,=\, \delta^2\,a \,-\, \delta\,\gamma\,b \,+\, \gamma^2\,c$, $b' \,=\, -2\delta\,\beta\,a \,+\, (\alpha\,\delta \,+\, \beta\,\gamma)\,b \,-\, 2\alpha\,\gamma\,c$ and $c' \,=\, \beta^2\,a \,-\, \alpha\,\beta\,b \,+\, \alpha^2\,c$. In [13], §17, p. 54, such numbers ω' and ω are called *equivalent (irrational) numbers* (*Det* $M \,=\, -1$ is also allowed there). This implies ([13], Satz 2.24, p. 55, attributed to *J. A. Serret*) that two irrational numbers have regular continued fractions which are identical starting with some partial quotient if and only if they are equivalent. Here we are dealing with (mixed, not purely) periodic regular continued fractions. For the relation to the *Pell* equations of special interest is the irrational number, called ω_p which belongs to the so called unique *principal form* [1], p. 26, $F_p \,:=\, \left[1,\, b,\, -\frac{D\,-\,b^2}{4}\right]$ ($b \,=\, b(D)$ will be determined uniquely) which defines the *principle period* (or cycle) of which it is an element. The irrational number $-\omega_p$ belongs to the form with a sign flip in a and c, *i.e.*, to $\left[-1,\, b,\, \frac{D\,-\,b^2}{4}\right]$.

Lemma 2: Principal reduced form [1, b, c] of D

The indefinite form $F_p \,=\, \left[1,\, b,\, -\frac{D\,-\,b^2}{4}\right]$ of discriminant D is unique and, with $f(D) \,=\, \left\lceil\sqrt{D}\right\rceil$, it is given by

$$b \,=\, b(D) \,=\, \begin{cases} \mathrm{f(D) - 2} & \text{if } D \text{ and } f(D) \text{ have the same parity,} \\[2ex] \mathrm{f(D) - 1} & \text{if } D \text{ and } f(D) \text{ have opposite parity.} \end{cases} \tag{5}$$

Later these two cases will be called case I and case II. See A226134 for these $b(D)$ values for D from A079896.

Proof: This follows from the definition for a reduced form in the version I) [14]. $0 \,<\, b \,\leq\, f(D)$ and because $2\,|a| \,=\, 2\,\cdot\,1$ and $|c| \,\geq\, 1$ one has $f(D) \,-\, 2 \,\leq\, b \,<\, f(D)$. Therefore, either $b \,=\, f(D) \,-\, 2$ or $b \,=\, f(D) - 1$. The parities of b and D have to be equal from the definition of D. Therefore, if the parity of D and $f(D)$ is the same (either even or odd), b has to be $f(D) - 2$, and in the other case $b \,=\, f(D) - 1$. $\square$

Examples for principal forms

$D \,=\, 5, f(D) \,=\, 3:\;\; [1,\,1,\,-1]$; $D \,=\, 28, f(D) \,=\, 6:\;\; [1,\,4,\,-3]$; $D \,=\, 20, f(D) \,=\, 5:\;\; [1,\,4,\,-1]$; $D \,=\, 13, f(D) \,=\, 4:\;\; [1,\,3,\,-1]$. They belong to the cases I, I, II and II, respectively.

The corresponding quadratic irrational depends only on D and it will be called $\omega_p(D) \,=\, \dfrac{-b(D) \,+\, \sqrt{D}}{2}$, where $b(D)$ is given in eq. (5) for the two cases I and II.

Lemma 3: $|\boldsymbol{\omega_p}(\mathbf{D})|$

For each discriminant $D(n) \,=$ A079896(n) , $n \,\geq\, 0$, the quadratic irrational $\omega_p(D(n))$ satisfies

$$\begin{aligned} \frac{1}{2} \;&<\; \omega_p(D(n)) \,<\, 1 \text{ if } D \text{ and } f(D) \text{ have the same parity, case I },\\ 0 \;&<\; \omega_p(D(n)) \,<\, \frac{1}{2} \text{ if } D \text{ and } f(D) \text{ have of opposite parity, case II }. \end{aligned} \tag{6}$$

Proof: This follows immediately from the two cases of *Lemma 2*. $\square$

Examples for values of $\boldsymbol{\omega_p}(D)$

Case I: $\omega_p(D(8)) \,=\, \omega_p(28) \,=\, -2 \,+\, \sqrt{7} \approx 0.645751311$ (Maple 13 [9], 10 digits).

Case II: $\omega_p(D(5)) \;=\; \omega_p(20) \;=\; -2 \,+\, \sqrt{5} \;=\; -3 \,+\, 2\,\Phi$, with the golden section $\Phi \;:=\; \dfrac{1 \,+\, \sqrt{5}}{2}$. $\omega_p(20) \,\approx\, 0.236067977$. Note that $<1, \Phi>$ is the basis for integers in the quadratic number field $\mathbb{Q}(\sqrt{5})$.

Lemma 4: $\omega_{\mathbf{p}}(\mathbf{D})$ is integer in a certain $\mathbb{Q}(\sqrt{\mathbf{m}}\,)$

Define $m(n) := sqfp(D(n)) =$A226693(n) as the square-free part of $D(n) =$A079896(n), $n \geq 0$. See A226693. Then $\omega_p(D(n)) \in \mathbb{Z}(\sqrt{m(n)}\,)$, the ring of integers of the real quadratic field $\mathbb{Q}(\sqrt{m(n)}\,)$ with basis $< 1, \omega(n) >$, where $\omega(n) = \sqrt{m(n)}$ if $m(n) \equiv 2$, or $3\,(mod\,4)$, and $\omega(n) = \dfrac{1 + \sqrt{m(n)}}{2}$ if $m(n) \equiv 1\,(mod\,4)$.

In [1] one finds in ch. 6, pp. 87-106 an account on quadratic number fields. For the basis of the ring of integers in $\mathbb{Q}(\sqrt{m}\,)$ (m is called there d) see *Proposition 6.6.* on p. 91. The ring of integers $\mathbb{Z}(\sqrt{m}\,)$ in $\mathbb{Q}(\sqrt{m}\,)$ is there, on p. 92, called $\mathcal{O}(\sqrt{d}\,)$. For more on this subject see, *e.g.*, *Hardy* and *Wright* [5].

Proof: The square-free part of $D =$A079896(n) is called $m(n) =$A226693(n) or sometimes (by abuse of notation) $m(D)$.

i) Consider first the case $D \equiv 0\,(mod\,4)$, not a square. This is the sequence 4∗A000037. The square-free part $m(D)$ of a $D = 4\,k$, $k = 1, 2, \ldots$, not a square, coincides with the square-free part of k, not a square, which is found as sequence A002734. D and b have always the same parity, which is here even, hence $b = 2\,b'$ and $\omega_p(D) = \dfrac{-2\,b' + 2\,a\,\sqrt{m(D)}}{2}$ with the positive integer $a := \dfrac{k(n)}{sqfp(k(n))}$. This is then an element of $\mathbb{Z}(\sqrt{m(D)}\,)$, also if $m(D) \equiv 1\,(mod\,4)$, which can happen, *e.g.*, for $D = 20$ and $m(D) = 5$. The coefficients in the basis $< 1, \omega(n) >$ are $(-(b' + a), 2\,a)$ if $m(n) \equiv 1\,(mod\,4)$ and $(-b', a)$ if $m(n) \equiv 2$ or $3\,(mod\,4)$.

ii) If $D \equiv 1\ (mod\,4)$ then $b = 2\,b' + 1$ (odd with D). The Ds are shown in A077425 and their square-free parts m in A226165. We will show that $m \equiv 1\,(mod\,4)$, and then the basis element is $\omega(n) = \dfrac{1 + \sqrt{m(n)}}{2}$. The quadratic irrational $\omega_p(D)$ turns out to be of the form $\dfrac{-(2\,b' + 1) + a\,\sqrt{m}}{2}$, with integer b' and an odd positive integer $a = 2\,a' + 1$. This will then produce the $\mathbb{Z}(\sqrt{m}\,)$ integer $\omega_p(D) = -(a' + b' + 1) + a\,\omega(\sqrt{m}\,)$. The proof that $m \equiv 1\,(mod\,4)$, follows from the prime number factorization of the odd D. Because it is $1\,(mod\,4)$ the number of odd primes of the type $3\,(mod\,4)$ with an odd exponent has to be even (including 0). Otherwise the $mod\,4$ arithmetic would produce a 3 instead of a 1. Then $m(D) = sqfp(D)$ will have, besides possible factors of distinct primes of the type $1\,(mod\,4)$, only an even number of distinct type $3\,(mod\,4)$ primes, therefore, $m(D) \equiv 1\ (mod\,4)$. $\square$

For the following topic of periods and continued fractions one will need the result that the $L-$tuple of t-values of the $\mathbf{R}$ matrices for the principal period alternates in sign, starting negative.

Lemma 5: $\vec{t}_p(D)$ entries for principal period alternate

The $L-$tuple for the principal period $\vec{t}_p(D) = (t_{p,1}, t_{p,2}, \ldots, t_{p,L(D)})$ satisfies $t_{p,1} < 0$, $t_{p,2} > 0$, ..., $t_{p,L(D)-1} < 0$, $t_{p,L(D)} > 0$.

Example: $\vec{t}_p(17) = (-1, 1, -3, 1, -1, 3)$. $L(17) = 2 \cdot 3 = 6$.

Proof: We omit the index p. Remember that all forms of a period are reduced. That $t_1 < 0$ is clear from *Lemma 1* because the principal reduced form $F_p(D) = F_0(D)$ has negative c-component. Therefore the t_1-value of the matric $\mathbf{R}(t_1)$ leading to the half-reduced right neighbour $F_1(D)$ has to be negative too, because of *Lemma 1*. The a component of $F_1(D)$ is the (negative) c component of $F_0(D)$, hence its c-component has to be positive due to the first statement in eq. (3), and t_2 has to be positive due to *Lemma 1*, *etc*. This shows again, that the length of the period L is even, because $F_L(D) = F_0(D)$, hence t_{L+1} is again negative, and t_L has to be positive. $\square$

The main topic of these notes is now the direct relationship between the principal period of length $L(D) = 2\,l(D)$ of discriminant D in the indefinite case and the regular continued fraction (*r.c.f.*) expansion of $\omega_p(D)$. Here we depart slightly from [1], sect. 3.3., p. 35ff., which starts (on p. 40) with the irrational for the non-reduced forms $\left(1,\, 0,\, -\dfrac{D}{4}\right)$ for even D, and $\left(1,\, 1,\, \dfrac{1-D}{4}\right)$ for odd D. We start with the (reduced) principal form F_p of D and its irrational $\omega_p(D)$. The identification of the partial quotients of the *r.c.f.* and the real numbers Z_j in the continued fraction algorithm will also differ from [1].

The principal period of indefinite discriminant D, starting with the principal form $F_p(D) = F_0(D)$, is obtained from applying repeatedly the equivalence transformation matrix $\mathbf{R}(t_j)$ (see above for the definition of this matrix) with the uniquely defined t_j. One can end this process when one reaches the original form after L steps, when $F_L(D) = F_0(D)$.

$$F_p(D) = F_0(D) \overset{\mathbf{R}(t_1)}{\sim} F_1(D) \overset{\mathbf{R}(t_2)}{\sim} F_2(D) \cdots F_{L-1}(D) \overset{\mathbf{R}(t_L)}{\sim} F_L(D) = F_0(D) \, . \tag{7}$$

The quadratic irrational $\omega_j(D)$ corresponding to the forms $F_j(D)$, for $j = 1, 2, ..., L, L = 2\,l$, is obtained from $\omega_{j-1}(D)$ like in eq. (4) with $\mathbf{M} = \mathbf{R}^{-1}(t_j)$. Here $\omega_0(D) = \omega_p(D)$. This corresponding chain of relations is denotes as follows.

$$\omega_p(D) = \omega_0(D) \overset{\mathbf{R}^{-1}(t_1)}{\sim} \omega_1(D) \overset{\mathbf{R}^{-1}(t_2)}{\sim} \omega_2(D) \cdots \omega_{L-1}(D) \overset{\mathbf{R}^{-1}(t_L)}{\sim} \omega_L(D) = \omega_0(D) \, . \tag{8}$$

Because $\mathbf{R}^{-1}(t_j) = Matrix([[t, 1], [-1, 0]])$ one has

$$-\,\omega_j(D) = t_j + \frac{1}{\omega_{j-1}(D)} \, , \; j = 1, 2, ..., L(D), \, L = 2\,l(D) \, . \tag{9}$$

Example: Chain of irrationals for $\mathbf{D = 28}$

The principal period is [1, 4, −3], [−3, 2, 2], [2, 2, −3], [−3, 4, 1]. $L(28) = 4$, and the $t-$tuple is $\vec{t}(28) = (-1, 1, -1, 4)$. The quadratic irrationals are

$$\omega_p(28) = \omega_0(28) = -2 + \sqrt{7}, \; \omega_1(28) = \frac{1 - \sqrt{7}}{3}, \; \omega_2(28) = \frac{-1 + \sqrt{7}}{2}, \; \omega_3(28) = \frac{2 - \sqrt{7}}{3} \, . \tag{10}$$

Note that only $\omega_p(D)$ has been proven to be an integer in $\mathbb{Q}(\sqrt{m(D)}\,)$.

Now the connection to the *r.c.f.* algorithm for $\omega_p(D)$ becomes clear, because the algorithm for the periodic *r.c.f.* $\omega_p(D) = [0, \overline{d_1, d_2, ..., d_{k(D)}}]$ (we always use the primitive period of length $k(D)$, the smallest sequence of positive integers d_j needed for the period, not multiples of this sequence). We omitted the arguments D for the partial quotients d_j. If one writes $\omega_p(D) = [0, d_1, d_2, ..., d_{n-1}, X_n]$, for $n = 1, 2, ...$ (using the notation of [1], sect. 3.3) we have for $n = 1\,, \omega_p = [0, X_1] = 1/X_1 := Z_0$ and the following recurrence.

Regular continued fraction recurrence for $\mathbf{Z_j}$ and determination of $\mathbf{d_j}$

$$X_j = \frac{1}{Z_{j-1}} \, , \;\; Z_j = \frac{1}{Z_{j-1}} - \left\lfloor \frac{1}{Z_{j-1}} \right\rfloor \, , \; Z_0 = \omega_p(D) = \omega_0(D) \text{ and } d_j = \left\lfloor \frac{1}{Z_{j-1}} \right\rfloor . \tag{11}$$

Example: $\omega_p(17) = Z_0 = \frac{-3 + \sqrt{17}}{2} \approx 0.561552813$. $d_1 = \left\lfloor \frac{1}{Z_0} \right\rfloor = 1$. $Z_1 \approx .780776406$, $d_2 = \left\lfloor \frac{1}{Z_1} \right\rfloor = 1$. $Z_2 \approx .280776407$, $d_3 = \left\lfloor \frac{1}{Z_1} \right\rfloor = 3$, *etc.* Here one will find the primitive *r.c.f.* period $(1, 1, 3)$. Note that one needs appropriate precision for the real numbers Z_j if one wants to compute large indexed d_js correctly. Observe that the primitive period length 3 coincides here with one half of the length $L = 6$ of the principal period for $D = 17$. This is not always the case. See the later *Lemma 6*.

This Z_j-recurrence $-Z_j = d_j - \dfrac{1}{Z_{j-1}}$ can obviously be mapped to the one for the periodic $\omega_j(D)$ chain, eq. (9), by identifying

$$\omega_j(D) = (-1)^j \, Z_j(D) \, , \;\; t_j = (-1)^j \, d_j \, , \; j = 0, 1, ... \tag{12}$$

Note that the d_js are indeed positive due to *Lemma 5*.

In order to compare the even length $L = 2l$ of the principal period of D with the length of the primitive period of the *r.c.f.* for $\omega_p(D)$, called $k(D)$ above, one observes that the reduced form $\hat{F}_p(D) = [-1, b(D), -c(D)]$ corresponding to $F_p(D) = [1, b(D), c(D)]$ (*via* sign flip of the outer components of $F_p(D)$) has the associated quadratic irrational $\hat{\omega}_p(D) = -\omega_p(D)$ due to $a = -1$, and because also the c sign has been changed, D remains invariant. Then it is clear that whenever this form $\hat{F}_p(D)$ appears in the principal period then the pattern of t values in the periodic ω_j-chain has to repeat with opposite signs, and this also happens in the principal period of forms. Therefore in this case $\hat{F}_p(D)$ will appear at one position after the middle of the period, *i.e.*, $F_{L/2=l}(D) = \hat{F}_p(D)$. We state this as a *Lemma*.

Lemma 6: Case $\hat{\mathbf{F}}_{\mathbf{p}}(\mathbf{D})$ in the principal period

If $\hat{F}_p(D) = [-1, b(D), -c(D)]$ is a member of the principal period of D of length $L(D) = 2l(D)$ which starts with the principal form $F_p(D) = [1, b(D), c(D)] = F_0(D)$ then $\hat{F}_p(D) = F_{L/2}$, and $t_{L(D)/2+k}(D) = -t_k$ for $k = 1, ..., L(D)/2$.

It is clear that in the case when $\hat{F}_p(D)$ is in the principal period (together with $F_p(D)$) then the *r.c.f.* recurrence corresponding to the ω_j chain, will produce the d_js of the primitive period already for $j = 1, 2 ..., l, l = \frac{L}{2}$. In this case the length of the primitive period of the *r.c.f.* $\omega_p(D)$, called $k(D)$, equals l. If, on the other hand, $\hat{F}_p(D)$ is not a member of the principal period then one has to consider the whole $\omega_p(D)$ chain of even length L to obtain the primitive period of the *r.c.f.* for $\omega_p(D)$. The length $L/2$ of the principal period of $D(n)$ is found under A226166(n), $n \geq 0$.

The connection to the solvability of the *Pell* eqs. $x^2 - D\,y^2 = +4$ for all indefinite discriminants D follows from its connection to so-called *automorphs* [1], sect. 3.2 (or [14], p. 106 , where this is called *automorphic substitution*). These are (nontrivial) equivalence transformations which transform a form $F(\mathbf{A}, \vec{x})$ into itself: $F(\mathbf{A}', \vec{x}') = F(\mathbf{A}, \vec{x})$, or $F'(D) = F(D)$ for short. Such a transformation exists certainly for the reduced principal form, because $F_L(D) = F_0(D) = F_p(D)$. The corresponding transformation matrix is $\mathbf{R}_L(D) := \mathbf{R}(t_1) \cdot \mathbf{R}(t_2) \cdots \mathbf{R}(t_L)$ (we omitted the argument D for the t_js). *E.g.*, $D = 17$, $L(17) = 6$, $\vec{t}(17) = (-1, 1, -3, 1, -1, 3)$ with $\mathbf{R}_6 = Matrix([[-9, -32], [-16, -57]])$ (the theorem on the determinant of products of matrices ensures that $Det\,\mathbf{R}_6 = +1$).

Under an automorph the quadratic irrational transforms also into itself: $\omega' = \omega$. We follow [1], pp. 31, 32. Due to eq. (4) this leads to the quadratic eq. $\gamma\,\omega^2 + (\delta - \alpha)\,\omega - \beta = 0$. Because ω is a solution of the eq. $a\,\omega^2 + b\,\omega + c = 0$, and the form F is supposed to be primitive, there exits an integer k such that $\gamma = k\,a$, $\delta - \alpha = k\,b$ and $-\beta = k\,c$. This implies for $D\,k^2 = (b^2 - 4\,a\,c)\,k^2 = (\delta - \alpha)^2 + 4\,\gamma\,\beta$, or, if also $\alpha\,\delta - \gamma\,\beta = +1$ is used,

$$(\alpha + \delta)^2 - D\left(\frac{\gamma}{a}\right)^2 = +4\,. \tag{13}$$

This is how a solution for the *Pell* $+4$ equation is guaranteed for each indefinite discriminant D and for each automorph $\mathbf{M}$ of some form of D via eq. (4) with $\omega' = \omega$. We will take the principal period of D which has the automorphic matrix $\mathbf{R_L}(D)$. This matrix has a simple structure, as shown in [14], p. 117, due to $\mathbf{R}(t) = Matrix([[0, -1], [1, t]])$.

Lemma 7: $\mathbf{R_n}(\vec{t})$ matrix [14]

$$\mathbf{R}_n(\vec{t}) := \mathbf{R}(t_1) \cdot \mathbf{R}(t_2) \cdots \mathbf{R}(t_n) = \begin{pmatrix} r_{n-1} & r_n \\ s_{n-1} & s_n \end{pmatrix},\ n = 1, 2, \ldots, \tag{14}$$

where $\{r_n\}$ and $\{s_n\}$ satisfy the three term recurrences

$$\begin{aligned} r_n &= t_n\,r_{n-1} - r_{n-2}\,, \quad \text{with the inputs } r_{-1} = 1,\ r_0 = 0\,, \\ s_n &= t_n\,r_{n-1} - s_{n-2}\,, \quad \text{with the inputs } s_{-1} = 0,\ s_0 = 1\,. \end{aligned} \tag{15}$$

Proof: The inputs fit for $\mathbf{R}_1(\vec{t}) = \mathbf{R}(t_1)$ because $r_1 = -1$ and $s_1 = t_1$. The recurrence $\mathbf{R}_n(\vec{t}) = \mathbf{R}(t_n) \cdot \mathbf{R}_{n-1}(\vec{t})$, for $n \geq 2$, leads then, with the Ansatz eq. (14), to the recurrences of eq. (15). $\square$

Note: Such three term recurrences have a combinatorial interpretation in terms of so called *Morse code* polynomials. See [4], p. 302, on *Euler*'s continuants and *Morse* code, and a comment under A084950.

For a discussion of the *Pell* +4 equation and the relation to automorphic substitutions see also [14], §32, p. 121ff. There the important result is given (Satz 83. p. 120) that each automorphic substitution matrix $\mathbf{M}$ for an indefinite form F of discriminant D, $F \overset{\mathbf{M}}{\sim} F$, is of the form $\mathbf{M} = \pm(\mathbf{R}_L(D))^k$, with some integer k.

This leads now to the fundamental solutions of the *Pell* +4 equation from eq, (13) because the entries of the $\mathbf{R}_n$ matrix are related to the convergents of *r.c.f.*s, called in [1], p. 35, P_n and Q_n for the numerator and denominator, respectively. Here we use the nth convergent of the *r.c.f.* $[0, d_1, ..., d_n] = \frac{P_n}{Q_n}$, (*i.e.*, *Buell*'s a_js are our d_js, and $d_0 = 0$).

$$\begin{aligned} P_n &= d_n\, P_{n-1} + P_{n-2}\,, \quad \text{with the inputs } \ P_{-1} = 1,\ P_0 = 0\,,\\ Q_n &= d_n\, Q_{n-1} + Q_{n-2}\,, \quad \text{with the inputs } \ Q_{-1} = 0,\ Q_0 = 1\,. \end{aligned} \tag{16}$$

Lemma 8: Identification of $\mathbf{P_n}$ and $\mathbf{Q_n}$ from $\mathbf{r_n}$ and $\mathbf{s_n}$

With the imaginary unit $i = \sqrt{-1}$ one has

$$P_n = i^{n\,(n+1)}\, r_n\,, \quad Q_n = i^{n\,(n+1)}\, s_n\,, \quad n \geq 1\,. \tag{17}$$

Proof: Elementary, by comparing eq. (16) with eq. (15).

2.1 Solutions of the Pell +4 equations

The matrix $\mathbf{M}$ used in eq. (4) with $\omega' = \omega = \omega_p(D)$ is now $\mathbf{R}_L^{-1}(\vec{t})$ for the principal period of D. $L = L(D)$ is the length of the principal period, and the matrix elements in eq. (4) are $\alpha = s_L$, $\delta = r_{L-1}$, $\beta = -r_L$ and $\gamma = -s_{L-1}$. Because for the principal form $a = 1$ we have the following proposition:

Proposition 1: Fundamental positive solution of the Pell +4 equation for each D

For each discriminant $D = D(n) =$ A079896(n) of indefinite binary quadratic forms the fundamental positive solution $(X(D), Y(D))$ of the *Pell* equation $x^2 - D\,y^2 = +4$ is, with $L(D) = 2\,l(D)$ the length of the principal period of D,

$$\begin{aligned} X(D) &= |\,s_{L(D)} + r_{L(D)-1}\,| = Q_{2\,l(D)} + P_{2\,l(D)-1}\,,\\ Y(D) &= |\,s_{L(D)-1}\,| = Q_{2\,l(D)-1}\,, \end{aligned} \tag{18}$$

where P_n and Q_n are defined by the three term recurrences eq. (16) in terms of the partial quotients d_j of the *r.c.f.* $\omega_p(D)$.

Proof: In eq. (13) $\alpha + \delta = (s_L(D) + r_{L(D)-1}) = (-1)^{l(D)\,(2\,l(D)+1)}\,(Q_{2\,l(D)} + P_{2\,l(D)-1})$ from eq. (17). We are taking the positive solution for $X(D)$, hence the first assertion follows. Similarly, $\gamma = -s_{L(D)-1} = -(-1)^{l(D)\,(2\,l(D)-1)}\,Q_{2\,l(D)-1}$, hence the positive value of the second assertion is obtained. The claim is that this is also the solution with the smallest positive values for Y (then also X). Here we can use proposition 83 (Satz 83) of [14] and the connection between the Pell +4 equation and the automorphs. Application of the (proper) automorphs $\mathbf{R}^k_{L(D)}$, with positive or negative k values means to consider non-primitive principal periods, and the $Y = |\,\gamma\,|$ element of this matrix product is always $|r_{|k|\,L(D)}| = P_{|k|\,L(D)}$. But the $\{P_n\}$ (and $\{Q_n\}$) sequences are (not necessary strictly) monotone (from their recurrences). Thus the $k = 1$ case provides the solution with the smallest positive Y-value. $\square$

See *Table 2* for the fundamental solutions $(X(D(n)), Y(D(n))$, for these *Pell* +4 equations, for $D(n) =$ A079896(n), $n = 0, 1, ..., 40$.

Example: Fundamental solution of the Pell +4 equation for $\mathbf{D\,=\,17}$

The first convergents of the *r.c.f.* for $\omega_p(17) = \frac{-3+\sqrt{17}}{2}$ for $n = 0, 1 ..., 6$ are obtained from $[0, 1, 1, 4, 5, 9, 32]$ for P and $[1, 1, 2, 7, 9, 16, 57]$ for Q. $(X(17), Y(17)) = (57 + 9, 16) = (66, 16) = 2*(33, 8)$ which is an improper fundamental solution, obtained from the positive proper solution of the *Pell* +1 equation for $D = 17$ which is $(33, 8)$. The proper (non-negative) fundamental solution is, of course, $(0, 1)$ (note that $gcd(0, n) = n$). Similarly, the improper (non-negative) fundamental solution of the *Pell* +4 equation is, of course $(2, 0)$. There is no proper fundamental solution for $D = 17$. See also [8].

All other positive solutions of the *Pell* +4 equation can be obtained from the fundamental positive one in the following way (see [14], eq. (149), p. 123). Positive powers of the inverse automorph $\mathbf{R}^{-1}_{L(D)}$ are associated with these solutions, playing the rôle of $\mathbf{M}$ in eq.(4) for $\omega' = \omega$. Because of $Det\left(\mathbf{R}^{-1}_{L(D)}\right)^k = +1$ (form the product theorem for *Det*) one finds the recurrence $\left(\mathbf{R}^{-1}_{L(D)}\right)^k + \left(\mathbf{R}^{-1}_{L(D)}\right)^{k-2} = \left(\mathbf{R}^{-1}_{L(D)}\right)^{k-1} \cdot \left(\mathbf{R}^{-1}_{L(D)} + \mathbf{R}_{L(D)}\right) = \left(\mathbf{R}^{-1}_{L(D)}\right)^{k-1} \cdot \mathbf{1}_2\,(s_{L(D)} + r_{L(D)-1})$. Because from eq. (17) one finds (we omit here the arguments D, and $L = 2\,l$) $s_L + r_{L-1} = (-1)^l\,X$ and $\alpha_k + \delta_k = s_{k\,L} + r_{k\,L-1} = (-1)^{k\,l}\,x_k$ with the positive $x_k = Q_{k\,L} + P_{k\,L-1}$, the recurrence $(-1)^{k\,l}\,x_k = (-1)^l\,X\,(-1)^{(k-1)\,l}\,x_{k-1} - (-1)^{(k-2)\,l}\,x_{k-2}$ follows. Similarly, for $\gamma_k = -s_{k\,L-1} = -(-1)^{k\,l}\,Q_{k\,L-1}$ and $y_k = Q_{k\,L-1}$ one has $-(-1)^{k\,l}\,y_k = (-1)^l\,X\,(-)(-1)^{(k-1)\,l}\,y_{k-1} - (-)(-1)^{(k-2)\,l}\,y_{k-2}$. After adjusting the inputs with $(x_0, y_0) = (2, 0)$ (the trivial improper solution) and $(x_1, y_1) = (X, Y)$ (counting backwards to find (x_1, y_1)) this results in the following *proposition*.

Proposition 2: All positive solutions of $\mathbf{x^2 + D\,y^2\,=\,+4}$ [14]

$$\begin{aligned} x_k(D) &= X(D)\,x_{k-1}(D) - x_{k-2}(D)\,,\ k \geq 1,\ x_{-1}(D) = X(D),\ x_0(D) = 2\,, \\ y_k(D) &= X(D)\,y_{k-1}(D) - y_{k-2}(D)\,,\ k \geq 1,\ y_{-1}(D) = -Y(D),\ y_0(D) = 0\,. \end{aligned} \tag{19}$$

The positive solutions are $(x_k(D), y_k(D))$, $k \geq 1$.

This is related to Chebyshev $S-$ and $T-$polynomials (for their coefficient table see A049310) in the following way.

$$\begin{aligned} x_k(D) &= 2\,S_k(X(D)) - X(D)\,S_{k-1}(X(D)),\ \ k \geq 1\,, \\ &= S_{k-1}(X(D)) - S_{k-2}(X(D)) = 2\,T_k\left(\frac{X(D)}{2}\right) \\ y_k(D) &= Y(D)\,S_{k-1}(X(D)),\ \ k \geq 1\,. \end{aligned} \tag{20}$$

Here $T_k(x)$ is a Chebyshev $T-$polynomial (see A053120 for the coefficients).

Note that proper and improper fundamental solutions may appear among these pair of sequences $(x_k(D), y_k(D))$, *E.g.*, $D = 5$ has improper solutions generated from $(2, 0)$, as well as proper ones generated from two different fundamental proper solutions $(3, 1)$ and $(7, 3)$. $D = 12$ has the positive solutions $(4, 1)$, $(14, 4)$, $(52, 15)$, $(194, 56)$, ... (2*A001075 and A001353) alternating between proper and improper solutions.

2.2 Solution of the Pell −4 equations

Solutions of the *Pell* −4 equation are related to equivalence transformation matrices $\mathbf{M}$ which yield in eq. (4) $\omega' = -\omega$. This is because then the quadratic equation becomes $\gamma\,\omega^2 + (\delta + \alpha)\,\omega + \beta = 0$. Because $a\,\omega^2 + b\,\omega + c = 0$, with $gcd(a, b, c) = 1$, one has now $\gamma = k\,a$, $\delta - \alpha = k\,b$ and $\beta = k\,c$. This implies for $D\,k^2 = (b^2 - 4\,a\,c)\,k^2 = (\delta + \alpha)^2 - 4\,\gamma\,\beta$, or, if also $\alpha\,\delta - \gamma\,\beta = +1$ is used,

$$(\alpha - \delta)^2 - D\left(\frac{\gamma}{a}\right)^2 = -4\,. \tag{21}$$

If $\omega(D)$ is the quadratic irrational for the form $F = [a, b, c]$ of discriminant D then $-\omega(D)$ belongs to the form $\hat{F} = [-a, b, -c]$ which we have met above. Because D has to stay invariant, this is the only way to produce this sign flip in ω. We refrain from giving the general equivalence transformation which belongs to $F \overset{\mathbf{M}}{\sim} \hat{F}$. For those D for which $\hat{F}_p$ is in the principal period it is clear from *Lemma 6* that this matrix is $\mathbf{R}_{L(D)/2} = \mathbf{R}_{l(D)}$. From its inverse one can read off the fundamental solution $\tilde{X}(D) = |\alpha - \delta| = |s_{l(D)} - r_{l(D)-1}| = Q_{l(D)} - (-1)^{l(D)} P_{l(D)-1}$ (because from the recurrences (16) follows by induction $Q_n \geq P_n \geq P_{n-1}$, $n \geq 0$). Similarly $\tilde{Y}(D) = |-s_{l(D)-1}| = Q_{l(D)-1}$ is the smallest positive integer solution. (See the discussion on fundamental solutions given for the *Pell* $+4$ case which applies here *mutatis mutandis*). This leads to the following *proposition*.

Proposition 3: Fundamental positive integer solution of the Pell -4 equation for specific D

For each discriminant $D = D(n) =$ A079896(n) of indefinite binary quadratic forms for which the form $\hat{F}_p = [-1, b(D), -c(D)]$ is a member of the fundamental period starting with $F_p = [1, b(D), c(D)]$ the fundamental positive solution $(\tilde{X}(D), \tilde{Y}(D))$ of the *Pell* equation $x^2 - D\,y^2 = -4$ is, with $L(D) = 2\,l(D)$ the length of the principal period of D,

$$\begin{aligned} \tilde{X}(D) &= Q_{l(D)} - (-1)^{l(D)} P_{l(D)-1} \,, \\ \tilde{Y}(D) &= Q_{l(D)-1} \,, \end{aligned} \tag{22}$$

where P_n and Q_n are defined by the three term recurrences eq. (16).

Again, the general positive integer solutions of the *Pell* $-\mathbf{4}$ equation can be found from $\left(\tilde{X}(D), \tilde{Y}(D)\right)$ and also $X(D)$ from the *Pell* $+\mathbf{4}$ case, by application of positive integer powers k of the inverse matrix $\mathbf{R}^{-1}_{L(D)}$. For the resulting three term recurrences, using the above given recurrence for $\left(\mathbf{R}^{-1}_{L(D)}\right)^k$, we need besides the inputs $\tilde{x}_1 = \tilde{X}$ and $\tilde{y}_1 = \tilde{Y}$ the one for $\tilde{x}_0$ and $\tilde{y}_0$. Here we have to take the solution $(-\tilde{X}, \tilde{Y})$, because this is the next smallest solution with nonnegative Y (there is, of course, no solution with $\tilde{Y} = 0$). Again, the inputs $(\tilde{x}_{-1}, \tilde{y}_{-1})$ are computed backwards and become $(-\tilde{X}\,(X+1), \tilde{Y}\,(X-1))$. This leads us to

Proposition 4: All positive integer solutions of the Pell -4 equation for specific D

For each discriminant $D = D(n) =$ A079896(n) of indefinite binary quadratic forms for which the form $\hat{F}_p = [-1, b(D), -c(D)]$ is a member of the fundamental period starting with $F_p = [1, b(D), c(D)]$ the positive integer solutions of $x^2 - D\,y^2 = -4$ are

$$\begin{aligned} \tilde{x}_k(D) &= X(D)\,\tilde{x}_{k-1}(D) - \tilde{x}_{k-2}(D)\,,\ k \geq 1,\ \tilde{x}_{-1}(D) = -\tilde{X}\,(X+1),\ \tilde{x}_0(D) = -\tilde{X}\,, \\ \tilde{y}_k(D) &= X(D)\,\tilde{y}_{k-1}(D) - \tilde{y}_{k-2}(D)\,,\ k \geq 1,\ \tilde{y}_{-1}(D) = \tilde{Y}\,(X-1),\ \tilde{y}_0(D) = \tilde{Y}\,. \end{aligned} \tag{23}$$

For these D values 5, 8, 13, 17, 20, 29, 37, 40, 41, ..., found under A226696, the positive solutions are $(\tilde{x}_n(D), \tilde{y}_n(D))$, $n \geq 1$.

This is related to Chebyshev $S-$polynomials in the following way.

$$\begin{aligned} \tilde{x}_k(D) &= \tilde{X}(D)\,(-S_k(X(D)) + (X(D)+1)\,S_{k-1}(X(D))) \\ &= \tilde{X}(D)\,(S_{k-1}(X(D)) + S_{k-2}(X(D)))\,,\ \ k \geq 1\,, \\ \tilde{y}_k(D) &= \tilde{Y}(D)\,(S_k(X(D)) - (X(D)-1)\,S_{k-1}(X(D))) \\ &= \tilde{Y}(D)\,(S_{k-1}(X(D)) - S_{k-2}(X(D)))\ \ k \geq 1\,. \end{aligned} \tag{24}$$

The pre-factors show, that the *Pell* -4 equation is solved with $\left(\tilde{X}(D)\,\tilde{x}'_k(D), \tilde{Y}(D)\,\tilde{y}'_k(D)\right)$ (where $\tilde{x}'_k(D)$ and $\tilde{y}'_k(D)$ can be read off eq. (24)), and whenever $\tilde{X}(D)$ and $\tilde{Y}(D)$ are even, one actually solves the *Pell* -1 equation for this D with $(\frac{\tilde{X}(D)}{2}\,\tilde{x}'_k(D), \frac{\tilde{Y}(D)}{2}\,\tilde{y}'_k(D))$, $k = 1$. See $D = 17, 37, 41, 65, 73, 89, 97, 101, \ldots$

Again, proper and improper fundamental solutions may appear among these pair of sequences $(\tilde{x}_k(D), \tilde{y}_k(D))$, *E.g.*, $D = 5$: the pairs come from three sources, one proper fundamental (positive) solution $(1, 1)$ with the proper descendants $(29, 13)$, $(521, 233)$, $(9349, 4181)$, ..., another proper fundamental solution $(11, 5)$ with descendants $(199, 89)$, $(3571, 1597)$, ... (here $(1, -1)$ with non-positive entries belongs), and the improper fundamental solution $(4, 2)$ with descendants $(76, 34)$, $(1364, 610)$, $(24476, 10946)$, ... Altogether this produces the $(\tilde{x}_k(5), \tilde{y}_k(5))$ sequences (A002878$(k-1)$, A001519(k)), $k \geq 1$.

See *Table 2* for the fundamental solutions $\left(\tilde{X}(D(n)), \tilde{Y}(D(n)\right)$, for these *Pell* -4 equations, for $D(n) =$ A079896(n), $n = 0, 1, ..., 40$.

Note: Another approach to find all positive solutions of the *Pell* equations is given in [13], §27, pp. 92-95 and [11], ch. VI, 58., pp. 204-212.

3 Fundamental units in the number field $\mathbb{Q}(\sqrt{\mathbf{m}}\,)$

For this topic see besides [1], sect. 6.2, pp. 89-93, also *Hardy-Wright*, sect. 14.4, p. 207ff., *Hasse* [6] and also [15], [16] and the program [10].

The units in $\mathbb{Q}(\sqrt{m}\,)$, $m \neq 1$, square-free (see A005117), are the integers which divide 1: ε is a unit if and only if $\varepsilon \in \mathbb{Z}(\sqrt{m}\,)$ and there exists a $\gamma \in \mathbb{Z}(\sqrt{m}\,)$ such that $1 = \varepsilon\,\gamma$. *I.e.*, ε has in the ring $\mathbb{Z}(\sqrt{m}\,)$ a (multiplicative) inverse $\varepsilon^{-1} = \gamma$. The units form a (multiplicative) commutative (*Abelian*) group $\mathbf{E}$. A subgroup of order 2 is the cyclic group $C_2 = \{-1, +1\}$ formed from the only units of $\mathbb{Q}$. These are called trivial units. For an element $\alpha = \frac{a+b\sqrt{m}}{2} \in \mathbb{Q}(\sqrt{m}\,)$ the *norm* N is defined as $N(\alpha) := \alpha\,\overline{\alpha}$ with $\overline{\alpha} = \frac{a-b\sqrt{m}}{2}$, *i.e.*, $N(\alpha) = \frac{a^2-b^2\,m}{4}$. It satisfies $N(\alpha\,\beta) = N(\alpha)\,N(\beta)$ (multiplicativity) and vanishes precisely for $\alpha = 0$. This norm can also be negative. The trace S (from *Spur*) is $S(\alpha) := \alpha + \overline{\alpha} = a$. Integers in $\mathbb{Q}(\sqrt{m}\,)$ (*i.e.*, elements of $\mathbb{Z}(\sqrt{m}\,)$) are the elements α which satisfy $N(\alpha) \in \mathbb{Z}$ and $S(\alpha) \in \mathbb{Z}$ (*i.e.*, the monic polynomial which has a root α has coefficients from $\mathbb{Z}$; it is an integer monic polynomial). If one uses for $\mathbb{Z}(\sqrt{m}\,)$ the basis $< 1, \omega >$, with $\omega = \dfrac{1+\sqrt{m}}{2}$ if $m \equiv 1\,(mod\,4)$ and $\omega = \sqrt{m}$ if $m \equiv 2$ or $3\,(mod\,4)$ (note the different definition of δ in [1], p. 92), then an integer is written as $\alpha = x\,1 + y\,\omega$, with x, $y \in \mathbb{Z}$ (we omit the argument m for ω and also the basis element 1 later on). The relation to a, b is $a = 2\,x + y$ and $b = y$ (*i.e.*, $a \pm b \equiv 0\,(mod\,2)$), in the first case, and $a = 2\,x$, $b = 2\,y$ in the second case. $\overline{\omega} = \dfrac{1-\sqrt{m}}{2}$ resp. $-\sqrt{m}$ for the two cases. This means that $\overline{\omega} = 1 - \omega$ or $\overline{\omega} = -\omega$ if $m \equiv 1\,(mod\,4)$ or $m \equiv 2$ or $3\,(mod\,4)$, respectively. Also $\omega + \overline{\omega} = 1$ or 0 and $\omega - \overline{\omega} = \sqrt{m}$ or $2\sqrt{m}$ for these two cases. Because with a unit $\varepsilon = x + y\,\omega$, with x and y rational integers (*i.e.*, from $\mathbb{Z}$), one finds

$$\varepsilon^{-1} = \frac{1}{x+y\,\omega} = \frac{1}{N(\varepsilon)}\,(x + y\,\overline{\omega}) = \frac{1}{N(\varepsilon)}\begin{cases}(x+y) - y\,\omega & \text{if } m \equiv 1\,(mod\,4)\,,\\ x - y\,\omega & \text{if } m \equiv 2 \text{ or } 3\,(mod\,4)\ .\end{cases} \tag{25}$$

This shows that ε is a unit of $\mathbb{Q}(\sqrt{m}\,)$ if and only if it is from $\mathbb{Z}(\sqrt{m}\,)$ and $N(\varepsilon) = \pm 1$ ($\varepsilon^{-1} \in \mathbb{Z}(\sqrt{m}\,)$). The non trivial units come in quadruples ε, ε^{-1}, $-\varepsilon$, $-\varepsilon^{-1}$ if $N(\varepsilon) = +1$ and ε, $-\varepsilon^{-1}$, $-\varepsilon$, ε^{-1} if $N(\varepsilon) = -1$. Uniqueness within each quadruple is achieved by demanding $\varepsilon > 1$ (the order from $\mathbb{R}$ is used here). See [6], p. 287, for a graphical interpretation using right angled hyperbolas for $N(\varepsilon) = \pm 1$.

Using the explicit form of $N(\varepsilon)$ this norm requirement becomes for $\varepsilon = x + y\,\omega$, x, $y \in \mathbb{Z}$,

$$(2\,x + y)^2 - m\,y^2 = \pm 4 \quad if\ m \equiv 1\,(mod\,4)\,, \tag{26}$$

$$x^2 - m\,y^2 = \pm 1 \quad if\ m \equiv 2 \text{ or } 3\,(mod\,4)\ . \tag{27}$$

The above given relation between a, b and x, y shows that in order to have an integer $\varepsilon = \dfrac{a+b\sqrt{m}}{2}$ one needs $a - b \equiv 0\,(mod\ 2)$ (a, b of the same parity) if $m \equiv 1\,(mod\,4)$, and a as well as b even if

$m \equiv 2$ or $3\,(mod\,4)$. This representation of ε complicates the *Pell* issue and we will not use a and b here.

Sometimes, like in [6], one uses instead of m the discriminant $d = d(1, \omega) := (Det\,Matrix([[1, 1][\omega, \overline{\omega}]]))^2 = (\overline{\omega} - \omega)^2 = m$ if $m \equiv 1\,(mod\,4)$ or $4\,m$ if $m \equiv 2$ or $3\,(mod\,4)$. In this case an $\varepsilon = \dfrac{u + v\sqrt{d}}{2}$ with rational integers u, v, and $N(\varepsilon) = \pm 1$ yields the *Pell* equations $u^2 - d\,v^2 = \pm 4$. We will consider here the above given version.

The *fundamental unit* ε_1 of $\mathbb{Q}(\sqrt{m}\,)$ is defined to be the smallest unit with $\varepsilon_1 > 1$ which means the smallest positive integers $2\,x + y$ and y in eq. (26) (note that x may vanish) and x and y in eq. (27), *i.e.*, the fundamental solutions of the *Pell* equations. Then all other units are given by $\varepsilon = (-1)^{\nu}\,\varepsilon_1^k$ with $\nu = 0$ or 1 and $k \in \mathbb{Z}$ (*e.g.*, [6], p. 288, VIIIb).

Examples for $\mathbf{m = 1\,(mod\,4)}$:

See *Table 2* for the fundamental *Pell* ± 4 solutions.

$m = 5$, the *Pell* -4 equation has the smallest positive solution, namely (1, 1), *i.e.*, $y = 1$, $x = 0$, therefore $\varepsilon_1(5) = \omega(5)$, which is the golden section $\Phi = \dfrac{1 + \sqrt{5}}{2}$.

$m = 13$, the *Pell* -4 equation has the smallest positive solution, namely the improper (3, 1), *i.e.*, $y = 1$, $x = 1$, therefore $\varepsilon_1(13) = 1 + \omega(13)$, which is $\dfrac{3 + \sqrt{13}}{2}$.

$m = 17$, the *Pell* -4 equation has the smallest positive solution, namely the improper (8, 2), *i.e.*, $y = 2$, $x = 3$, therefore $\varepsilon_1(17) = 3 + 2\,\omega(17)$, which is $4 + \sqrt{17}$.

$m = 21$, the *Pell* $+4$ equation has the fundamental solution (5, 1), There is no solution of the *Pell* -4 equation. *I.e.*, $y = 1$, $x = 2$, therefore $\varepsilon_1(21) = 2 + \omega(21)$, which is $\dfrac{5 + \sqrt{21}}{2}$.

$m = 29$, the smallest fundamental solution is (5, 1), coming from the *Pell* -4 equation. *I.e.*, $y = 1$, $x = 2$, therefore $\varepsilon_1(29) = 2 + \omega(29)$, which is $\dfrac{5 + \sqrt{29}}{2}$.

$m = 33$, the *Pell* $+4$ equation has the improper fundamental solution (46, 8), There is no solution of the *Pell* -4 equation. *I.e.*, $y = 8$, $x = 19$, therefore $\varepsilon_1(33) = 19 + 8\,\omega(33)$, which is $23 + 4\sqrt{33}$.

Examples for $\mathbf{m = 2\,(mod\,4)}$:

In this case there is, of course, no discriminant $D = m$, but such m values appear as square-free parts of certain Ds. Here one uses the known facts about the ± 1 *Pell* equations.

$m = 2$, the smallest fundamental solution is (1, 1) coming from the *Pell* -1 equation (or from the improper *Pell* -4 solution (2, 2) divided by 2). *I.e.*, $x = 1$, $y = 1$, therefore $\varepsilon_1(2) = 1 + \omega(2)$, which is $1 + \sqrt{2}$.

$m = 6$, the smallest fundamental solution is (5, 2) coming from the *Pell* $+1$ equation (or from the improper *Pell* $+4$ solution (10, 4) divided by 2). There is no solution of the *Pell* -1 equation (also none of the -4 equation). Therefore $\varepsilon_1(6) = 5 + 2\sqrt{6}$.

$m = 10$, the smallest fundamental solution is (3, 1) coming from the *Pell* -1 equation (or from the improper *Pell* -4 solution (6, 2) divided by 2). Therefore $\varepsilon_1(10) = 3 + \sqrt{10}$.

$m = 14$, the smallest fundamental solution is (15, 4) coming from the *Pell* $+1$ equation (or from the improper *Pell* $+4$ solution (30, 8) divided by 2). There is no solution of the *Pell* -1 equation (also none of the -4 equation). Therefore $\varepsilon_1(14) = 15 + 4\sqrt{14}$.

$m = 22$, the smallest fundamental solution is (197, 42) coming from the *Pell* $+1$ equation (or from the improper *Pell* $+4$ solution (394, 84) divided by 2). There is no solution of the *Pell* -1 equation (also none of the -4 equation). Therefore $\varepsilon_1(22) = 197 + 42\sqrt{22}$.

Examples for $\mathbf{m = 3\,(mod\,4)}$:

The same remark as above for the $m = 2\,(mod\,4)$ case applies here as well.

$m = 3$, the smallest fundamental solution is (2, 1) coming from the *Pell* +1 equation (there are no solutions for the −1 case). Therefore, $\varepsilon_1(3) = 2 + \sqrt{3}$.

$m = 7$, the smallest fundamental solution is (8, 3) coming from the *Pell* +1 equation (there are no solutions for the −1 case). Therefore, $\varepsilon_1(7) = 8 + 3\sqrt{7}$.

$m = 11$, the smallest fundamental solution is (10, 3) coming from the *Pell* +1 equation (there are no solutions for the −1 case). Therefore, $\varepsilon_1(11) = 10 + 3\sqrt{11}$.

$m = 15$, the smallest fundamental solution is (4, 1) coming from the *Pell* +1 equation (there are no solutions for the −1 case). Therefore, $\varepsilon_1(15) = 4 + \sqrt{15}$.

$m = 19$, the smallest fundamental solution is (170, 39) coming from the *Pell* +1 equation (there are no solutions for the −1 case). Therefore, $\varepsilon_1(19) = 170 + 39\sqrt{19}$.

$m = 23$, the smallest fundamental solution is (24, 5) coming from the *Pell* +1 equation (there are no solutions for the −1 case). Therefore, $\varepsilon_1(23) = 24 + 5\sqrt{23}$.

$m = 31$, the smallest fundamental solution is (1520, 273) coming from the *Pell* +1 equation (there are no solutions for the −1 case). Therefore, $\varepsilon_1(31) = 1520 + 273\sqrt{31}$.

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Table legends

Table 1

Discriminant $D = D(n) =$ A079896(n), $n = 0, 1, ..., 50$. $f = f(D(n)) = \left\lfloor \sqrt{D(n)} \right\rfloor$. I indicates that D and f have the same parity, II indicates that D and f have opposite parity. $m = m(n) = sqfp(D(n)) =$ A226693(n), the square-free part of $D(n)$. F_p is the principal form of $D(n)$. The quadratic irrational associated to F_p is ω_p, which is given in the basis $< 1, \omega(m(n)) >$ and in a decimal approximation. See also the fourth column in *Table 2* where its explicit form is given. *Pell* -4 and $+4$ indicates which of the two types of *Pell* equations have solutions. *r.c.f.* gives the periodic regular continued fractions, where overlining marks the (primitive) period. Finally, $\vec{t}$ lists the t-values for the transformations in the principal period, starting with the principal form F_p for each discriminant $D(n)$, see eq. (7) and the definition of $\mathbf{R}(t)$.

Table 2

Discriminant $D = D(n) =$ A079896(n), $n = 0, 1, ..., 40$. $L = L(D(n))$ is the length of the principal period. ω_p is the quadratic irrational associated to the principal form F_p (see *Table 1*, the sixth column, and also the next column). The convergents of the periodic regular continued fraction of ω_p are in the column ω_p cvgts. The (positive) fundamental solution of the *Pell* $+4$ equation with $D(n)$ is given in column *Pell* (X, Y) (see eq. (18), and similarly $(\tilde{X}, \tilde{Y})$ is the (positive) fundamental solution of the *Pell* -4 equation with $D(n)$, if existent (see eq. (22)). *no. sol.* stands for 'no solution exists'. The A-numbers [12] for the general positive solutions for both cases are listed in the columns (x_k, y_k) and $(\tilde{x}_k, \tilde{y}_k)$. The offset for k is here 1.

Table 1: Principal forms of discriminant D, their irrationals, continued fractions, and Pell equations

n	D	f	I / II	m	$\mathbf{F_p}$	$\boldsymbol{\omega}_p$	Pell -4, $+4$	r.c.f.	$\tilde{\mathbf{t}}$
0	**5**	3	I	5	$[1,\ 1,\ -1]$	$(-1,1),\ \approx 0.618033988$	$-4,\ +4$	$[0;\overline{1}]$	$(-1,\ 1)$
1	**8**	3	II	2	$[1,\ 2,\ -1]$	$(-1,1),\ \approx 0.414213562$	$-4,\ +4$	$[0;\overline{2}]$	$(-2,\ 2)$
2	**12**	4	I	3	$[1,\ 2,\ -2]$	$(-1,1),\ \approx 0.732050808$	$+4$	$[0;\overline{1,2}]$	$(-1,\ 2)$
3	**13**	4	II	13	$[1,\ 3,\ -1]$	$(-2,1),\ \approx 0.302775638$	$-4,\ +4$	$[0;\overline{3}]$	$(-3,\ 3)$
4	**17**	5	I	17	$[1,\ 3,\ -2]$	$(-2,1),\ \approx 0.561552813$	$-4,\ +4$	$[0;\overline{1,1,3}]$	$(-1,\ 1,\ -3,\ 1,\ -1,\ 3)$
5	**20**	5	II	5	$[1,\ 4,\ -1]$	$(-2,1),\ \approx 0.236067977$	$-4\ +4$	$[0;\overline{4}]$	$(-4,\ 4)$
6	**21**	5	I	21	$[1,\ 3,\ -3]$	$(-2,1),\ \approx 0.791287848$	$+4$	$[0;\overline{1,3}]$	$(-1,\ 3)$
7	**24**	5	II	6	$[1,\ 4,\ -2]$	$(-2,1),\ \approx 0.449489743$	$+4$	$[0;\overline{2,4}]$	$(-2,\ 4)$
8	**28**	6	I	7	$[1,\ 4,\ -3]$	$(-2,\ 1),\ \approx 0.645751311$	$+4$	$[0;\overline{1,1,1,4}]$	$(-1,\ 1,\ -1,\ 4)$
9	**29**	6	II	29	$[1,\ 5,\ -1]$	$(-3,\ 1),\ \approx 0.192582404$	$-4,\ +4$	$[0;\overline{5}]$	$(-5,\ 5)$
10	**32**	6	I	2	$[1,\ 4,\ -4]$	$(-2,\ 2),\ \approx 0.828427124$	$+4$	$[0;\overline{1,4}]$	$(-1,\ 4)$
11	**33**	6	II	33	$[1,\ 5,\ -2]$	$(-3,\ 1),\ \approx 0.372281324$	$+4$	$[0;\overline{2,1,2,5}]$	$(-2,\ 1,\ -2,\ 5)$
12	**37**	7	I	37	$[1,\ 5,\ -3]$	$(-3,\ 1),\ \approx 0.541381265$	$-4,\ +4$	$[0;\overline{1,1,5}]$	$(-1,\ 1,\ -5,\ 1,\ -1,\ 5)$
13	**40**	7	II	10	$[1,\ 6,\ -1]$	$(-3,\ 1),\ \approx 0.162277660$	$-4,\ +4$	$[0;\overline{6}]$	$(-6,\ 6)$
14	**41**	7	I	41	$[1,\ 5,\ -4]$	$(-3,\ 1),\ \approx 0.701562118$	$-4,\ +4$	$[0;\overline{1,2,2,1,5}]$	$(-1,\ 2,\ -2,\ 1,\ -5,\ 1,\ -1,\ 2,\ -1,\ 5)$
15	**44**	7	II	11	$[1,\ 6,\ -2]$	$(-3,\ 1),\ \approx 0.316624790$	$+4$	$[0;\overline{3,6}]$	$(-3,\ 6)$
16	**45**	7	I	5	$[1,\ 5,\ -5]$	$(-4,\ 3),\ \approx 0.854101966$	$+4$	$[0;\overline{1,5}]$	$(-1,\ 5)$
17	**48**	7	II	3	$[1,\ 6,\ -3]$	$(-3,\ 2),\ \approx 0.464101616$	$+4$	$[0;\overline{2,6}]$	$(-2,\ 6)$
18	**52**	8	I	13	$[1,\ 6,\ -4]$	$(-3,\ 1),\ \approx 0.605551275$	$-4,\ +4$	$[0;\overline{1,1,1,1,6}]$	$(-1,\ 1,\ -1,\ 1,\ -6,\ 1,\ -1,\ 1,\ -1,\ 6)$
19	**53**	8	II	53	$[1,\ 7,\ -1]$	$(-4,\ 1),\ \approx 0.140054944$	$-4,\ +4$	$[0;\overline{7}]$	$(-7,\ 7)$
20	**56**	8	I	14	$[1,\ 6,\ -5]$	$(-3,\ 1),\ \approx 0.741657387$	$+4$	$[0;\overline{1,2,1,6}]$	$(-1,\ 2,\ -1,\ 6)$
21	**57**	8	II	57	$[1,\ 7,\ -2]$	$(-4,\ 1),\ \approx 0.274917218$	$+4$	$[0;\overline{3,1,1,1,3,7}]$	$(-3,\ 1,\ -1,\ 1,\ -3,\ 7)$
22	**60**	8	I	15	$[1,\ 6,\ -6]$	$(-3,\ 1),\ \approx 0.872983346$	$+4$	$[0;\overline{1,6}]$	$(-1,\ 6)$
23	**61**	8	II	61	$[1,\ 7,\ -3]$	$(-4,\ 1),\ \approx 0.405124838$	$-4,\ +4$	$[0;\overline{2,2,7}]$	$(-2,\ 2,\ -7,\ 2,\ -2,\ 7)$
24	**65**	9	I	65	$[1,\ 7,\ -4]$	$(-4,\ 1),\ \approx 0.531128874$	$-4,\ +4$	$[0;\overline{1,1,7}]$	$(-1,\ 1,\ -7,\ 1,\ -1,\ 7)$
25	**68**	9	II	17	$[1,\ 8,\ -1]$	$(-4,\ 1),\ \approx 0.123105626$	$-4,\ +4$	$[0;\overline{8}]$	$(-8,\ 8)$

Table 1 (cont'd): Principal forms of discriminant D, their irrationals, continued fractions, and Pell equations

n	**D**	**f**	**I / II**	**m**	$\mathbf{F_p}$	$\boldsymbol{\omega_p}$	**Pell** $-4, +4$	**r.c.f.**	$\tilde{\mathbf{t}}$
26	**69**	9	I	69	$[1, 7, -5]$	$(-4, 1), \approx 0.653311932$	$+4$	$[0;\overline{1,1,1,7}]$	$(-1, 1, -1, 7)$
27	**72**	9	II	2	$[1, 8, -2]$	$(-4, 3), \approx 0.242640686$	$+4$	$[0;\overline{4,8}]$	$(-4, 8)$
28	**73**	9	I	73	$[1, 7, -6]$	$(-4, 1), \approx 0.772001872$	$-4, +4$	$[0;\overline{1,3,2,1,1,2,3,1,7}]$	$(-1, 3, -2, 1, -1, 2, -3, 1, -7,$ $1, -3, 2, -1, 1, -2, 3, -1, 7)$
29	**76**	9	II	19	$[1, 8, 3]$	$(-4, 1), \approx 0.358898944$	$+4$	$[0;\overline{2,1,3,1,2,8}]$	$(-2, 1, -3, 1, -2, 8)$
30	**77**	9	I	77	$[1, 7, -7]$	$(-4, 1), \approx 0.887482194$	$+4$	$[0;\overline{1,7}]$	$(-1, 7)$
31	**80**	9	II	5	$[1, 8, -4]$	$(-4, 2), \approx 0.472135954$	$+4$	$[0;\overline{2,8}]$	$(-2, 8)$
32	**84**	10	I	21	$[1, 8, -5]$	$(-4, 1), \approx 0.582575695$	$+4$	$[0;\overline{1,1,2,1,1,8}]$	$(-1, 1, -2, 1, -1, 8)$
33	**85**	10	II	85	$[1, 9, -1]$	$(-5, 1), \approx 0.109772228$	$-4, +4$	$[0;\overline{9}]$	$(-9, 9)$
34	**88**	10	I	22	$[1, 8, -6]$	$(-4, 1), \approx 0.690415760$	$+4$	$[0;\overline{1,2,4,2,1,8}]$	$(-1, 2, -4, 2, -1, 8)$
35	**89**	10	II	89	$[1, 9, -2]$	$(-5, 1), \approx 0.216990566$	$-4, +4$	$[0;\overline{4,1,1,1,1,4,9}]$	$(-4, 1, -1, 1, -1, 4, -9,$ $4, -1, 1, -1, 1, -4, 9)$
36	**92**	10	I	23	$[1, 8, -7]$	$(-4, 1), \approx 0.795831523$	$+4$	$[0;\overline{1,3,1,8}]$	$(-1, 3, -1, 8)$
37	**93**	10	II	93	$[1, 9, -3]$	$(-5, 1), \approx 0.321825380$	$+4$	$[0;\overline{3,9}]$	$(-3, 9)$
38	**96**	10	I	6	$[1, 8, -8]$	$(-4, 2), \approx 0.898979486$	$+4$	$[0;\overline{1,8}]$	$(-1, 8)$
39	**97**	10	II	97	$[1, 9, -4]$	$(-5, 1), \approx 0.424428901$	$-4, +4$	$[0;\overline{2,2,1,4,4,1,2,2,9}]$	$(-2, 2, -1, 4, -4, 1, -2, 2, -9$ $2, -2, 1, -4, 4, -1, 2, -2, 9)$
40	**101**	11	I	101	$[1, 9, -5]$	$(-5, 1), \approx 0.524937810$	$-4, +4$	$[0;\overline{1,1,9}]$	$(-1, 1, -9, 1, -1, 9)$
41	**104**	11	II	26	$[1, 10, -1]$	$(-5, 1), \approx 0.099019514$	$-4, +4$	$[0;\overline{10}]$	$(-10, 10)$
42	**105**	11	I	105	$[1, 9, -6]$	$(-5, 1), \approx 0.623475385$	$+4$	$[0;\overline{1,1,1,1,1,9}]$	$(-1, 1, -1, 1, -1, 9)$
43	**108**	11	II	3	$[1, 10, -2]$	$(-5, 3), \approx 0.196152424$	$+4$	$[0;\overline{5,10}]$	$(-5, 10)$
44	**109**	11	I	109	$[1, 9, -7]$	$(-5, 1), \approx 0.720153255$	$-4, +4$	$[0;\overline{1,2,1,1,2,1,9}]$	$(-1, 2, -1, 1, -2, 1, -9,$ $1, -2, 1, -1, 2, -1, 9)$
45	**112**	11	II	7	$[1, 10, -3]$	$(-5, 2), \approx 0.291502622$	$+4$	$[0;\overline{3,2,3,10}]$	$(-3, 2, -3, 10)$
46	**113**	11	I	113	$[1, 9, -8]$	$(-5, 1), \approx 0.815072905$	$-4, +4$	$[0;\overline{1,4,2,2,4,1,9}]$	$(-1, 4, -2, 2, -4, 1, -9,$ $1, -4, 2, -2, 4, -1, 9)$
47	**116**	11	II	29	$[1, 10, -4]$	$(-5, 1), \approx 0.385164807$	$-4, +4$	$[0;\overline{2,1,1,2,10}]$	$(-2, 1, -1, 2, -10,$ $2, -1, 1, -2, 10)$
48	**117**	11	I	13	$[1, 9, -9]$	$(-6, 3), \approx 0.908326912$	$+4$	$[0;\overline{1,9}]$	$(-1, 9)$
49	**120**	11	II	30	$[1, 10, -5]$	$(-5, 1), \approx 0.477225575$	$+4$	$[0;\overline{2,10}]$	$(-2, 10)$
50	**124**	12	I	31	$[1, 10, -6]$	$(-5, 1), \approx 0.567764363$	$+4$	$[0;\overline{1,1,3,5,3,1,1,10}]$	$(-1, 1, -3, 5, -3, 1, -1, 10)$

Table 2: Discriminant D, L(D), $\boldsymbol{\omega_p}$ convergents, and Pell ± 4 equations

n	D	L	ω_p	ω_p cvgts	Pell (X, Y)	Pell $(\tilde{X}, \tilde{Y})$	(x_k, y_k)	$(\tilde{x}_k, \tilde{y}_k)$
0	**5**	2	$\frac{-1+\sqrt{5}}{2}$	$\left[\frac{0}{1}, \frac{1}{1}, \frac{1}{2}\right]$	(3, 1)	(1, 1)	(A005248, A001906)	(A002878$(k-1)$, A001519)
1	**8**	2	$-1+\sqrt{2}$	$\left[\frac{0}{1}, \frac{1}{2}, \frac{2}{5}\right]$	(6, 2)	(2, 1)	($2 \cdot$ A001541, $2 \cdot$ A001109)	($2 \cdot$ A002315$(k-1)$, A001653)
2	**12**	2	$-1+\sqrt{3}$	$\left[\frac{0}{1}, \frac{1}{1}, \frac{2}{3}\right]$	(4, 1)	no sol.	($2 \cdot$ A001075, A001353)	no solutions
3	**13**	2	$\frac{-3+\sqrt{13}}{2}$	$\left[\frac{0}{1}, \frac{1}{3}, \frac{3}{10}\right]$	(11, 3)	(3, 1)	(A057076, $3 \cdot$ A004190$(k-1)$)	($3 \cdot$ A097783$(k-1)$, A078922)
4	**17**	6	$\frac{-3+\sqrt{17}}{2}$	$\left[\frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{4}{7}, \frac{5}{9}, \frac{9}{16}, \frac{32}{57}\right]$	(66, 16)	(8, 2)	($2 \cdot$ A099370, $16 \cdot$ A097316$(k-1)$)	($8 \cdot$ A078989$(k-1)$, $2 \cdot$ A078988$(k-1)$)
5	**20**	2	$-2+\sqrt{5}$	$\left[\frac{0}{1}, \frac{1}{4}, \frac{4}{17}\right]$	(18, 4)	(4, 1)	($2 \cdot$ A023039, $4 \cdot$ A049660$(k-1)$)	($4 \cdot$ A049629$(k-1)$, A007805$(k-1)$)
6	**21**	2	$\frac{-3+\sqrt{21}}{2}$	$\left[\frac{0}{1}, \frac{1}{1}, \frac{3}{4}\right]$	(5, 1)	no sol.	(A003501, A004254)	no solution
7	**24**	2	$-2+\sqrt{6}$	$\left[\frac{0}{1}, \frac{1}{2}, \frac{4}{9}\right]$	(10, 2)	no sol.	($2 \cdot$ A001079, $2 \cdot$ A004189)	no solution
8	**28**	4	$-2+\sqrt{7}$	$\left[\frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{9}{14}\right]$	(16, 3)	no sol.	($2 \cdot$ A001081, $3 \cdot$ A077412$(k-1)$)	no solution
9	**29**	2	$\frac{-5+\sqrt{29}}{2}$	$\left[\frac{0}{1}, \frac{1}{5}, \frac{5}{26}\right]$	(27, 5)	(5, 1)	(A090248, $5 \cdot$ A097781$(k-1)$)	($5 \cdot$ A097834$(k-1)$, A097835(k-1))
10	**32**	2	$-2+\sqrt{2}$	$\left[\frac{0}{1}, \frac{1}{1}, \frac{4}{5}\right]$	(6, 1)	no sol.	($2 \cdot$ A001541, A001109)	no solution
11	**33**	4	$\frac{-5+\sqrt{33}}{2}$	$\left[\frac{0}{1}, \frac{1}{2}, \frac{1}{3}\ \frac{3}{8}, \frac{16}{43}\right]$	(46, 8)	no sol.	($2 \cdot$ A174748$(k+1)$, $2 \cdot$ A174772$(k+1)$)	no solution
12	**37**	6	$\frac{-5+\sqrt{37}}{2}$	$\left[\frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{6}{11}, \frac{7}{13}, \frac{13}{24}, \frac{72}{133}\right]$	(146, 24)	(12, 2)	($2 \cdot$ A174747$(k+1)$, $2 \cdot$ A174775$(k+1)$)	($12 \cdot$ A097729$(k-1)$, $2 \cdot$ A097730$(k-1)$)
13	**40**	2	$-3+\sqrt{10}$	$\left[\frac{0}{1}, \frac{1}{6}, \frac{6}{37}\right]$	(38, 6)	(6, 1)	($2 \cdot$ A078986, $6 \cdot$ A078987$(k-1)$)	($6 \cdot$ A097314$(k-1)$, A097315$(k-1)$)
14	**41**	10	$\frac{-5+\sqrt{41}}{2}$	$\left[\frac{0}{1}, \frac{1}{1}, \frac{2}{3}, \frac{5}{7}, \frac{7}{10}, \frac{40}{57}, \frac{47}{67}, \frac{134}{191}, \frac{315}{449}, \frac{449}{640}, \frac{2560}{3649}\right]$	(4098, 640)	(64, 10)	($2 \cdot$ A174752$(k+1)$, $2 \cdot$ A174778$(k+1)$)	($64 \cdot$ A226694$(k-1)$, $10 \cdot$ A226695$(k-1)$)
15	**44**	2	$-3+\sqrt{11}$	$\left[\frac{0}{1}, \frac{1}{3}, \frac{6}{19}\right]$	(20, 3)	no sol.	($2 \cdot$ A001085, $3 \cdot$ A075843)	no solution
16	**45**	2	$\frac{-5+3\sqrt{5}}{2}$	$\left[\frac{0}{1}, \frac{1}{3}, \frac{5}{6}\right]$	(7, 1)	no sol.	(A056854, A004187)	no solution

continued

Table 2 (cont'd): Discriminant D, L(D), ω_p convergents, and Pell ± 4 equations

n	**D**	**L**	ω_p	ω_p **cvgts**	**Pell** $(\mathbf{X},\ \mathbf{Y})$	**Pell** $(\tilde{\mathbf{X}},\ \tilde{\mathbf{Y}})$	$(\mathbf{x_k},\ \mathbf{y_k})$	$(\tilde{\mathbf{x}}_\mathbf{k},\ \tilde{\mathbf{y}}_\mathbf{k})$
17	**48**	2	$-3 + 2\sqrt{3}$	$\left[\frac{0}{1}, \frac{1}{2}, \frac{6}{13}\right]$	(14, 2)	no sol.	$(2\cdot$ A011943$(k+1)$, $2\cdot$ A007655$(k+1))$	no solution
18	**52**	10	$-3 + \sqrt{13}$	$\left[\frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{20}{33}, \frac{23}{38}, \frac{43}{71}, \frac{66}{109}, \frac{109}{180}, \frac{720}{1189}\right]$	(1298, 180)	(36, 5)	$(2\cdot$ A114047, A075871$)$	$(2\cdot$ A202155, A202156$)$
19	**53**	2	$\frac{-7+\sqrt{53}}{2}$	$\left[\frac{0}{1}, \frac{1}{7}, \frac{7}{50}\right]$	(51, 7)	(7, 1)	(A099368, $7\cdot$ A097837$(k-1))$	$(7\cdot$ A097836$(k-1)$, A097838$(k-1))$
20	**56**	4	$-3 + \sqrt{14}$	$\left[\frac{0}{1}, \frac{1}{1}, \frac{2}{3}, \frac{3}{4}, \frac{20}{27}\right]$	(30, 4)	no sol.	$(2\cdot$ A068203, $4\cdot$ A097313$(k-1))$	no solution
21	**57**	6	$\frac{-7+\sqrt{57}}{2}$	$\left[\frac{0}{1}, \frac{1}{3}, \frac{1}{4}, \frac{2}{7}, \frac{3}{11}, \frac{11}{40}, \frac{80}{291}\right]$	(302, 40)	no sol.	$(2\cdot$ A174759$(k+1)$ $2\cdot$ A175015$(k+1))$	no solution
22	**60**	2	$-3 + \sqrt{15}$	$\left[\frac{0}{1}, \frac{1}{1}, \frac{6}{7}\right]$	(8, 1)	no sol.	$(2\cdot$ A001091, A001090$)$	no solution
23	**61**	6	$\frac{-7+\sqrt{61}}{2}$	$\left[\frac{0}{1}, \frac{1}{2}, \frac{2}{5}, \frac{15}{37}, \frac{32}{79}, \frac{79}{195}, \frac{585}{1444}\right]$	(1523, 195)	(39, 5)	(A226669, $3\cdot 5\cdot 13\cdot$ A226700$)$	$(39\cdot$ A266701$(k-1)$, $5\cdot$ A266702$(k-1))$
24	**65**	6	$\frac{-7+\sqrt{65}}{2}$	$\left[\frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{8}{15}, \frac{9}{17}, \frac{17}{32}, \frac{128}{241}\right]$	(258, 32)	(16, 2)	$(2\cdot$ A176368$(k-1)$, $2\cdot$ A176369$(k-1))$	$(2\cdot$ A097736$(k-1)$, $16\cdot$ A097735$(k-1))$
25	**68**	2	$-4 + \sqrt{17}$	$\left[\frac{0}{1}, \frac{1}{8}, \frac{8}{65}\right]$	(66, 8)	(8, 1)	$(2\cdot$ A099370$(k-1)$, $8\cdot$ A097316$)$	$(8\cdot$ A078989, A078988$(k-1)$
26	**69**	4	$\frac{-7+\sqrt{69}}{2}$	$\left[\frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{15}{23}\right]$	(25, 3)	no sol.	(A090733, $3\cdot$ A097780$(k-1))$	no solution
27	**72**	2	$-4 + 3\sqrt{2}$	$\left[\frac{0}{1}, \frac{1}{4}, \frac{8}{33}\right]$	(34, 4)	no sol.	$(2\cdot$ A056771, $4\cdot$ A029547$(k-1))$	no solution
28	**73**	18	$\frac{-7+\sqrt{73}}{2}$	$\left[\frac{0}{1}, \frac{1}{1}, \frac{3}{4}, \frac{7}{9}, \frac{10}{13}, \frac{17}{22}, \frac{44}{57}, \frac{149}{193}, \frac{193}{250}, \frac{1500}{1943}, \frac{1693}{2193}, \frac{6579}{8522}, \frac{14851}{19237}, \frac{21430}{27759}, \frac{36281}{46996}, \frac{93992}{121751} \frac{318257}{412249}, \frac{412249}{534000}, \frac{3204000}{4150249}\right]$	(4562498, 534000)	(2136, 250)	$(2\cdot$ A176382, $2\cdot$ A176384$(k+1))$	$(2^3\cdot 3\cdot 89\cdot$ A227039$(k-1)$, $2\cdot 5^3\cdot$ A227040$(k-1))$
29	**76**	6	$\frac{-4+\sqrt{19}}{2}$	$\left[\frac{0}{1}, \frac{1}{2}, \frac{1}{3}, \frac{4}{11}, \frac{5}{14}, \frac{14}{39}, \frac{117}{326}\right]$	(340, 39)	no sol.	$(2\cdot$ A114048$(k+1)$, A174765$(k+1))$	no solution

continued

Table 2 (cont'd): Discriminant D, L(D), ω_p convergents, and Pell ± 4 equations

n	D	L	ω_p	ω_p cvgts	Pell (X, Y)	Pell $(\tilde{X}, \tilde{Y})$	(x_k, y_k)	$(\tilde{x}_k, \tilde{y}_k)$
30	**77**	2	$\frac{-7+\sqrt{77}}{2}$	$\left[\frac{0}{1}, \frac{1}{1}, \frac{7}{8}\right]$	(9, 1)	no sol.	(A056918, A018913)	no solution
31	**80**	2	$-4 + 2\sqrt{5}$	$\left[\frac{0}{1}, \frac{1}{2}, \frac{8}{17}\right]$	(18, 2)	no sol.	$(2 \cdot$ A023039, $2 \cdot$ A049660)	no solution
32	**84**	6	$-4 + \sqrt{21}$	$\left[\frac{0}{1}, \frac{1}{1}, \frac{3}{5}, \frac{4}{7}, \frac{7}{12}, \frac{60}{103}\right]$	(110, 12)	no sol.	$(2 \cdot$ A114049, A174745$(k+1))$	no solution
33	**85**	2	$\frac{-9+\sqrt{85}}{2}$	$\left[\frac{0}{1}, \frac{1}{9}, \frac{9}{82}\right]$	(83, 9)	(9, 1)	(A099373, $9 \cdot$ A097839)	$(9 \cdot$ A097840$(k-1)$, A097841$(k-1)$
34	**88**	6	$-4 + \sqrt{22}$	$\left[\frac{0}{1}, \frac{1}{1}, \frac{9}{13}, \frac{20}{29}, \frac{29}{42}, \frac{252}{365}\right]$	(394, 42)	no sol.	$(2 \cdot$ A114050$(k+1)$, A174766$(k+1))$	no solution
35	**89**	14	$\frac{-9+\sqrt{89}}{2}$	$\left[\frac{0}{1}, \frac{1}{4}, \frac{1}{5}, \frac{2}{9}, \frac{3}{14}, \frac{5}{23}, \frac{23}{106}, \frac{212}{977}, \frac{871}{4014}, \frac{1083}{4991}, \frac{1954}{9005}, \frac{3037}{13996}, \frac{4991}{23001}, \frac{23001}{106000}, \frac{212000}{977001}\right]$	(1000002, 106000)	(1000, 106)	$(2 \cdot$ A277110, $2 \cdot 53000 \cdot$ A277111)	$(2 \cdot 500 \cdot$ A227137, $2 \cdot 53 \cdot$ A227138)
36	**92**	4	$-4 + \sqrt{23}$	$\left[\frac{0}{1}, \frac{1}{1}, \frac{3}{4}, \frac{4}{5}, \frac{35}{44}\right]$	(48, 5)	no sol.	$(2 \cdot$ A114051, A174767$(k+1))$	no solution
37	**93**	2	$\frac{-9+\sqrt{93}}{2}$	$\left[\frac{0}{1}, \frac{1}{3}, \frac{9}{28}\right]$	(29, 3)	no sol.	(A090251, $3 \cdot$ A097782$(k+1))$	no solution
38	**96**	2	$-4 + 2\sqrt{6}$	$\left[\frac{0}{1}, \frac{1}{1}, \frac{8}{9}\right]$	(10, 1)	no sol.	$(2 \cdot$ A001079$(k-1)$, A004189)	no solution
39	**97**	18	$\frac{-9+\sqrt{97}}{2}$	$\left[\frac{0}{1}, \frac{1}{2}, \frac{2}{5}, \frac{3}{7}, \frac{14}{33}, \frac{59}{139}, \frac{73}{172}, \frac{205}{483}, \frac{483}{1138}, \frac{4552}{10725}, \frac{9587}{22588}, \frac{23726}{55901}, \frac{333131}{78489}, \frac{156978}{369857}, \frac{661225}{1557917}, \frac{818203}{1927774} \frac{2297631}{5413465}, \frac{5413465}{12754704}, \frac{51018816}{120205801}\right]$	(125619266, 12754704)	(11208, 1138)	$(2 \cdot$ A227150$(k-1)$, $2^4 \cdot 3 \cdot 467 \cdot 569 \cdot$ A227151$(k-1))$	$(2^3 \cdot 3 \cdot 467 \cdot$ A227274$(k-1)$, $2 \cdot 569 \cdot$ A227274$(k-1))$
40	**101**	6	$\frac{-4+\sqrt{101}}{2}$	$\left[\frac{0}{1}, \frac{1}{2}, \frac{10}{19}, \frac{11}{21}, \frac{21}{40}, \frac{200}{381}\right]$	(402, 40)	(20, 2)	$(2 \cdot$ A227152$(k-1)$, $40 \cdot$ A097740$(k-1))$	$(20 \cdot$ A097741$(k-1)$, $2 \cdot$ A097742$(k-1))$