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Note on a− and z−sequences of Sheffer number triangles for certain generalized Lah numbers

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Abstract

The so-called $a-$ and $z-$sequences for *Sheffer* number triangles provide a recurrence for each entry in terms of those of the preceding row. The $a-$ and $z-$sequences for the *Sheffer* triangles of the generalized *Lah* numbers, called $L[d, a]$, are considered.

For each *Sheffer* number triangle $\mathbf{S} = (g(t), f(t))$ with exponential generating functions (*e.g.f.* s) $g(t) = \sum_{n=0}^{\infty} g_k \frac{t^k}{k!}$, where $g_0 = 1$, and $f(t) = t\,\hat{f}(t)$, with $\hat{f}(t) = \sum_{n=0}^{\infty} \hat{f}_k \frac{t^k}{k!}$, where $\hat{f}_0 \neq 0$, one can give a recurrence for the entries of the first column of $\mathbf{S}$, in matrix notation $S(n, 0)$ in terms of maximal n entries of the preceding row, *i.e.*,

$$S(n,\,0) \,=\, n \sum_{j=0}^{n-1} z_j\, S(n-1,\,j), \quad \textit{for} \;\; n \,\geq\, 1, \;\; \text{with } S(0,\,0) \,=\, 1\,, \tag{1}$$

where the z-sequence has *e.g.f.*

$$Ez(y) \,:=\, \sum_{j=0}^{\infty} z_j \frac{y^j}{j!} \,=\, \frac{1}{f^{[-1]}(y)} \left(1 \,-\, \frac{1}{g(f^{[-1]}(y))}\right). \tag{2}$$

Here the compositional inverse of f is denoted by $f^{[-1]}$.
The recurrence for the other entries of the lower triangular (infinite demensional) *Sheffer* matrix $\mathbf{S}$ is given by the $a-$sequence.

$$S(n,\,m) \,=\, \frac{n}{m} \sum_{j=0}^{n-m} \binom{m-1+j}{j} a_j\, S(n-1,\,m-1+j), \quad \textit{for} \;\; n \,\geq\, 1,\, m \,\geq\, 1\,, \tag{3}$$

with *e.g.f.*

$$Ea(y) \,:=\, \sum_{j=0}^{\infty} a_j \frac{y^j}{j!} \,=\, \frac{y}{f^{[-1]}(y)}\,. \tag{4}$$

For details see the *W. Lang* link [1], part 2, where also the references [5],[3],[6] are given, which, however, refer to the *Riordan* triangle case.

The generalized non-negative *Lah* number triangles $\mathbf{L}[d, a]$, with $d \in \mathbb{N}$ and $a = 0$ if $d = 1$, and $\gcd(d,\,a) = 1$, *i.e.*, $a \in RRS(d)$, the smallest positive restricted residue system modulo d, have been proposed in [2], section 2, C) 4. We do not repeat here the properties of these lower triangular (infinite dimensional) triangles $\mathbf{L}[d, a]$ as transition matrices between $[d, a]-$generalizations of certain rising and falling factorials but concentrate on the $a-$ and $z-$sequences.

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The first instances are $\mathbf{L}[1,0] = \mathbf{L} =$ A271703, $\mathbf{L}[2,1] =$ A286724, $\mathbf{L}[3,1] =$ A290596, $\mathbf{L}[3,2] =$ A290598, $\mathbf{L}[4,1] =$ A290604, $\mathbf{L}[4,3] =$ A292219.

The *Sheffer* structure is

$$\mathbf{L}[d,a] = \left(\frac{1}{(1 - d\,t)^{\frac{2\,a}{d}}},\, \frac{t}{1 - d\,t}\right). \tag{5}$$

Thus the compositional inverse of f is

$$f^{[-1]}(y) = \frac{y}{1 + d\,y}. \tag{6}$$

The *e.g.f.* of the $a-$sequence is therefore $Ea(d;y) = 1 + d\,y$, with the sequence $a(d) = \{1,\, d,\, repeat(0)\}$. This means that the recurrence is always of the three term type (see also [2], eq. (143)):

$$L(d,a;n,m) = \frac{n}{m}\,L(d,a;n-1,m-1) + n\,d\,L(d,a;n-1,m), \quad \text{for } n \in \mathbb{N},\, m = 1,\, 2, ...,\, n. \tag{7}$$

As mentioned above this recurrence has to be used in connection wit the one from the $z-$sequence for the $m = 0$ column, to be discussed now.

For the $z-$sequence the analysis becomes more involved. The *e.g.f.* is (see [2], eq. (142))

$$\begin{aligned} Ez(d,a;y) &= (1 + d\,y)\,\frac{1}{y}\left[1 - \left(1 - \frac{d\,y}{1 + d\,y}\right)^{\frac{2\,a}{d}}\right], \\ &= (1 + d\,y)\,\frac{1}{y}\left[1 - (1 + d\,y)^{-\frac{2\,a}{d}}\right]. \end{aligned} \tag{8}$$

Lemma 1: Series $\mathbf{Ez(d,a;y)}$

$$\begin{aligned} Ez(d,a;y) &= 2\,a + \sum_{k=1}^{\infty} \frac{y^k}{k!}\,(-1)^k\,Z(d,a;k)\,, \quad \text{with} \\ Z(d,a;k) &= \frac{1}{k+1}\prod_{j=0}^{k}(2\,a + (j-1)\,d)\,. \end{aligned} \tag{9}$$

Proof:

$$\begin{aligned} \frac{1}{y}\left[1 - (1 + d\,y)^{-\frac{2\,a}{d}}\right] &= \frac{1}{y}\left[1 - \sum_{k=0}^{\infty}\left(\frac{2\,a}{d}\right)^{\overline{k}}\frac{(-d\,y)^k}{k!}\right] \\ = -\sum_{k=1}^{\infty}\left(\frac{2\,a}{d}\right)^{\overline{k}}(-d)^k\,\frac{y^{k-1}}{k!} &= \sum_{k=0}^{\infty}\left(\frac{2\,a}{d}\right)^{\overline{k+1}} d^{k+1}\,\frac{(-1)^k}{k+1}\,\frac{y^k}{k!}\,, \end{aligned} \tag{10}$$

with the rising factorial $x^{\overline{k}} := \prod_{j=0}^{k-1}(x + j)$ if $k \in \mathbb{N}$, and $x^{\overline{0}} := 1$.

This has to be multiplied with $(1 + d\,y)$ producing the leading term $2\,a$, and the coefficient of $\frac{y^k}{k!}$, $k \geq 1$, becomes

$$\begin{aligned} &\left(\frac{2\,a}{d}\right)^{\overline{k+1}}(-1)^k\,\frac{d^{k+1}}{k+1} + d\left(\frac{2\,a}{d}\right)^{\overline{k}}(-1)^{k-1}\,\frac{d^k}{k}\,k \\ &= (-1)^k\,\frac{1}{k+1}\prod_{j=0}^{k-1}(2\,a + d\,j)\cdot[(2\,a + d\,k) - d\,(k+1)] = \frac{(-1)^k}{k+1}\prod_{j=0}^{k}(2\,a + d\,(j-1)). \end{aligned} \tag{11}$$

$\square$

The $z(d,a)$ sequence has therefore the entries

$$z(d,a;k) \;=\; \begin{cases} 2\,a\,, & \text{for } k \,=\, 0 \\ (-1)^k\, Z(d,a;k)\,, & \text{for } k \,\geq\, 1 \end{cases} \tag{12}$$

Examples:

1) $z(1,0;k) \,=\, 0$, for $k \,\geq\, 0$.

2) $z(2,1;0) \,=\, 2,\; z(2,1;k) \,=\, 0$, for $k \,\geq\, 1$.

3) $z[3,1] \,=\, \{2,\, \frac{2}{2},\, -\frac{2\cdot5}{3},\, \frac{2\cdot5\cdot8}{4},\, -\frac{2\cdot5\cdot8\cdot11}{5},\, \frac{2\cdot5\cdot8\cdot11\cdot14}{6},\, ...\} \,=\, \{2,\, 1,\, -\frac{10}{3},\, 20,\, -176,\, \frac{6160}{3},\, ...\}$.

4) $z[3,2] \,=\, \{4,\, -\frac{4}{2},\, \frac{4\cdot7}{3},\, -\frac{4\cdot7\cdot10}{4},\, \frac{4\cdot7\cdot10\cdot13}{5},\, -\frac{4\cdot7\cdot10\cdot13\cdot16}{6},\, ...\} \,=\, \{4,\, -2,\, \frac{28}{3},\, -70,\, 728,\, -\frac{29120}{3},\, ...\}$.

5) $z[4,1] \,=\, \{2,\, \frac{2\cdot2}{2},\, -\frac{2\cdot2\cdot6}{3},\, \frac{2\cdot2\cdot6\cdot10}{4},\, -\frac{2\cdot2\cdot6\cdot10\cdot14}{5}\, \frac{2\cdot2\cdot6\cdot10\cdot14\cdot18}{6},\, ...\}$
$\qquad = \{2,\, 2,\, -8,\, 60,\, -672,\, 10080,\, ...\} \,=\, 2*\{1,\, 1\,, -4\,, 30,\, -336,\, 5040,\, ...\}$.

6) $z[4,3] \,=\, \{6,\, -\frac{2\cdot6}{2},\, \frac{2\cdot6\cdot10}{3},\, -\frac{2\cdot6\cdot10\cdot14}{4},\, \frac{2\cdot6\cdot10\cdot14\cdot18}{5},\, -\frac{2\cdot6\cdot10\cdot14\cdot18\cdot22}{6},\, ...\}$
$\qquad = \{6,\, -6,\, 40,\, -420,\, 6048,\, -110880,\, ...\} \,=\, 2*\{3,\, -3,\, 20,\, -210,\, 3024,\, -55440,\, ...\}$.

The third and fourth example shows that some entries become fractional, with powers of 3 in the denominator.

See the instances $z(3,1;k) \,=\,$ A290599(k)/A038500$(k+1)$, $z(3,2;k) \,=\,$ A290603(k)/A038500$(k+1)$, $z(4,1;k) \,=\, 2*$A292220(k) and $z(4,3;k) \,=\, 2*$A292221(k).

The question is for which $k\,+\,1$ values, with $k \,\geq\, 1$, $z(d,a;k)$ (in lowest terms) is fractional, *i.e.*, which $k\,+\,1$ values do not divide the numerator

$$N(z(d,a;k)) \;=\; (-1)^k \prod_{j=0}^{k} (2\,a \,+\, (j-1)\,d) \;=:\; (-1)^k\, P(d,a;k) \tag{13}$$

$$\;=\; (-1)^k\, \mathrm{sign}(2\,a - d)\, |d \,-\, 2\,a| \prod_{j=1}^{k} (2\,a \,+\, (j-1)\,d). \tag{14}$$

For this analysis we state two trivial Lemmata.

Lemma 2:

The product $P(d,a;k)$ has all numbers $2\,a \pmod d$ from the interval $[2\,a, P_{\max}(d,a;k)]$, with $P_{\max}(d,a;k) \,:=\, 2\,a \,+\, (k-1)\,d \,=\, d\,(k+1) \,-\, 2\,(d-a)$ as factors. In addition the (sometimes negative or vanishing) number $2\,a \,-\, d$ is a factor.

This is obvious from the definition of $P(d,a;k)$ in eq. (14).

The number $|d \,-\, 2\,a|$ (0 only for $d \,=\, 2$ because $\gcd(d,\,a) \,=\, 1$) will be called 'the first number' in $P(d,a;k)$ (even if $\mathrm{sign}(2\,a \,-\, d)$ is negative). Note that if $2\,a \,-\, d$ is not positive then this first number $d \,-\, 2\,a$ is congruent to $2\,a$ modulo d only for $[d,a] \,=\, [1,0]$, $[2,1]$ or $[4,1]$.

Corollary 1: Even d, factors

If $d \,=\, 2\,D$, $D \,\in\, \mathbb{N}$, then $P(2\,D,a;k)$ has all numbers $a \pmod D$ from the interval $[a, \overline{P}_{\max}(2\,D,a;k)]$, with $\overline{P}_{\max}(2\,D,a;k) \,:=\, a \,+\, (k-1)\,D \,=\, D\,(k+1) \,-\, 2\,D \,+\, a$ as factors. In addition there is a negative factor $-2^{k+1}\,(D-a)$. The first (positive) number is then $2^{k+1}\,(D-a)$.

Lemma 3:

$P(d,a;k) \;\equiv\; (2\,a)^{k+1} \pmod d$, *for* $k \,\geq\, 1$.

This is also clear because $P(d,a;k)$ has $k+1$ factors, each $2\,a \pmod d$.

Corollary 2: Even d, congruence

For $d = 2\,D$, $D \in \mathbb{N}$, $\overline{P}(2\,D,a;k) := \frac{P(2\,D,a;k)}{2^{k+1}} \equiv a^{k+1} \pmod{D}$ for $k \geq 1$.

To find out which numbers appear in the denominators of $z(d,a;k)$, for $k \geq 1$, one looks first at the power sequence $\{(2\,a)^{k+1}\}_{k>=1} \pmod{d}$. Such modular power sequences always become (or are already) periodic because the sequence has more than d terms, but there are only d possible values from $RS_{>}(d)$ (the smallest non-negative residue system modulo d). See the *Table* for $d = 1, ..., 10$, and the restricted a values.

Proposition 1: Denominators of $\mathbf{z(3,a;k)}$

z(3,a;k) has for $a = 1, 2$ the denominator (in lowest terms) A038500$(k+1)$ for $k \geq 0$, *i.e.*, the highest power of 3 in $k+1$.

Proof:

For $k = 0$ this is trivial; the denominator is 1.

i) In the cases $[d,a] = [3,1]$ and $[3,2]$ it is clear that 3 cannot divide $P(3,1;k)$ for $k \geq 1$, because $P(3,1;k) \equiv 1\,2 \pmod{3}$ depending on the parity of k. The same holds for $P(3,2;k) \equiv 1 \pmod{3}$. Therefore, in the prime factorization of $k+1$ one can separate the powers of the prime 3 which will never divide $P(3,a;k)$, hence $Num(z(3,a;k))$, for $a = 1, 2$. This explains why the denominator of $z(3,a;k)$ has certainly a factor A038500$(k+1)$, the highest power of 3 in $k+1$. But at this stage of the argument there could remain more factors of $k + 1$ in the denominator.

ii) Therefore one has to show that powers of primes congruent to 1 or 2 modulo 3 of $k+1$ always divide $P(3,a;k)$. For this we consider the factorization $\overline{k+1} = \frac{k+1}{3^{e3}} = \prod_{j=1}^{M1} p_{1,j}^{e1,j} \prod_{j=1}^{M2} p_{2,j}^{e2,j} = P1 \cdot P2$, with $P1 \equiv 1 \pmod{3}$ the product from powers of prime $1 \pmod{3}$ (A002476), and $P2$ the product from powers of primes $2 \pmod{3}$ (A003627). $P2$ is $\equiv 1 \pmod{3}$ or $\equiv 2 \pmod{3}$ if $\sum_{j=1}^{M2} e_{2,j}$ is even or odd, respectively. We omit the arguments (d) of the factors Pi.

If $d = 3$, $a = 1$ *Lemma 2* shows that all numbers $2 \pmod{3}$ from 2 to $P_{\max}(3,1;k) = (k+1)+2\,(k-1) = 3\,(k+1) - 4$ appear as factors in $P(3,1;k)$. Therefore in the second case, when $P2 = 2 \pmod{3}$, we show that $P2 \leq 3\,(k+1) - 4$. This is clear because $3\,(k+1) - 4 \geq 3\,\overline{k+1} - 4 \geq 3\,P2 - 4 \geq P2$, *i.e.*, $2\,P2 \geq 2$. which is trivial because $P2 \geq 2$.

In the first case, when $P2 = 1 \pmod{3}$, this number $P2$ will be shown to appear as a factor in $P(3,1;k)$ as the $2 \pmod{3}$ number $2 \cdot P2$. Namely $P_{\max}(3,1;k) = 3\,(k+1) - 4 \geq 3\,\overline{k+1} - 4 \geq 3\,P2 - 4 \overset{!}{\geq} 2 \cdot P2$, *i.e.*, $P2 \overset{!}{\geq} 4$. This is trivial because the smallest prime in $P2$ is 2 and the exponent sum is now even, hence $P2 \geq 4$.

For $P1$ one also shows that the $2 \pmod{3}$ number $2 \cdot P1$ is always a factor of $P(3,1;k)$ because, as above, this reduces to $P1 \overset{!}{\geq} 4$ which is trivial because the smallest prime in $P1$ is 7. This concludes the proof for $a = 1$.

iii) If $a = 2$ then all numbers congruent to 1 modulo 4 from 4 to $P_{\max}(3,2;k) = 3\,(k+1) - 2$ appear in $P(3,2;k)$, and one can proceed as above. If $P2 = 2 \pmod{3}$ one shows that the $1 \pmod{3}$ number $2\,P2$ is smaller than $3\,(k+1) - 2$. This results in the trivial inequality $P2 \geq 2$. In the other case for which $P2 = 1 \pmod{3}$ this reduces to the inequality $2\,P2 \geq 2$. The same appears for $P1$: $2\,P1 \geq 2$ is satisfied because $P1 \geq 7$. □

This method can now be generalized to higher $[d,a]$ cases. Observe that in the proof we did not use the first number in $P(d,a;k)$. It will turn out that in general also this number will have to be used for small numbers $k+1$ for certain a values.

Proposition 2: Integer sequences $\{\mathbf{z(4,a;k)}\}_{\mathbf{k\geq 0}}$

The denominators of $z(4,a;k)$ (in lowest terms) are for $a = 1, 3$ always 1. (This holds also for the not considered case $a = 2$, with $z(4,2;0) = 4$, and $z(4,2;k) = 0$, for $k \geq 1$.)

Proof:

For $k = 0$ this is clear: $z(4,a;0) = 2a$.

$P(4,a;k) \equiv 0 \pmod 4$, $k \geq 1$, for $a = 1$ and 3 (*Lemma 3*). Also $\overline{P}(4,a;k) \equiv 1 \pmod 2$ for these a values (*Corollary 1*).

Each of the three prime power factors of $k+1 = P1 \cdot P2 \cdot P3$ (powers of primes modulo 4) can be shown to be a factor of $P(4,a;k)$. There is no $P0$ and $P2 = 2^{e2}$. $P_{\max}(4,1;k) = 4(k+1) - 6$ or $P_{\max}(4,3;k) = 4(k+1) - 2$, and $\overline{P}_{\max}(4,a;k) = 2(k+1) - 4 + a$ for the odd numbers in $\overline{P}(4,a;k)$. For $P2 = 2^{e2}$ this is trivial, because $P(4,a;k)$ has at least the factors 2^{k+1}, and $2^{k+1} \geq 2^{2^{e2}} \geq 2^{e2}$.

Here the above used procedure will not work for for 2 in $P2 \equiv 2 \pmod 4$, the odd exponent sum case, for all $k \geq 1$ if $a = 1$. But for this 2 the first number of $P(4,1;k)$ namely $|4-2| = 2$ comes to help, for $k = 1$. This standard procedure would run as follows. $P2 \equiv 2 \pmod 4$ or $\equiv 0 \pmod 4$, therefore we look for $2P2$ or $P2$, respectively, in $P(4,a;k) \equiv 0 \pmod 4$. This leads in the first case, with odd exponent sum of $P2$, to $2P2 \geq 6$ or ≥ 2 for $a = 1$ or $a = 3$, respectively, and in the other case to $3P2 \geq 6$ or ≥ 2. Because in the first case $P2 \geq 2$ the case $a = 1$ is not satisfied. In the second case, with $P2 \geq 4$, there is no problem. Thus one has to treat for $a = 1$ the case $k+1 = 2$ separately, using the first number 2 in $P(4,1;1)$, as announced.

$P1$, the powers of primes $\equiv 1 \pmod 4$ (A002144), and $P3$, the powers of primes $\equiv 3 \pmod 4$ (A002145), are both $\equiv 1 \pmod 2$ (odd). Now one looks for $P1$ and $P3$ in $\overline{P}(4,a;k) \equiv 1 \pmod 2$. This is successful because $\overline{P}_{\max}(4,a;k) \geq 2(k+1) - 4 + a \geq 2Pi - 4 + a$, and one proves $2Pi - 4 + a \geq Pi$, for $i = 1, 3$. For $a = 1$ and $a = 3$ this becomes $Pi \geq 3$ and $Pi \geq 1$, respectively. This is trivial for both Pi because they are are ≥ 5 or ≥ 3 for $i = 1$ or 3, respectively. □

To finish we discuss the case $d = 9$ in order to explain the general procedure in a more involved show piece.

Proposition 3: $\{\mathbf{z(9,a;k)}\}_{\mathbf{k\geq 0}}$

The denominators of $z(9,a;k)$ for $a = 1, 2, 4, 5, 7, 8$ are all A038500$(k+1)$, the highest power of 3 in $k+1$.

Proof

(i) $P(9,a;k)$ cannot have a divisor 3 because otherwise (from *Lemma 3*) $(2a)^{k+1} \equiv 0 \pmod 3$, due to $P(9,a;k) = (2a)^{k+1} + 9K = 3L$ with integers K and L. This means that $a^{k+1} \equiv 0 \pmod 3$, which implies that 3 has to divide a, contradicting $\gcd(9,a) = 1$. (In passing: the sequences $\{z(9,a;k)\}_{k>=0}$ for $a = 3$ and $a = 6$ are integer ones.). Thus the factor $P3 = 3^{e3}$ in the prime number factorization of $k+1$ can never divide $P(9,a;k)$, and this highest power of 3 in $k+1$ certainly remains in the denominator of $z(9,a;k)$.

As above one has to show that each of the other factors in $k+1$, *i.e.*, $P1 \cdot P2 \cdot P4 \cdot P5 \cdot P7 \cdot P8$, divide $P(9,a;k)$ (whose modulo 9 congruence property depends on a (see the *Table*)). $P6$ is not present. For these prime sequences see A061237, A061238, A061239, A061240, A061241, A061242.

$P_{\max}(9,a;k) = 9(k+1) - 2(9-a)$, and the proofs first check the standard estimates like above.

(ii) Each factor Pi is analysed for the possible mod 9 subfactors. $P1 \equiv 1 \pmod 9$, $P2 \equiv 2, 4, 8, 7, 5, 1 \pmod 9$, $P4 \equiv 4, 7, 1 \pmod 9$, $P5 \equiv 5, 7, 8, 4, 2, 1 \pmod 9$, $P7 \equiv 7, 4, 1 \pmod 9$, $P8 \equiv 8, 1 \pmod 9$. For each case the sum of the exponents of the relevant primes mod 9 satisfy a certain congruence condition with the modulus given by the period of $\{i^q\}_{q\geq 1}$ for Pi. *E.g.*, if in $k+1$ the factor $P2 \equiv 7 \pmod 9$ then $\sum_{j=1}^{M2} e2j = 4 \pmod 6$. This subfactor of $P2$ will be abbreviated as $P2_{4(6)}$.

(iii) Let each specific $n \pmod 9$ instance of these Pi sub-factors be collected in $Q(9,\, n)$. *E.g.*, $Q(9,4) = \{P2_{2(6)},\, P4_{1(3)},\, P5_{4(6)},\, P7_{2(3)}\}$, with the index giving the sum of the exponents modulo the periods in bracket. Each $Q(9,\, n)$ is treated as a representative for its Pi sub-factors for all possible a values. A number $m(9,n,a)$ is determined such that $m(9,n,a)\, Q(9,\, n) \equiv 2\,a \pmod 9$. Thus $Q(9,\, 2) = \{P2_{1(6)},\, P5_{2(6)}\}$ is multiplied by $m(9,2,a) = a$. $Q(9,\, 1) = \{P1,\, P2_{6(6)},\, P4_{3(3)},\, P5_{6(6)},\, P7_{3(3)},\, P8_{2(2)}\}$ is multiplied by $m(9,1,a) = 2\,a \pmod 9$, *i.e.*, 2, 4, 8, 1, 5, 7. $Q(9,4)$ needs $m(9,4,a) = 5,\, 1,\, 2,\, 7,\, 8,\, 4$, for the relevant rising $2\,a$ values. $Q(9,\, 5)$ needs $m(9,5,a) = 4,\, 8,\, 7,\, 2,\, 1,\, 5$. $Q(9,\, 7)$ needs $m(9,7,a) = 8,\, 7,\, 5,\, 4,\, 2,\, 1$, and $Q(9,\, 8)$ needs $m(9,8,a) = 7,\, 5,\, 1,\, 8,\, 4,\, 2$.

(iv) Each of these $2\,a \pmod 9$ numbers $m(9,n,a)\, Q(9,\, n)$ is thus $\geq 2\,a$, and one checks whether $m(9,n,a)\, Q(9,\, n) \leq P_{\max}(9,a;k)$ with $P_{\max}(9,a;k) = 9\,(k+1) - 2\,(9-a)$, *i.e.*, $9\,(k+1) - 2\,(9-a) \geq 9\, Q(9,\, n) - 2\,(9-a) \geq m(d,n,a)\, Q(9,\, n)$.

$$(9 - m(d,n,a))\, Q(9,\, n) \geq 2\,(9-a). \tag{15}$$

This inequality is checked for each $n = 1, 2, 4, 5, 7, 8$ and a with the same values. Sometimes certain low $k+1$ values have to be excluded for certain a values and one has to treat such cases separately.

The case $n = 2$ is trivial because $m(9,2,a) = a$ and $Q(9,\, 2) \geq 2$.

For $n = 1$ one has $\{7, 5, 1, 8, 4, 2\}\, Q(9,\, 1) \geq 2 \cdot \{8, 7, 5, 4, 2, 1\}$ (six separate inequalities for each corresponding sequence entry on both sides). With $Q(9,\, 1) \geq 19$ (from P1 with exponent sum 1) this is satisfied for each a.

For $Q(9,\, 4)$ this becomes $\{4, 8, 7, 2, 1, 5\}\, Q(9,\, 4) \geq 2 \cdot \{8, 7, 5, 4, 2, 1\}$, satisfied because $Q(9,\, 4) \geq 4$ (from $P2$ with exponent sum 2).

For $Q(9,\, 5) = \{P2_{5(6)},\, P5_{1(6)}\}$ one finds $\{5, 1, 2, 7, 8, 4\}\, Q(9,\, 5) \geq 2 \cdot \{8, 7, 5, 4, 2, 1\}$ with $Q(9,\, 5) \geq 5$ (from $P5$ with exponent sum 1). But now this does not hold for $k \geq 1$ and $a = 2$ because $Q(9,\, 5) \ngeq 14$ for $k + 1 = 5$. In fact, $P(9,2;4) = (-5) \cdot 4 \cdot 13 \cdot 22 \cdot 31$ and this is another instance where the first number, here 5, is needed to complete the $Q(9,\, 5)$ proof for $k + 1 = 5$ from $P5$. All six inequalities are satisfied for $k + 1 \geq 23$ (the next entry in $Q(9,\, 5)$, also from $P5$ with exponent sum 1) without invoking the first number 5 from $P(9,2;4)$.

For $Q(9,\, 7) = \{P2_{4(6)},\, P4_{2(3)},\, P5_{2(6)},\, P7_{1(3)}\}$ one has $\{1, 2, 4, 5, 7, 8\}\, Q(9,\, 7) \geq 2 \cdot \{8, 7, 5, 4, 2, 1\}$ with $Q(9,\, 7) \geq 7$ (from P_7). Here the case $a = 1$ is not satisfied for all relevant k, because $7 \ngeq 16$. Again, for $k + 1 = 7$ one needs the first number 7 of $P(9,1;6) = (-7) \cdot 2 \cdot 11 \cdot 20 \cdot 29 \cdot 38 \cdot 47$. The other numbers in $Q(9,\, 7)$ which are ≥ 16 (from $P2$ with exponent sum 4) satisfy eq. (15).

For $Q(9,\, 8) = \{P2_{3(6)},\, P5_{3(6)},\, P8_{1(2)}\}$ one has $\{2, 4, 8, 1, 5, 7\}\, Q(9,\, 8) \geq 2 \cdot \{8, 7, 5, 4, 2, 1\}$ with $Q(9,\, 8) \geq 8$ (from P_2 with exponent sum 3). All inequalities are satisfied. □

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OEIS [4] A-numbers:
A002144, A002145, A002476, A003627, A038500, A061237, A061238, A061239, A061240, A061241, A061242, A271703, A286724, A290596, A290598, A290599, A290603, A290604, A292219, A292220, A292221.

Table: $\{(2\,a)^{k+1}\}_{k>=1}\,(\mathrm{mod}\,d)$

d	a	$\{(2\,a)^{k+1}\}_{k>=1}\,(\mathrm{mod}\,d)$	denominators of $z(d, a; k)$
1	0	{repeat(0)}	1
2	1	{repeat(0)}	1
3	1	{repeat(1,2)}	A038500$(k+1)$
	2	{repeat(1)}	A038500$(k+1)$
4	1	{repeat(0)}	1
	3	{repeat(0)}	1
5	1	{repeat(4, 3, 1, 2)}	A060904$(k+1)$
	2	{repeat(1, 4)}	A060904$(k+1)$
	3	{repeat(1)}	A060904$(k+1)$
	4	{repeat(4, 2, 1, 3)}	A060904$(k+1)$
6	1	{repeat(4, 2)}	A038500$(k+1)$
	5	{repeat(4)}	A038500$(k+1)$
7	1	{repeat(4, 1, 2)}	A268354$(k+1)$
	2	{repeat(2, 1, 4)}	A268354$(k+1)$
	3	{repeat(1, 6)}	A268354$(k+1)$
	4	{repeat(1)}	A268354$(k+1)$
	5	{repeat(2, 6, 4, 5, 1, 3)}	A268354$(k+1)$
	6	{repeat(4, 6, 2, 3, 1, 5)}	A268354$(k+1)$
8	1	{repeat(4, 0)}	1
	3	{repeat(4, 0)}	1
	5	{repeat(4, 0)}	1
	7	{repeat(4, 0)}	1
9	1	{repeat(4, 8, 7, 5, 1)}	A038500$(k+1)$
	2	{repeat(7, 1, 4)}	A038500$(k+1)$
	4	{repeat(1, 8)}	A038500$(k+1)$
	5	{repeat(1)}	A038500$(k+1)$
	7	{repeat(7, 8, 4, 2, 1, 5)}	A038500$(k+1)$
	8	{repeat(4, 1, 7)}	A038500$(k+1)$
10	1	{repeat(4, 8, 6, 2)}	A060904$(k+1)$
	3	{repeat(6)}	A060904$(k+1)$
	7	{repeat(6, 4)}	A060904$(k+1)$
	9	{repeat(4, 2, 6, 8)}	A060904$(k+1)$