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Collatz Trees from Vaillant-Delarue Maps

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1) The *Vaillant-Delarue* [1] function $\text{fs} : \mathbb{O} \to \mathbb{O}$, $2\,m+1 \mapsto \text{fs}(2\,m+1) \ =: \ a(m)$, for $m \in \mathbb{N}_0$, s with the positive odd numbers $\mathbb{O}$, is defined by (in [1] fs is named f_s)

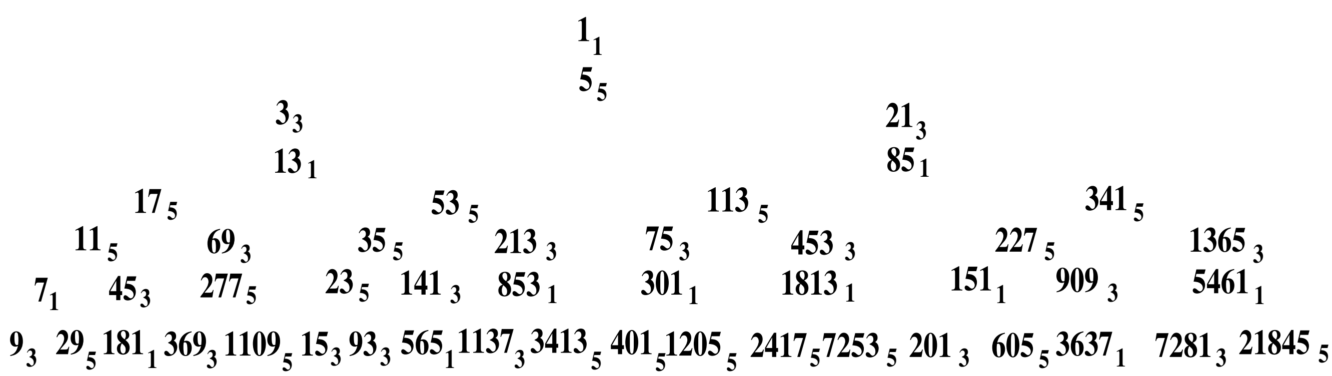
$$a(m) \ = \ \text{fs}(2\,m+1) \ := \begin{cases} \frac{2+3\,m}{2} & \text{if} \quad m \ \equiv \ 0\,(mod\,4)\,, \\ 2 \ + \ 3\,m & \text{if} \quad m \ \equiv \ 1\,\text{or}\,3\,(mod\,4)\,, \\ \frac{m}{2} & \text{if} \quad m \ \equiv \ 2\,(mod\,4)\,. \end{cases} \tag{1}$$

This sequence is given in OEIS [2] as A324036(m).

The levels of the corresponding tree CfsTree are given by the sets $S(l)$ defined recursively by

$$\begin{aligned} S(0) \ &= \ \{1\}, \\ S(1) \ &= \ \{5\}, \\ S(l) \ &= \ \{o \in \mathbb{O}\,|\,\text{fs}(o) \in S(l-1)\}, \text{ for } l \ = \ 2,\, 3,\, \ldots\,. \end{aligned} \tag{2}$$

Collatz Tree for Vaillant–Dellarue fs–function CfsTree $_7$



The indices indicate the congruence classes modulo 6 of the vertex labels. The out-degree of each vertex $3\,(mod\,6)$ is 1, the other vertices have out-degree 2. The branches are not depicted but should be clear from the positions of the labels. See also Figure 1 of [1].
The number of vertices on level l is $L(l) = 2\,L(l-1) \ - \ L_3(l-1)$, for $l \ = \ 2,\, 3,\, \ldots$, with L_3 the number of vertices with klables congruent to 3 modulo 6, and $L(0) \ = \ 1 \ = \ L(1)$. The list of $L(l)$, for $l \ = \ 0,\, 1,\, \ldots,\, 20$ is (see A324039):

$$[1,\, 1,\, 2,\, 2,\, 4,\, 8,\, 11,\, 19,\, 31,\, 50,\, 88,\, 146,\, 241,\, 407,\, 675,\, 1118,\, 1871,\, 3102,\, 5175,\, 8633,\, 14394]. \tag{3}$$

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The number $L_3(l)$, for $l = 0, 1, ..., 20$, is given by (see A324040):

$$[0, 0, 2, 0, 0, 5, 3, 7, 12, 12, 30, 51, 75, 139, 232, 365, 640, 1029, 1717, 2872, 4789]. \tag{4}$$

For $m \in S(l) \to m' \in S(l+1)$ the successor formula is, for $l = 1, 2, \ldots$, (m is here odd and not m from eq. (1))

$$m'(m) := \begin{cases} 4\,m + 1 & \text{if } m \equiv 3\,(mod\,6)\,, \\ \frac{4\,m-1}{3} \text{ and } 4\,m + 1 & \text{if } m \equiv 1\,(mod\,6)\,, \\ \frac{2\,m-1}{3} \text{ and } 4\,m + 1 & \text{if } m \equiv 5\,(mod\,6)\,. \end{cases} \tag{5}$$

Note that 1_1 does not obey this rule, The rule would lead to level $S(l) = \{1_1, 5_5\}$ which would imply a duplication of the tree.

The rephrased *Collatz* conjecture for the tree CfsTree is:

$$\forall o \in \mathbb{O}\; \exists\, l \in \mathbb{N}_0 : o \in S(l)\,. \tag{6}$$

Because the level sets $S(l)$ and $S(l')$ satisfy $S(l) \cap S(l') = \emptyset$ for $l \neq l'$ the $\exists$ symbol in the conjecture can be replaced by the a unique $\exists_1$ symbol.

The conjecture means then that CfsTree gives a permutation of $\mathbb{O}$ when read level by level.
For the CfsTree see A324038.

The successor formula eq. (5) leads to nine rules $Ss(i, j; D)$ with i and j from $\{1, 3, 5\}$ for the residue classes modulo 6 for $m \in S(l)$ and $m' \in S(l+1)$, respectively, for $l \in \mathbb{N}$, and $D \in \{l, r, v\}$ for the downwards directions left, right, vertical. The notation $m = (a, b)$ and $m' = (c, d)$ means that from $m = a + K\,b$ follows $m' = c + K\,d$ for $K \in \mathbb{N}_0$.

$$\begin{aligned}
Ss(1, 1; r):&\quad m = (1, 18) \xrightarrow{r} m' = (1, 24)\ (\equiv 1\,(mod\,6))\,,\\
Ss(1, 3; r):&\quad m = (7, 18) \xrightarrow{r} m' = (9, 24)\ (\equiv 3\,(mod\,6))\,,\\
Ss(1, 5; r):&\quad m = (13, 18) \xrightarrow{r} m' = (17, 24)\ (\equiv 5\,(mod\,6))\,,\\
Ss(1, 5; l):&\quad m = (1, 6) \xrightarrow{l} m' = (5, 24)\ (\equiv 5\,(mod\,6))\,,\\
\\
Ss(3, 1; v):&\quad m = (3, 6) \xrightarrow{v} m' = (13, 24)\ (\equiv 1\,(mod\,6))\,,\\
\\
Ss(5, 1; r):&\quad m = (11, 18) \xrightarrow{r} m' = (7, 12)\ (\equiv 1\,(mod\,6))\,,\\
Ss(5, 3; r):&\quad m = (5, 18) \xrightarrow{r} m' = (3, 12)\ (\equiv 3\,(mod\,6))\,,\\
Ss(5, 5; r):&\quad m = (17, 18) \xrightarrow{r} m' = (11, 12)\ (\equiv 5\,(mod\,6))\,,\\
Ss(5, 5; l):&\quad m = (5, 6) \xrightarrow{l} m' = (21, 24)\ (\equiv 5\,(mod\,6))\,.
\end{aligned} \tag{7}$$

Because $m = 1 \in A(0)$ is not considered for these rules, $K \geq 1$ for $Ss(1, 1; r)$ and $Ss(1, 5; l)$.
$Ss(1, 3; r)$ is always followed by $Ss(3, 1; v)$, hence $m = (7, 18) \xrightarrow{rv} m'' = (37, 96)$.
$Ss(5, 3; r)$ is always followed by $Ss(3, 1; v)$, hence $m = (5, 18) \xrightarrow{rv} m'' = (13, 48)$.

Examples:

1) $Ss(1, 1; r)$ with $K = 1$: $m = 19_1 \in S(8) \xrightarrow{r} m' = 25_1 \in S(9)$ with a downwards right branch.
2) $S(1, 3; r)$ with $K = 0$: $m = 7_1 \in S(6) \xrightarrow{r} m' = 9_3 \in S(7)$.
3) $S(5, 5; l)$ with $K = 1$: $m = 11_5 \in S(5) \xrightarrow{l} m' = 45_3 \in S(6)$.
4) $S(1, 3; r)$ then $S(3, 1; v)$, with $K = 1$: $m = 25_1 \in S(9) \xrightarrow{rv} m'' = 133_1 \in S(11)$.

For the CfsTree there are also nine rules for the predecessors $m \in S(l)$ of $m' \in S(l+1)$, for $l = 1, 2, \ldots$. They are denoted by $Ps(i, j; D)$, with i and j from $\{1, 3, 5\}$ for the residue classes modulo 6 of odd m'

and odd m, respectively, and $D \in \{l, r, v\}$ gives the respectively upward left, right, vertical direction of the (not depicted) branches. The notation $m' = (a, b)$ from $m = (c, d)$ means that if $m' = a + b\,K$ then $m = c + d\,K$ for $K \in \mathbb{N}_0$, as long as $m' \geq 5$.

$$\begin{array}{lllll} Ps(1, 1; r): & m' = (1, 24) & \text{from} & m = (1, 18) & (\equiv 1\,(mod\,6))\,, \\ Ps(1, 3; v): & m' = (13, 24) & \text{from} & m = (3, 6) & (\equiv 3\,(mod\,6))\,, \\ Ps(1, 5; r): & m' = (7, 12) & \text{from} & m = (11, 18) & (\equiv 5\,(mod\,6))\,, \\ & & & & \\ Ps(3, 1; r): & m' = (9, 24) & \text{from} & m = (7, 18) & (\equiv 1\,(mod\,6))\,, \\ Ps(3, 5; l): & m' = (21, 24) & \text{from} & m = (5, 6) & (\equiv 5\,(mod\,6))\,, \\ Ps(3, 5; r): & m' = (3, 12) & \text{from} & m = (5, 18) & (\equiv 5\,(mod\,6))\,, \end{array} \tag{8}$$

$$\begin{array}{lllll} Ps(5, 1; l): & m' = (5, 24) & \text{from} & m = (1, 6) & (\equiv 1\,(mod\,6))\,, \\ Ps(5, 1; r): & m' = (17, 24) & \text{from} & m = (13, 18) & (\equiv 1\,(mod\,6))\,, \\ Ps(5, 5; r): & m' = (11, 12) & \text{from} & m = (17, 18) & (\equiv 5\,(mod\,6))\,. \end{array} \tag{9}$$

In $Ps(1, 1; r)$ $K \geq 1$ because $l \geq 1$, hence $m' \geq 5$.
If also level $l = 0$ is considered then there is an exception for $Ps(5, 1; l)$ for $m' = 5$ ($K = 0$) because the upwards direction to 1 from 5 is vertical (v), not left (l).

Examples:

1) $Ps(1, 1; r)$ with $K = 1$: $m' = 25_1 \in S(9)$ from $m = 19_1 \in S(8)$, upwards to the right direction.
2) $Ps(3, 1; r)$ with $K = 0$: $m' = 9_3 \in S(7)$ from $m = 7_1 \in S(6)$, upwards to the right direction.
3) $Ps(5, 5; r)$ with $K = 0$: $m' = 11_5 \in S(5)$ from $m = 17_5 \in S(4)$,upwards to the right direction.

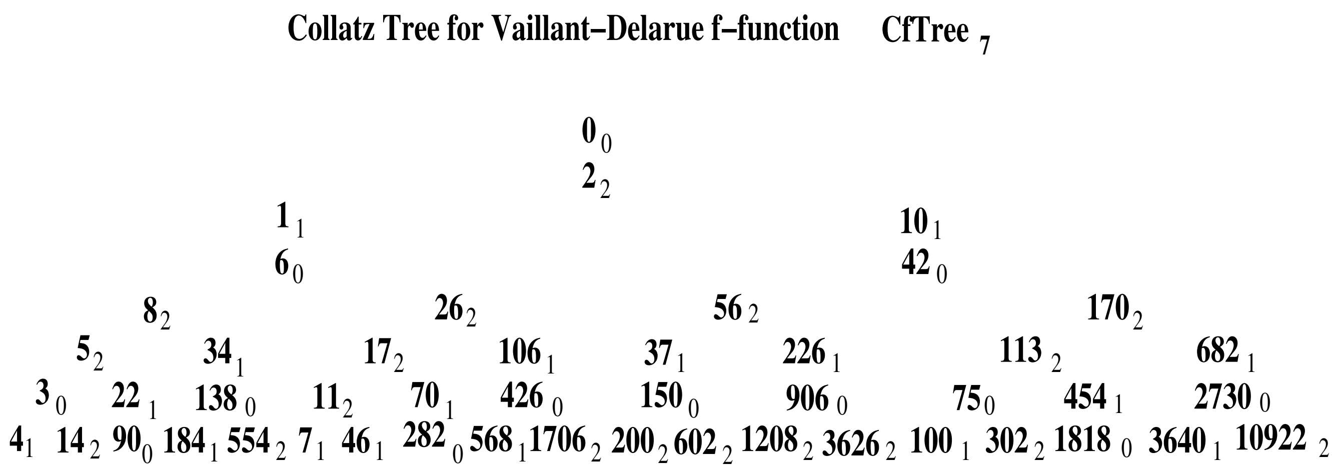
2) The *Vaillant-Delarue* function $f := \mathbb{N}_0 \to \mathbb{N}_0$ is defined by

$$f(m) := \begin{cases} \frac{3\,m+1}{2} & \text{if } m \equiv 1\,(mod\,2)\,, \\ \frac{3\,m}{4} & \text{if } m \equiv 0\,(mod\,4)\,, \\ \frac{m-2}{4} & \text{if } m \equiv 2\,(mod\,4)\,. \end{cases} \tag{10}$$

This function is given in A324245.

The levels of the tree CfTree are given by the set $A(l)$ defined recursively by

$$\begin{aligned} A(0) &= \{0\}, \\ A(1) &= \{2\}, \\ A(l) &= \{m \in \mathbb{N}_0 \,|\, f(m) \in A(l-1)\}, \text{ for } l = 2, 3, \ldots. \end{aligned} \tag{11}$$



The indices indicate the congruence classes modulo 3. See also Figure 2 of [1].

The out-degree of each vertex with a label $1 (mod\, 3)$ is 1, the other vertices have out-degree 2. The number of vertices on level l is $L(l) = 2\,L(l-1) - L_1(l-1)$, for $l = 2, 3, \ldots,$ with L_1 the number of vertices with labels congruent to 1 modulo 3, and $L(0) = 1 = L(1)$. The list of $L(l)$, for $l = 0, 1, ..., 20$ has been given above. The number $L_1(l) = L_3(l)$ from above.

For $m \in A(l) \to m' \in A(l+1)$ the successor formula is, for $l = 1, 2, \ldots,$

$$m'(m) := \begin{cases} 2\,(2\,m + 1) & \text{if } \; m \equiv 1\,(mod\, 3)\,, \\ \frac{2\,m-1}{3} \text{ and } 2\,(2\,m + 1) & \text{if } \; m \equiv 2\,(mod\, 3)\,, \\ \frac{4\,m}{3} \text{ and } 2\,(2\,m + 1) & \text{if } \; m \equiv 0\,(mod\, 3)\,. \end{cases} \tag{12}$$

Note that 0_0 does not obey this rule. The rule would lead to level $A(1) = \{0_0,\, 2_2\}$ and a duplication of the tree would be present.

Therefore we omit in the following level $l = 0$, *i.e.*, the entry 0_0, and consider the tree starting with 2_2 on level $l = 1$.

The rephrased *Collatz* conjecture for the tree CfTree (for $l \geq 1$) is:

$$\forall m \in \mathbb{N}\; \exists\, l \in \mathbb{N} : m \in A(l)\;. \tag{13}$$

Because the level sets $A(l)$ and $A(l')$ satisfy $A(l) \cap A(l') = \emptyset$ for $l \neq l'$ the $\exists$ symbol in this conjecture can be replaced by the a unique $\exists_1$ symbol.

This conjecture means that the CfTree gives a permutation of $\mathbb{N}$ when read level by level, for $l \geq 1$.

For the CfTree see A324246.

The successor formula eq. (11) leads to nine rules $S(i, j; D)$ with i and j from $\{0, 1, 2\}$ for the residue classes modulo 3 for $m \in A(l)$ and $m' \in A(l+1)$, respectively, for $l \in \mathbb{N}$, , and $D \in \{l, r, v\}$ for the downwards directions left, right, vertical. The notation $m = (a, b)$ and $m' = (c, d)$ means that from $m = a + K\,b$ follows $m' = c + K\,d$ for $K \in \mathbb{N}_0$.

$$
\begin{array}{lllll}
S(0,\,0;r): & m\,=\,(0,\,9) & \xrightarrow{r} & m'\,=\,(0,\,12)\;(\equiv\,0\,(mod\,3))\,, & \\
S(0,\,1;r): & m\,=\,(3,\,9) & \xrightarrow{r} & m'\,=\,(4,\,12)\;(\equiv\,1\,(mod\,3))\,, & \\
S(0,\,2;r): & m\,=\,(6,\,9) & \xrightarrow{r} & m'\,=\,(8,\,12)\;(\equiv\,2\,(mod\,3))\,, & \\
S(0,\,2;l): & m\,=\,(0,\,3) & \xrightarrow{l} & m'\,=\,(2,\,12)\;(\equiv\,2\,(mod\,3))\,, & \\
 & & & & \\
S(1,\,0;v): & m\,=\,(1,\,3) & \xrightarrow{v} & m'\,=\,(6,\,12)\;(\equiv\,0\,(mod\,3))\,, & (14) \\
 & & & & \\
S(2,\,0;r): & m\,=\,(5,\,9) & \xrightarrow{r} & m'\,=\,(3,\,6)\;(\equiv\,0\,(mod\,3))\,, & \\
S(2,\,1;r): & m\,=\,(2,\,9) & \xrightarrow{r} & m'\,=\,(1,\,6)\;(\equiv\,1\,(mod\,3))\,, & \\
S(2,\,2;r): & m\,=\,(8,\,9) & \xrightarrow{r} & m'\,=\,(5,\,6)\;(\equiv\,2\,(mod\,3))\,, & \\
S(2,\,1;l): & m\,=\,(2,\,3) & \xrightarrow{l} & m'\,=\,(10,\,12)\;(\equiv\,1\,(mod\,3))\,. &
\end{array}
$$

Because $m\,=\,0\,\in\,A(0)$ is not considered, $K\,\geq\,1$ for $S(0,\,0;r)$ and $S(0,\,2;l)$.
$S(0,\,1;r)$ is always followed by $S(1,\,0;v)$, hence $m\,=\,(3,\,9)\xrightarrow{rv} m''\,=\,(18,\,48)$.
$S(2,\,1;r)$ is always followed by $S(1,\,0;v)$, hence $m\,=\,(2,\,9)\xrightarrow{rv} m''\,=\,(6,\,24)$.
$S(2,\,1;l)$ is always followed by $S(1,\,0;v)$, hence $m\,=\,(2,\,3)\xrightarrow{lv} m''\,=\,(42,\,48)$.

Examples:

1) $S(0,\,0;r)$ with $K\,=\,1$: $m\,=\,9_0\,\in\,A(8)\;\xrightarrow{r}\;m'\,=\,12_0\,\in\,A(9)$ with a downwards right branch.
2) $S(0,\,2;l)$ with $K\,=\,1$: $m\,=\,3_0\,\in\,A(6)\;\xrightarrow{l}\;m'\,=\,14_2\,\in\,A(7)$ with a downwards left branch.
3) $S(2,\,1;l)$ with $K\,=\,1$: $m\,=\,5_2\,\in\,A(4)\;\xrightarrow{l}\;m'\,=\,22_1\,\in\,A(5)$.
4) $S(2,\,1;l)$ then $S(1,\,0;v)$, with $K\,=\,1$: $m\,=\,5_2\xrightarrow{lv}\;m''\,=\,90_0$.

For the CfTree there are also nine rules for the predecessors $m\,\in\,A(l)$ of $m'\,\in\,A(l+1)$, for $l\,\in\,\mathbb{N}$. They are denoted by $P(i,j;D)$, with i and j from $\{0,\,1,\,2\}$ for the residue classes modulo 3 of m' and m, respectively, and $D\,\in\,\{l,\,r,\,v\}$ for the upwards directions left, right, vertical.
The notation is as above: $m'\,=\,(a,\,b)$ from $m\,=\,(c,\,d)$ means that if $m'\,=\,a\,+\,b\,K$ then $m\,=\,c\,+\,d\,K$ for $K\,\in\,\mathbb{N}_0$, as long as $m'\,\geq\,2$.

$$
\begin{array}{llll}
P(0,\,0;r): & m'\,=\,(0,\,12)\ \text{from}\ m\,=\,(0,\,9)\;(\equiv\,0\,(mod\,3))\,, & \\
P(0,\,1;v): & m'\,=\,(6,\,24)\ \text{from}\ m\,=\,(1,\,3)\;(\equiv\,1\,(mod\,3))\,, & \\
P(0,\,2;r): & m'\,=\,(3,\,6)\ \text{from}\ m\,=\,(5,\,9)\;(\equiv\,2\,(mod\,3))\,, & \\
 & & \\
P(1,\,0;r): & m'\,=\,(4,\,12\ \text{from}\ m\,=\,(3,\,9)\;(\equiv\,0,\,(mod\,3))\,, & \\
P(1,\,2;r): & m'\,=\,(1,\,6)\ \text{from}\ m\,=\,(2,\,9)\;(\equiv\,2,\,(mod\,3))\,, & \\
P(1,\,2;l): & m'\,=\,(10,\,12)\ \text{from}\ m\,=\,(2,\,3)\;(\equiv\,2,\,(mod\,3))\,, & \\
 & & (15) \\
P(2,\,0;r): & m'\,=\,(8,\,12)\ \text{from}\ m\,=\,(6,\,9)\;(\equiv\,0\,(mod\,3))\,, & \\
P(2,\,0;l): & m'\,=\,(2,\,12)\ \text{from}\ m\,=\,(0,\,3)\;(\equiv\,0\,(mod\,3))\,, & \\
P(2,\,2;r): & m'\,=\,(5,\,16)\ \text{from}\ m\,=\,(8,\,9)\;(\equiv\,2\,(mod\,3))\,. & \\
 & & (16)
\end{array}
$$

If also level $l\,=\,0$ is considered there is an an exception for rule $P(2,\,0;l)$ for $m'\,=\,2$ $(K\,=\,0)$ because the direction to 0 from 2 of level $l\,=\,1$ is vertical (v), not left (l).

Examples:

1) $P(1,\, 0; r)$ with $K = 0$: $m' = 4_1 \in A(6)$ leading to $m = 3_0 \in A(5)$ in the right direction.

2) $P(2,\, 0; r)$ with $K = 0$: $m' = 8_2 \in A(4)$ to $m = 6_0 \in A(3)$ in the right direction.

3) $P(2,\, 2; r)$ with $K = 0$: $m' = 5_2 \in A(5)$ to $m = 8_2 \in A(4)$ in the right direction.

Instead of the level entries m one can use words w over the alphabet $\{l, r, v\}$ (left, right, vertical, respectively) for the paths of the branches starting with $2 \in A(1)$ downwards to the level containing m. For example, the path to $m = 200 \in A(7)$ is encoded as word $lvrrvr$ of length 6, or backwards from 200 to 2 by $rvrrvl$.

In general, $m \in A(l)$ is mapped to a word $w(m)$ of length $\#w(m) = l$, for $l = 1, 2, \ldots$, starting with l or r.

References

[1] Nicolas Vaillant and Philippe Delarue, The hidden face of the $3x + 1$ problem. Part I: Intrinsic algorithm, April 26, 2019

http://nini-software.fr/site/uploads/arithmetics/collatz/Intrinsic%203x+1%20V2.01.pdf.

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[2] The On-Line Encyclopedia of Integer Sequences (2010), published electronically at http://oeis.org.

OEIS A-numbers related to the Vaillant and Delarue paper: A072197, A324036, A324037, A324038, A324039, A324040, A324245, A324246.
