Karlsruhe
December 10 2014

On lattice point circles for the Archimedean tiling $(3, 4, 6, 4)$

Wolfdieter L a n g [1]

Abstract

The lattice points belonging to the semi-regular planar tiling, using tiles of the shape of regular 3-, 4-, and 6-gons, lie on circles centered at any point of a hexagon with increasing radii. The square of these radii, which are integers in the real quadratic number field $\mathbb{Q}(\sqrt{3})$, are computed. The circular disk sequence and its first differences, the number of lattice points on the circles, are given.

1. Introductory remarks

The semi-regular Archimedean tiling (tessellation) denoted by $(3, 4, 6, 4)$ is considered [2, 1, 4, 5, 6]. The corresponding planar lattice has vertices of out-degree 4 with edges of length 1 (in some length units *l.u.*), dividing a circular disk into sectors of degree $\frac{\pi}{3}$, $\frac{\pi}{2}$, $\frac{2\pi}{3}$ and $\frac{\pi}{2}$. This means that each of the hexagon vertices is surrounded by an equilateral triangle, a square, a hexagon and again a square in this order (the sense of the rotation does not matter), hence the name $(3, 4, 6, 4)$, indicating these regular n-gons.

In this note we compute the coordinates of the points of this lattice and the increasing radii of the circles around any of the hexagon vertices, which hit, by and by, all lattice points. It is found that the square of these radii $R_{\rm hit}(n)$, $n = 0, 1, ..., \infty$, called $R2_{\rm hit}(n)$, are integers in the real quadratic number field $\mathbb{Q}(\rho(3))$ with $\rho(3) = \sqrt{3}$, the smaller of the two diagonal lengths of the hexagon (see [3]).

2. The point lattice for $(3, 4, 6, 4)$

The unit cell is defined by the two equal length column vectors $\vec{E}_1 = \left(1 + \sqrt{3}, 0\right)^\top$ and $\vec{E}_2 = \left(\frac{1+\sqrt{3}}{2}, \frac{3+\sqrt{3}}{2}\right)^\top$. The parallelogram of this unit cell is shown in *Figure 1* in green. It has six 'atoms', namely $P_0 = (0, 0)$, $P_1 = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, $P_2 = \left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right)$, $P_3 = \left(1 + \frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, $P_4 = \left(1 + \frac{\sqrt{3}}{2}, \frac{3}{2}\right)$ and $P_5 = (1 + \sqrt{3}, 2)$, with Cartesian coordinates referring to the x-direction of E_1 and the orthogonal $y-$axis. The corresponding column vectors (starting at the origin $O = (0, 0)$) are $\vec{P}_j$ for $j = 0, 1, ..., 5$. The general lattice point has then vector $\vec{P}(k, l, j) = \vec{P}_j + \vec{P}(k, l)$ with $\vec{P}(k, l) = k\,\vec{E}_1 + l\,\vec{E}_2$, with integers k and l. See *Figure 1* for all lattice points belonging to the cells (k, l) with $k = -2, -1, 0, 1$ and $l = -2, -1, 0, 1$. Note that on the right hand side and the at the top the boundaries of the cells have not been drawn (they belong to the next cells).

Proposition 1: Square radii $\mathbf{R2(k, l, j)} \in \mathbb{Q}(\sqrt{\mathbf{3}})$

The squares of the radii $R2(k, l, j) := \vec{P}(k, l, j) \cdot \vec{P}(k, l, j)$ *are integer elements of the real quadratic number field* $\mathbb{Q}(\sqrt{3})$*, i.e.,* $R2(k, l, j) = a(k, l, j) + b(k, l, j)\sqrt{3}$*, for* $j = 0, 1, ..., 5$ *and* k*,* l *(rational)*

[1] wolfdieter.lang@partner.kit.edu, http://www.itp.kit.edu/~wl/

integers. The rational and irrational parts are given as follows, using $Q(k,l) := 2\,(k^2 + k\,l + l^2)$.

$$\begin{aligned}
a(k,l,0) &= 2\,b(k,l,0), \text{ with } b(k,l,0) = Q(k,l),\\
a(k,l,1) &= 2\,Q(k,l) + 3\,k + 3\,l + 1,\;\; b(k,l,1) = Q(k,l) + k + l,\\
a(k,l,2) &= 2\,Q(k,l) + 3\,(k + 2\,l + 1),\;\; b(k,l,2) = Q(k,l) + k + 2\,l,\\
a(k,l,3) &= 2\,Q(k,l) + 5\,k + 4\,l + 2,\;\; b(k,l,3) = Q(k,l) + 3\,k + 2\,l + 1,\\
a(k,l,4) &= 2\,Q(k,l) + 5\,k + 7\,l + 4,\;\; b(k,l,4) = Q(k,l) + 3\,k + 3\,l + 1,\\
a(k,l,5) &= 2\,Q(k,l) + 2\,(4\,k + 5\,l + 4),\;\; b(k,l,5) = Q(k,l) + 2\,(2\,k + 2\,l + 1).
\end{aligned}$$

Proof: Elementary by calculating $R2(k,l,j)$ for $j = 0, 1, ..., 5$. □

Proposition 2: Mirror symmetry of the lattice

The lattice is mirror symmetric under the transformation $x = \rightarrow -x$. *The precise relation can be stated as follows.*

$$\hat{P}(k,l,j) = P(-m(k,l,j), l, \pi(j)), \quad \text{for } k \geq \left\lfloor \frac{l + n(j)}{2} \right\rfloor,$$

where $\hat{P}$ *denotes the mirror point of* P *for a point on the right half-plane* ($x \geq 0$). *Here* $m(k,l,j) = k + l$ *for* $j = 0$, $= k + l + 1$, *for* $j = 1, 2, 3, 4$ *and* $= k + l + 2$ *for* $j = 5$. *Moreover* $n(j) = 0$ *for* $j = 0, 1, 2, 3$ *and* $= 1$ *for* $j = 4, 5$.

Proof: The symmetry of the lattice points with respect to $x \rightarrow -x$ is obvious for each strip of cells belonging to a given $l \in \mathbb{Z}$. The precise determination of the mirror point of $P(k,l,j)$, for $x \geq 0$, is more intricate, but can be handled for each j and the first nonnegative l, as well as the first negative ones. Then the results for all l follow by periodicity. □

Example 1: The point $P(0,0,5)$ has mirror point $\hat{P} = (-2,0,5)$ because $k \geq 0$ ($n(5) = 1$), $\pi(5) = 5$ and $-m(0,0,5) = -(0 + 0 + 2) = -2$. Similarly $\hat{P}(-1,0,5) = P(-1,0,5)$ because this point is on the $y-$ axis. Some other examples: $\hat{P}(0,1,1) = P(-2,1,3)$, $\hat{P}(1,-1,2) = P(-1,-1,4)$ and $\hat{P}(0,-2,5) = P(0,-2,5)$.

3. Circles

Now consider a circular disk of radius R centered around the origin O (any fixed hexagon vertex) and ask which cells (k,l) are needed in order that all points within the R-disk are reached. For this consider the lattice constant $a = 1 + \sqrt{3}$, and the constant $\hat{a} = \sin\left(\frac{\pi}{6}\right) = \frac{3 + \sqrt{3}}{2}$ which is the height of a each strip (or layer) l, and also the width of each layer k. It is then clear that $k_{\max}(R) = \left\lfloor \frac{R}{\hat{a}} \right\rfloor$, and the same results holds for $l_{\max}(R)$. If we let k run from $-(1 + k_{max}(R)), ..., 0, ..., k_{\max}(R)$ and similarly l from $-(1 + l_{max}(R)), ..., 0, ..., l_{\max}(R)$ certainly all points within radius R will be covered. One can exclude all j values for points of the outer cells (not covered completely by the $R-$disk) which lead to a distance from the origin which is larger than R. Then only the vertices on the $R-$circle and in its interior survive. The number of lattice points covered by the R-disk is called $n(R)$. A computation (thanks to Maple13) leads to the following sequence if one considers increasing nonnegative R values (equidistant radii).

[1, 5, 14, 29, 49, 73, 103, 140, 183, 233, 293, 361, 428, 495, 569, 649, 741, 841, 943, 1051, ...].

The first difference sequence which counts the new points which are covered when the radius is enlarged one unit is [1, 4, 9, 15, 20, 24, 30, 37, 43, 50, 60, 68, 67, 67, 74, 80, 92, 100, 102, 108, 118, ...]. See *Figure 2* for $R = 0, 1, ..., 4$.

A more interesting problem is to find the increasing radii, starting with 0 for the point at the origin O, searching iteratively (one of) the points with smallest distance to the origin which have not been covered by circular disks found before. This will define the so called lattice disk sequence for the considered tiling.

For this one computes the distances of all points covered by a circular disk of radius R and sorts them, keeping track of the corresponding lattice points. There will occur multiple entries in the list of radii, telling how many points lie on the corresponding radius. The sequence of radii $R_{\rm hit}(n)$ found this way, is the square root of $R2_{\rm hit}(n)$, the corresponding squares of the radii. These squares have been shown above to be integers in the real quadratic number field $\mathbb{Q}(\sqrt{3}\,)$, namely $R2_{\rm hit}(n) = a(n) + b(n)\sqrt{3}$ with nonnegative integer pairs $[a(n), b(n)]$. The beginning of this sequence of pairs is, up to $R = 10$ with 85 pairs,

[[0, 0], [1, 0], [2, 0], [3, 0], [2, 1], [4, 0], [4, 1], [4, 2], [5, 2], [6, 3], [8, 2], [8, 3], [7, 4], [8, 4], [10, 3], [10, 4], [10, 5], [13, 4], [14, 4], [11, 6], [12, 6], [13, 6], [15, 6], [14, 8], [16, 7], [16, 8], [17, 8], [16, 9], [19, 8], [20, 8], [22, 8], [19, 10], [20, 10], [20, 11], [24, 9], [23, 10], [21, 12], [25, 10], [22, 12], [23, 12], [28, 10], [26, 12], [26, 13], [28, 13], [31, 12], [28, 14], [32, 12], [28, 15], [28, 16], [30, 15], [32, 15], [34, 15], [35, 16], [32, 18], [33, 18], [38, 16], [34, 19], [36, 18], [38, 17], [37, 18], [40, 18], [37, 20], [38, 20], [43, 18], [40, 20], [44, 18], [40, 21], [46, 19], [41, 22], [43, 22], [46, 21], [44, 23], [48, 21], [43, 24], [50, 20], [44, 24], [48, 24], [49, 24], [50, 24], [51, 24], [50, 25], [52, 26], [50, 28], [57, 24], [58, 24], ...].

The sequences of the rational and irrational parts a and b are given in A249870 and A249871, respectively. The corresponding sequence for the norms $N(R2_{\rm hit}(n)) = (a(n) + b(n)\sqrt{3})\,(a(n) - b(n)\sqrt{3}) = a^2(n) - b^2(n)\,3$ is

[0, 1, 4, 9, 1, 16, 13, 4, 13, 9, 52, 37, 1, 16, 73, 52, 25, 121, 148, 13, 36, 61, 117, 4, 109, 64, 97, 13, 169, 208, 292, 61, 100, 37, 333, 229, 9, 325, 52, 97, 484, 244, 169, 277, 529, 196, 592, 109, 16, 225, 349, 481, 457, 52, 117, 676, 73, 324, 577, 397, 628, 169, 244, 877, 400, 964, 277, 1033, 229, 397, 793, 349, 981, 121, 1300, 208, 576, 673, 772, 873, 625, 676, 148, 1521, 1636, ...].

The associated values of the radii $R_{\rm hit}(n) = \sqrt{R2_{\rm hit}(n)}$ is (Maple 10 digits if not a positive integer)

[0, 1, 1.414213562, 1.732050808, 1.931851653, 2, 2.394170171, 2.732050808, 2.909312911, 3.346065215, 3.385867927, 3.632650881, 3.732050808, 3.863703305, 3.898224265, 4.114389776, 4.319751618, 4.464101616, 4.574735318, 4.625181602, 4.732050808, 4.836559195, 5.039077778, 5.277916867, 5.303240110, 5.464101616, 5.554854315, 5.620360955, 5.732050808, 5.818625822, 5.988021915, 6.026649822, 6.109051323, 6.249204661, 6.291935892, 6.349843154, 6.464101616, 6.505421438, 6.540994550, 6.616994008, 6.732050808, 6.839927609, 6.965390190, 7.107507334, 7.196152424, 7.228327006, 7.265301762, 7.347160140, 7.464101616, 7.482029278, 7.614509972, 7.744724793, 7.919142184, 7.948390689, 8.011049528, 8.106343991, 8.179790055, 8.196152424, 8.212482191, 8.256931303, 8.436641188, 8.464101616, 8.522969914, 8.612602077, 8.639503235, 8.670462187, 8.739168551, 8.883071842, 8.894105789, 9.005837983, 9.075960939, 9.156263899, 9.185481314, 9.196152424, 9.200055226, 9.250363204, 9.464101616, 9.516786190, 9.569180706, 9.621289903, 9.659258263, 9.850549273, 9.924586773, 9.928203232, 9.978437723, ...].

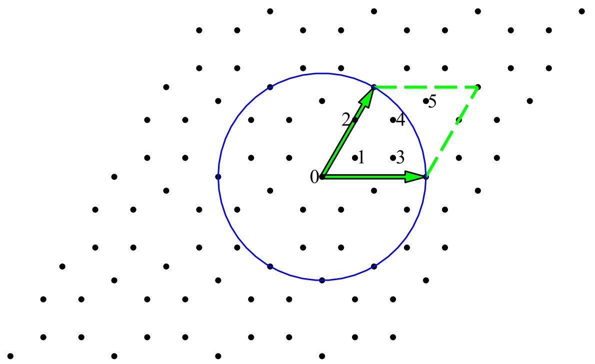

Figure 1: Elementary cell and lattice points with $R = 3$ *l.u.*

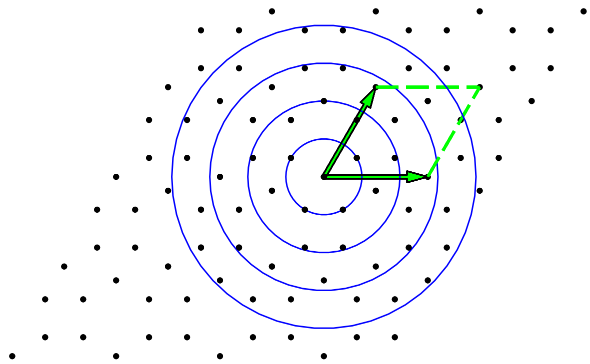

Figure 2: Equidistant radii up to $R = 4$ *l.u.*

4. Lattice disk sequence and its differences

It remains to determine the number of lattice points on each of the circles with radius $R_{\rm hit}(n)$. The cumulative numbers up to $R_{\rm hit}(85) = \sqrt{58 + 24\sqrt{3}} \approx 9.978437723$ are given by the lattice disk sequence

$$[1, 5, 7, 9, 13, 14, 18, 25, 29, 33, 35, 39, 43, 45, 49, 51,\\ 55, 57, 59, 63, 69, 73, 77, 79, 83, 89, 93, 97, 99, 101, 103,\\ 107, 109, 113, 117, 121, 123, 127, 129, 133, 134, 136, 140, 144,\\ 146, 158, 160, 164, 165, 169, 173, 177, 181, 183, 187, 189, 193,\\ 199, 203, 207, 209, 211, 213, 217, 219, 221, 225, 229, 233,\\ 237, 241, 245, 249, 251, 253, 255, 261, 265, 267, 271, 275,\\ 287, 289, 291, 293, ...].$$

This sequence is given in A251627.

The corresponding difference list, that is the list of lattice points on the increasing radii $R_{\rm hit}(n)$, $n = 0, 1, ...$ is

$$[1, 4, 2, 2, 4, 1, 4, 7, 4, 4, 2, 4, 4, 2, 4, 2, 4, 2, 2, 4,\\ 6, 4, 4, 2, 4, 6, 4, 4, 2, 2, 2, 4, 2, 4, 4, 4, 2, 4, 2, 4, 1,\\ 2, 4, 4, 2, 12, 2, 4, 1, 4, 4, 4, 4, 2, 4, 2, 4, 6, 4, 4, 2,\\ 2, 2, 4, 2, 2, 4, 4, 4, 4, 4, 4, 4, 2, 2, 2, 6, 4, 2, 4,\\ 4, 12, 2, 2, 2, \ldots].$$

This is found in A251628. See *Figure 3* for the first 6, and *Figure 4* for the first 10 circles hitting lattice points.

Proposition 3: Odd entries of A251628

Odd numbers of lattice points occur for circles whose squared radii belong to the $[a, b]$ *pairs i)* $[0, 0]$ *and ii)* $[4\,(3\,L^2 + 3\,L + 1),\ 2\,L\,(3\,L + 2)]$*, for* $L \in \mathbb{Z}$*,* $l = 2\,L$*.*

Proof: Odd points on a lattice circle arise from points on the $y-$axis which have no companion with opposite y-coordinate. On the $y-$axis lie only points with $j = 0$ and $j = 5$. The origin

O is the only $j = 0$ point which is of interest; all other $j = 0$ points have a companion. No $j = 5$ point has a companion. The radii for these $j = 5$ points are found to be $2 + L\,(2\,\hat{a})$ for $l = 2\,L \geq 0$, that is $(2 + 3\,L) + L\,\sqrt{3}$, and the squared radii have $a(L) = 4\,(3\,L^2 + 3\,L + 1)$ and $b(L) = 2\,L\,(2 + 3\,L)$. For negative $l = 2\,L$ one finds the radii $a - (L + 1)\,(2\,\hat{a})$, that is $-(3\,L + 2) - L\,\sqrt{3}$, and the squared radii coincide with the ones just found for nonnegative L values. $\square$

The rational parts $a(L)$ of the squared radii of the $j = 5$ points for $L \geq 0$ are [4, 28, 76, 148, 244, 364, 508, 676, 868, 1084, ...] which is $4 \cdot$ A003215, and for $L < 0$ the same result is obtained because $4\,(3\,(L - 1)^2 + 3\,(L - 1) + 1) = 4\,(3\,L^2 - 3\,L + 1)$. The irrational parts for $b(L)$ for $L \geq 0$ are [0, 5, 16, 33, 56, 85, 120, 161, 208, 261, ...] which is $2 \cdot$ A045944, and for $L < 0$ it is [2, 16, 42, 80, 130, 192, 266, 352, 450, 560, ... which is $2 \cdot$ A000567(L).

These $[a(L), b(L)]$ values correspond for $L = -4, -3, -2, -1, 0, 1, 2, 3$ to the odd entries for $n =$ 5, 7, 40, 48, 110, 123, 215, 233 of A251628$(n \geq 0)$. The first odd entry 1 for $n = 0$ belongs to the $j = 0$ case [0, 0].

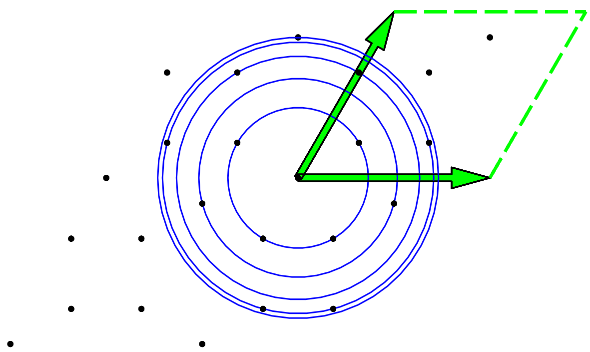

Figure 3: The first 6 point hitting circles

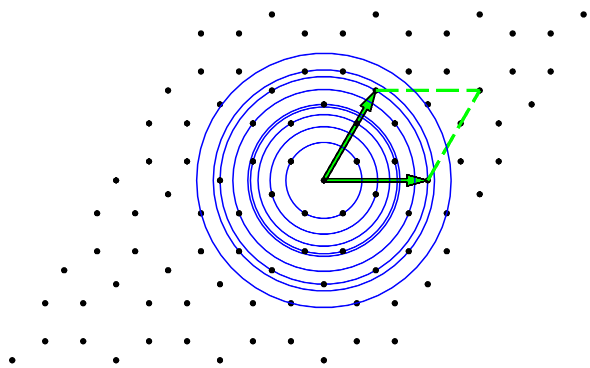

Figure 4: The first 10 point hitting circles

Acknowledgments

This computation was inspired by a construction proposed by Kival Ngaokrajang in A245094.

References

[1] Branko Gruenbaum and G. C. Shephard, Tilings and Patterns. W. H. Freeman, New York, 1987, frontispiece and statement 2.1.3, p. 59 , figure 2.1.1, p. 60 and figure 2.1.5 on p. 63.

[2] J. Kepler, Harmonices Mundi, Weltharmonik, übersetzt und eingeleitet von Max Caspar, R. Oldenbourg Verlag, München, 1990, p. 71, Figur *Ii* (XX. Satz).

Biblioteca Virtuale On-Line (Pisa):

http://bivio.filosofia.sns.it/bvWorkTOC.php?authorSign=KeplerJohannes&titleSign=Harmonices

(in Latin, liber II, XX propositio, 2., p. 76).

[3] W. Lang, The field $Q(2\cos(\pi/n))$, its Galois group, and length ratios in the regular n-gon, arXiv:1210.1018 [math.GR], http://arxiv.org/abs/1210.1018.

[4] Wikipedia, List of convex tilings.
https://en.wikipedia.org/wiki/List_of_convex_uniform_tilings.

[5] Wikipedia, Rhombitrihexagonal Tiling.
https://en.wikipedia.org/wiki/Rhombitrihexagonal_tiling.

[6] Wikipedia, Tiling by regular polygons, Archimedean, uniform or semiregular tilings.
http://en.wikipedia.org/wiki/Tiling_by_regular_polygons#Archimedean.2C_uniform_or_semiregu

Keywords: Archimedean tiling, rhombitrihexagonal tiling, covering circular disks and radii.

AMS MSC numbers: 52C20, 52C05

OEIS A-numbers: A000567, A003215, A045944, A249870, A249871, A251627, A251628 .