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Cantor's List of Real Algebraic Numbers of Heights 1 to 7

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Abstract

Cantor gave in his fundamental article [1] an elegant proof of the countability of real algebraic numbers based on a positive integer height, denoted by him as N, of integer and irreducible polynomials of given degree (denoted by him as n) with relative prime coefficients. The finite number of real algebraic numbers with given height he called $\varphi(N)$, and gave the first three instances.
Here we give a systematic list for the real algebraic numbers of height, which we denote by n, for $n = 1, 2, ..., 7$ and polynomials of degree k.

1 Cantor's proof

An *algebraic number* is a number ω which satisfies a non-constant polynomial equation

$$\sum_{j=0}^{k} a_j\, x^j \,=\, 0, \text{ for } x \,=\, \omega. \tag{1}$$

where $k \in \mathbb{N}$, and the $a_j \in \mathbb{Z}$, for $j \in \{0, 1, ..., k\}$. Without loss of generality one restricts to $a_k > 0$, and assumes that $\gcd(a_0, a_1, ..., a_k) = 1$ (relative prime coefficients). In order to avoid multiple counting of numbers ω (in general complex, but in the following real) the polynomials have to be irreducible (they do not factorize over $\mathbb{Z}$).

To prove that the set of such real algebraic numbers is countable, *i.e.*, that there is a bijective map between $\{\omega\}$ and $\mathbb{N}$, one has to list all such polynomials and restrict to their real roots, *i.e.*, discard all roots ω with non-vanishing imaginary part.

Cantor [1] managed this by considering first polynomials with only non-negative coefficients $|a_j|$, and later allows all possible sign arrangements but keeping $a_k \geq 1$. Define

$$K \,:=\, \sum_{j=0}^{k-1} |a_j| \,+\, a_k, \tag{2}$$

hence K is a positive integer. *Cantor* introduced instead a *height* $n \in \mathbb{N}$ satisfying, for $k \in \{1, 2, ..., n\}$,

$$n \,=\, K \,+\, (k \,-\, 1)\,. \tag{3}$$

For each height n there is a finite number of real roots (the polynomial of degree k has at most k distinct real roots), denoted $\Phi(n)$ (in [1] $\varphi(n)$) counted without roots which already appeared at lower heights. Thus one can give a list of the $\Phi(n, k)$ real algebraic numbers $\{\omega\}$ for each height n, and degree k sorted in a specific way. Then these lists with possible k values from 1 to n give, after concatenation, the list of $\Phi(n)$ entries for height n, for each n, for $n \geq 1$. Thus each real algebraic number ω, of degree k determines the composition, *i.e.*, K, hence n. Therefore each ω has a unique address $c \in \mathbb{N}$ (c for count), and for each $c \in N$ there is a unique ω by construction.

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Some remarks and implications depending on the height n and degree k follow.

(a) To find the unsigned polynomials of height n, *i.e.*, the coefficients in eq. (3), one uses *relatively prime compositions* (*i.e.*, partitions without regard of order and with only relatively prime parts). For the number of relatively prime partitions of K, with $K \in \mathbb{N}$, see A000837(K) (note that $\gcd(K) = K$). The corresponding number of relatively prime compositions is found in A000740.

These compositions (also the partitions) have positive integer parts, but here, except for a_k, the coefficients may also vanish. Therefore one uses sometimes also part 0 in the later tables for the entries in the composition column. There the combinations will be recorded by falling powers of x of the corresponding polynomial. As explained below such 0s will only be shown if no confusion can arise.

(b) Positive, negative or complex roots for signed polynomials.
Descartes' rule of signs (`https://mathworld.wolfram.com/DescartesSignRule.htm`) is useful to find the maximal number of (real) positive roots $\#r_+$ (counting multiplicities) or negative roots $\#r_-$ of a polynomial $P(k, x)$ (with real coefficients, but here with integer coefficients). Let the number of changes of signs in the coefficients be S, then $\#r_+ \in \{S, S-2, ..., 0 \text{ or } 1\}$ The range of values of $\#r_-$ of $P(k, x)$ is obtained from the one of $\#r_+$ of $\pm P(k, -x)$.
Concerning complex roots remember that for real polynomials they come in pairs of complex conjugate roots.
E.g., For $P(3, x) = x^3 + x^2 - x - 1$ and for $-P(3, -x) = x^3 - x^2 - x + 1$ one finds $S = 1$ and $S = 2$, respectively. Hence there is one r_+ and either $\#r_- = 0$ (the case $r_+, c, \overline{c}$) or $\#r_- = 2$, the case of three real roots. In fact, the given $P(3, x)$ factorizes over the integers $P(3, x) = (x - 1)(x - 1)^2$, and the second case applies (with multiplicity counting) $1, -1, -1$.

(c) Case $k = 1$ which has $K = n$.
(i) If $n = 1$ this gives the first real root $\omega(1) = 0$ because $a_0 = 0$, following from $a_1 \geq 1$. This leads to the unique polynomial $1\,x$, and the composition will be recorded as $[1]$, not $[1, 0]$ (the $+$ sign of a_k is never recorded).
(ii) If $n \geq 2$ one needs $|a_0| \geq 1$ because if $a_0 = 0$ the polynomial would give $n\,x$ (with no recorded $+$ sign, and $\gcd(n, 0) = n \neq 1$), repeating the root $\omega(1)$ already found for height $n = 1$. This means that both parts a_1 and $|a_0|$ in the compositions are always non-vanishing, *i.e.*, the number of parts is $m = 2$.
Example: For $n = 2$ one finds, with the composition $[1, 1]$ with $K = 2$ and $m = 2$ the unsigned polynomial $x + 1$ with root -1, and then the signed polynomial with root $+1$. These two roots will be recorded in this order: $\omega(2) = -1$, $\omega(3) = +1$.

(d) Case $k = 2$ with $K = n - 1$. *i.e.*, $a_2 \geq 1$ and $|a_0| \geq 1$ (no factorization).
(i) Compositions with $a_1 = 0$. For $n = 2$ this is impossible.
Cases $n \geq 3$: The candidates for relative prime compositions are ($a_1 = 0$ is omitted) $[n - q, q - 1]$ and $[q - 1, n - q]$. For real roots and no factorization over the integers, with $q \geq 2$ and $n \geq q + 1$, the unsigned version does not qualify because it leads to complex solutions.For the signed versions not both, $q - 1$ and $n - q$, can be squares (otherwise factorization occurs), and $\gcd(n - q, q - 1) = \gcd(q - 1, n - q + (q - 1)) = \gcd(q - 1, n - 1) = 1$. *I.e.*, $\gcd(\overline{q - 1}, n - 1) = 1$, where $\overline{m} =$ A007947(m) (the squarefree kernel of m). Note that always $n - q \neq q - 1$ (because equality is only possible for $q = 2$, and $n = 2\,q - 1 = 3$, but $2 - 1$ is a square, as well as $3 - 2$.
Example 1: $q = 2$, $\gcd(1, n-1) = 1$ for $n \geq 3$, and $n \neq c^2 + 2$, for $c \geq 1$. Thus for $n = 3, 6, 11, 18, ...$ no such compositions qualify. $n = 4, 5, 7, 8, 9, , 10, 12, \ldots$.
Example 2: $q = 13$, $\overline{12} = 2 \cdot 3$, $n \geq 14$: $\gcd(2 \cdot 3, n - 1) = 1$, *i.e.*, n is even and $n \equiv \{0, 2\} \pmod 3$. Thus $n = 14, 18 \pmod 6$. The real roots come from signed compositions $[1, -12]$, $[7, -12], ...$ and $[5, -12]$, $[11, -12], ...$, and the ones with interchanged positive numbers.

(ii) Compositions with $|a_1| \geq 1$.
The compositions lead to the signed triples $[q, s1\,a_1, s2\,(n - 1 - (q + a_1))]$, with $q = a_2 \geq 1$, $a_1 \geq 1$, $n \geq 2 + q + a_1$, and signs $s1$ and $s2$ from $\{+1, -1\}$.

Complex conjugate pairs of roots arise if $a_1^2 - s2\,4\,q\,N(n,\,q,\,a_1) \,<\, 0$ with $N(n,\,q,\,a_1) \,:=\, n{-}1{-}(q{+}a_1) \,\geq\, 1$.
If $a_1^2 \,-\, s2\,4\,q\,N(n,\,q,\,a_1) \,=\, 0$ then factorization over $\mathbb{Z}$ of the corresponding polynomial appears.
If $a_1^2 \,-\, s2\,4\,q\,N(n,\,q,\,a_1) \,>\, 0$ then also factorization over $\mathbb{Z}$ appears if $A(c) \,:=\, (a_1^2 \,-\, c^2)/(4\,s2\,q)$ becomes an integer number, for integer $c \geq 0$ and $A(c) \,=\, N(n,\,q,\,a_1)$. Otherwise the two real roots of the corresponding irreducible polynomial are the algebraic numbers

$$x_- \,=\, -\frac{1}{2\,q}\left(s1\,a_1 \,-\, \sqrt{a_1^2 \,-\, s2\,4\,q\,N(n,\,q,\,a_1)}\right),\ \text{and}\ \mathrm{x_+} \,=\, -\frac{1}{2\,\mathrm{q}}\left(\mathrm{s1\,a_1} \,+\, \sqrt{\mathrm{a_1^2 \,-\, s2\,4\,q\,N(n,\,q,\,a_1)}}\right). \tag{4}$$

Example: For $n \,=\, 4$ the composition is $[1,\,1,\,1]$. The signed triples with two real roots are $[1,\,+1,\,-1]$ and $[1,\,-1,\,-1]$. They are $-\varphi$ and $\varphi \,-\, 1$, where φ is the golden section A001622, and $-(\varphi \,-\, 1)$ and φ. repsectively. This means that $\Phi(4,\,2) \,=\, 2\cdot 2 \,+\, 2\cdot 2 \,=\, 8$, including the above considered $a_1 \,=\, 0$ cases.

(e) Case $k \,=\, n \,\geq\, 1$ with $K \,=\, 1$.
This can only appear for height 1, when $a_0 \,=\, 0$, giving the root $0 \,=\, \omega(1)$. Thus k runs at most from 1 to $n - 1$, for $n \,\geq\, 2$.

(f) Case $k \,=\, n - 1$, where $K \,=\, 2$, and this case cannot appear for $n \,\geq\, 3$.
This is because $a_n \,\geq\, 1$ and $|a_0| \,\geq\, 1$ (factorization otherwise), hence the polynomials are $x^{n-1} \,+\, 1$ or $x^{n-1} \,-\, 1$. The latter one is reducible ($n \,\geq\, 3$), it factorizes into $(x \,-\, 1)\,\sum_{j=0}^{n-2} x^j$. The first one has the negated roots of the second one if n is even. If $n \,=\, 2\,q \,+\, 1$, for $q \,\geq\, 1$, then $\omega^q \,=\, \mp i$, hence ω is not real. Thus only degrees $k \,\in\, \{1,\,2,\,...,\,n-2\}$ qualify if $n \,\geq\, 3$.

(g) Order of composition with 0 parts (belonging to the same partition, not regarding their 0 parts).
These compositions are ordered such that the polynomials appear with falling powers of x.
The first two instances appears for $n \,=\, 5$ and $k \,=\, 3$ ($K \,=\, 3$), *i.e.*, $[1,\,1,\,0,\,1]$ and $[1,\,0,\,1,\,1]$. Both qualify for all 4 sign assignments (the + sign of a_3 is not recorded).
Such 0 parts are recorded whenever the number m of (non-zero) parts of qualifying compositions of K is less than $k+1$, except for $m \,=\, 1$ (only for $n \,=\, 1$ possible), and for all $m \,=\, 2$. The number of (inner) 0 parts is $z \,=\, k+1 \,-\, m$, for $m \,\geq\, 3$.

(h) Order of the ωs for each composition with given sign assignment (called signature). The order of the ωs increase.

(i) Order of compositions for given n and k.
The order for the list of the (qualifying) compositions of K, for $n \,=\, K \,+\, k \,-\, 1$, possibly with 0 parts, obeys the order of the underlying relatively prime partitions of K and given n and k (parts of partitions are ordered anti-lexicographically) with rising number of (positive) parts m. Compositions, not regarding 0 parts, belonging to like m partitions are ordered anti-lexicographically (*e.g.*, [3, 1, 1] before [2, 2, 1]).

(j) Order of the signature for a given composition.
If there are different signatures for a given composition (possibly with zero parts) the order is in general with the first ωs increasing. The exception is if it possible to arrange the order such that a $-|\omega|$ can be followed immediately by $|\omega|$. *E.g.*, $\omega(40)$, then $\omega(41)$ (not $\omega(42)$).

2 The list of real algebraic numbers of height 1 to 7

The list of the first 291 ωs, numerated by $c = 1, ,2, ..., 291$, is given in 7 tables:
Table 1: $\omega(1), ..., \omega(43)$
Table 2: $\omega(44), ..., \omega(83)$
Table 3: $\omega(84), ..., \omega(125)$
Table 4: $\omega(126), ..., \omega(167)$
Table 5: $\omega(168), ..., \omega(211)$
Table 6: $\omega(212), ..., \omega(257)$
Table 7: $\omega(258), ..., \omega(291)$.

The start is $\omega(0) = 0$ for $n = 1$ and $k = 1$, with $K = 1$, and the sign is not recorded (see remark **(c) (i)**).
We give some examples for the compositions illustrating the use (or omission) of 0 parts, their order and the order of the signatures (signs) for a given composition.

Example 1: Not recorded 0 parts.
For $n = 1$ see **(c) (i)**.
For all $n \geq 4$ and $k = 2$ and partitions with number of parts $m = 2$, like $\omega(12)$.
Example 2: Not recorded compositions.
(i) For all $n \geq 2$ and $k = n$, because $K = 1$ (see remark (**e**)).
(ii) For all ≥ 3 and $k = n - 1$. See remark (**f**).
(iii) If for all signatures either the polynomials have pairs of complex conjugate roots or the polynomials factorize over the integers (the reducible case).
The instance of a composition where for all signatures the ωs are complex cannot occur, because of *Descartes'* rule of signs (see (**b**)): start with all signs + and assume only complex roots, then change the sign of the last coefficient, to obtain a positive real root.
For $n = 6$ and $k = 2$ the compositions [4, 1] and [1, 4] have two distinct pairs of complex conjugate roots for sign $(+)$, and lead to factorization for sign $(-)$.
For $n = 7$ and $k = 2$ the composition [1, 2, 3] has a pair of complex conjugate roots for each of the signs $(+, +)$ and $(-, +)$, and factorization occurs for signs $(+, -)$ and $(-, -)$. The same applies to the compositions [2, 3, 1], [3, 1, 2] and [3, 2, 1].
For $n = 7$ and $k = 4$ the compositions with two 0 parts [2, 1, 0, 0, 1], [2, 0, 1, 0, 1] and [2, 0, 0, 1, 1] lead each to a pair of complex conjugate ωs for each of the signs $(+.+)$ and $(-, +)$, and factorization occurs for signs $(+, -)$ and $(-, -)$. The same applies to the compositions [1, 1, 0, 0, 2], [1, 0, 1, 0, 2] and [1, 0, 0, 1, 2].

Example 3: Not recorded signatures for compositions.
This happens if for recorded composition not all signatures are shown. Complex conjugat pairs may show up or factorization occurs.
For many compositions of order $k = 2$ with two parts a complex conjugate pairs appear, like for sign $(+)$ with $n = 4$ compositions [2, 1] and [1, 2], or for $n = 5$ compositions [3, 1] and [1, 3].
One complex conjugate pair may also appear for $k = 2$ compositions with more than two parts, like for $n = 5$ with composition [1, 2, 1] and signatures $(-, +)$ and $(+, +)$.
For $k = 3$ one may find factorizations like for $n = 6$ and composition [2, 1, 0, 1] and signatures $(+, +)$ and $(-, -)$. For each case there is a real root found already for lower n and, a complex conjugate pair.

Example 4: Compositions with a given signature but not k real solutions.
Like for $n = 6$, $k = 4$, composition [2, 1] with sign $(-)$ and only two real roots (and a complex conjugate pair).

The number of ωs for given height n and degree k, *i.e.*, $\Phi(n, k)$, and the total number of real algebraic solution for n, *i.e.*, $\Phi(n)$, are given in *Table 9*. The total numbers $\Phi(n) =$ A362366(n) are, for $n = 1, 2, ... 7$, [1, 2, 4, 12, 28, 72, 172].

References

[1] G. Cantor, Über eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen, Journal f. d. reine u. angew. Math. 77 (1874) 258-262.

G. Cantor, Über unendliche, lineare Punktmannigfaltigkeiten, Arbeiten zur Mengenlehre aus den Jahren 1872-1884, Teubner Archiv zur Mathematik, Band 2, Teubner Verlagsgesellschft, Leipzig, 1984, 1. Auflage; darin [B] pp. 20-24.

G. Cantor, On a Property of the Class of all Real Algebraic Numbers, English translation by P. Grant: https://www.coursehero.com/file/150879541/Cantors1874Paperpdf

[2] The On-Line Encyclopedia of Integer Sequences^{TM}, published electronically at http://oeis.org. 2023.

OEIS[2] A-numbers: A000740, A000837, A007947, A362362, A362363, A362364, A362365, A362366.
For the other A-numbers see the seven tables.

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Table 1: Real algebraic numbers $\{\omega(c)\}_{c=1}^{43}$ of height n and degree k.

c	height n	degree k	composition	signs	ω	A-number
1	1	1	[1]	()	$\mathbf{0}$	A001477(0)
2	2	1	[1, 1]	(+)	$-\mathbf{1}$	−A001477(1)
3				(−)	$+\mathbf{1}$	+A001477(1)
4	3	1	[2, 1]	(+)	$-\frac{1}{2}$	−A020761
5				(−)	$+\frac{1}{2}$	+A020761
6			[1, 2]	(+)	$-\mathbf{2}$	−A001477(2)
7				(−)	$+\mathbf{2}$	+A001477(2)
8	4	1	[3, 1]	(+)	$-\frac{1}{3}$	−A010701
9				(−)	$+\frac{1}{3}$	+A010701
10			[1, 3]	(+)	$-\mathbf{3}$	−A001477(3)
11				(−)	$+\mathbf{3}$	+A001477(3)
12		2	[2, 1]	(−)	$-1/\sqrt{2}$	∓A010503
13					$+1/\sqrt{2}$	∓A010503
14			[1, 2]	(−)	$-\sqrt{2}$	−A002193
15					$+\sqrt{2}$	+A002193
16			[1, 1, 1]	(+, −)	$-\varphi$	−A001622
17					$(\varphi - 1)$	A094214
18				(−, −)	$-(\varphi - 1)$	−A094214
19					$+\varphi$	A001622
20	5	1	[4, 1]	(+)	$-\frac{1}{4}$	−A020773
21				(−)	$+\frac{1}{4}$	+A020773
22			[1, 4]	(+)	$-\mathbf{4}$	−A001477(4)
23				(−)	$+\mathbf{4}$	+A001477(4)
24			[3, 2]	(+)	$-\frac{2}{3}$	−A010722
25				(−)	$+\frac{2}{3}$	+A010722
26			[2, 3]	(+)	$-\frac{3}{2}$	−A152623
27				(−)	$+\frac{3}{2}$	+A152623
28		2	[3, 1]	(−)	$-1/\sqrt{3}$	−A020760
29					$+1/\sqrt{3}$	+A020760
30			[1, 3]	(−)	$-\sqrt{3}$	−A002194
31					$+\sqrt{3}$	+A002194
32			[1, 2, 1]	(+, −)	$-(\sqrt{2} + 1)$	−A014176
33					$+\sqrt{2} - 1$	+A188582
34				(−, −)	$-(\sqrt{2} - 1)$	−A188582
35					$+(\sqrt{2} + 1)$	+A014176
36		3	[2, 1]	(+)	$-(\frac{1}{2})^{\frac{1}{3}}$	−A270714
37				(−)	$+(\frac{1}{2})^{\frac{1}{3}}$	+A270714
38			[1, 2]	(+)	$-2^{\frac{1}{3}}$	−A002580
39				(−)	$+2^{\frac{1}{3}}$	+A002580
40			[1, 1, 0, 1]	(+, +)	$-1.4655712318...$	−A092526
41				(−, −)	$+1.4655712318...$	+A092526
42				(−, +)	$-0.7548776662...$	−A075778
43				(+, −)	$+0.7548776662...$	+A075778
tbc	...	...	...	...	...	...

Table 2: Continued: Real algebraic numbers $\{\omega(c)\}_{c=44}^{83}$ of height n and degree k.

c	height n	degree k	composition	signs	ω	A-number
44	5 cont.	3	$[1,0,1,1]$	$(-,+)$	$-1.3247179572\ldots$	−A060006
45				$(-,-)$	$+1.3247179572\ldots$	+A060006
46				$(+,+)$	$-0.6823278038\ldots$	−A263719
47				$(+,-)$	$+0.6823278038\ldots$	+A263719
48	6	1	$[5,1]$	$(+)$	$-\frac{1}{5}$	−A000038
49				$(-)$	$+\frac{1}{5}$	+A000038
50			$[1,5]$	$(+)$	-5	−A001477(5)
51				$(-)$	$+5$	+A001477(5)
52		2	$[3,2]$	$(-)$	$-\sqrt{\frac{2}{3}}$	−A157697
53					$+\sqrt{\frac{2}{3}}$	+A157697
54			$[2,3]$	$(-)$	$-\sqrt{\frac{3}{2}}$	−A115754
55					$+\sqrt{\frac{3}{2}}$	+A115754
56			$[3,1,1]$	$(+,-)$	$-\frac{1+\sqrt{13}}{6}$	−(A188943 − 1)
57					$+\frac{-1+\sqrt{13}}{6}$	+A356033
58				$(-,-)$	$-\frac{-1+\sqrt{13}}{6}$	−A356033
59					$+\frac{1+\sqrt{13}}{6}$	+(A188943 − 1)
60			$[1,3,1]$	$(+,-)$	$-\frac{3+\sqrt{13}}{2}$	−A098316
61					$\frac{-3+\sqrt{13}}{2}$	+A085550
62				$(+,+)$	$-(1+\varphi)$	−A104457
63					$-(2-\varphi)$	−A132338
64				$(-,-)$	$-\frac{-3+\sqrt{13}}{2}$	-A085550
65					$\frac{3+\sqrt{13}}{2}$	+A098316
66				$(-,+)$	$2-\varphi$	+A132338
67					$1+\varphi$	+A104457
68			$[1,1,3]$	$(+,-)$	$-\frac{1+\sqrt{13}}{2}$	−A209927
69					$+\frac{-1+\sqrt{13}}{2}$	+A223139
70				$(-,-)$	$-\frac{-1+\sqrt{13}}{2}$	−A223139
71					$+\frac{1+\sqrt{13}}{2}$	+A209927
72			$[2,2,1]$	$(+,-)$	$-\frac{1+\sqrt{3}}{2}$	−A332133
73					$+\frac{-1+\sqrt{3}}{2}$	+A152422
74				$(-,-)$	$-\frac{-1+\sqrt{3}}{2}$	−A152422
75					$+\frac{1+\sqrt{3}}{2}$	+A332133
76			$[2,1,2]$	$(+,-)$	$-\frac{1+\sqrt{17}}{4}$	−A188934
77					$+\frac{-1+\sqrt{17}}{4}$	+(A188485 − 1)
78				$(-,-)$	$-\frac{-1+\sqrt{17}}{4}$	−(A188485 − 1)
79					$+\frac{1+\sqrt{17}}{4}$	+A188934
80			$[1,2,2]$	$(+,-)$	$-(1+\sqrt{3})$	−A090388
81					$-1+\sqrt{3}$	+A160390
82				$(-,-)$	$1-\sqrt{3}$	−A160390
83					$1+\sqrt{3}$	+A090388
tbc	…	…	…	…	…	…

Table 3: Continued: Real algebraic numbers $\{\omega(c)\}_{c=84}^{125}$ of height n and degree k.

c	height n	degree k	Composition	signs	ω	A-number
84	6 cont.	3	[3, 1]	(+)	$-(\frac{1}{3})^{\frac{1}{3}}$	−A072365
85				(−)	$+(\frac{1}{3})^{\frac{1}{3}}$	+A072365
86			[1, 3]	(+)	$-3^{\frac{1}{3}}$	−A002581
87				(−)	$+3^{\frac{1}{3}}$	+A002581
88			[2, 1, 0, 1]	(−, +)	−0.6572981061...	−A089826
89				(+, −)	+0.6572981061...	+A089826
90			[2, 0, 1, 1]	(+, +)	−0.5897545123...	−A356031
91				(+, −)	+0.5897545123...	+A356031
92			[1, 2, 0, 1]	(+, +)	−2.2055694304...	−A356035
93				(−, −)	+2.2055694304...	+A356035
94			[1, 0, 2, 1]	(+, +)	−0.4533976515...	−A272874
95				(−, −)	+0.4533976515...	+A272874
96			[1, 1, 0, 2]	(+, +)	−1.6956207695...	−A289265
97				(−, −)	+1.6956207695...	+A289265
98			[1, 0, 1, 2]	(−, +)	−1.5213797068...	−A356030
99				(−, −)	+1.5213797068...	+A356030
100			[1, 1, 1, 1]	(+, −, +)	−1.8392867552...	−A058265
101				(−, −, −)	+1.8392867552...	+A058265
102				(−, +, +)	−0.5436890126...	−A192918
103				(+, +, −)	+0.5436890126...	+A192918
104	6	4	[2, 1]	(−)	$-(\frac{1}{2})^{\frac{1}{4}}$	−A228497
105					$+(\frac{1}{2})^{\frac{1}{4}}$	+A228497
106			[1, 2]	(−)	$-2^{\frac{1}{4}}$	−A010767
107					$+2^{\frac{1}{4}}$	+A010767
108			[1, 1, 0, 0, 1]	(+, −)	−1.3802775690...	−A086106
109					+0.8191725133...	+A230151
110				(−, −)	−0.8191725133...	−A230151
111					+1.3802775690...	+A086106
112			[1, 0, 1, 0, 1]	(−, −)	$-\sqrt{\varphi}$	−A139339
113					$\sqrt{\varphi}$	+A139339
114				(+, −)	$-\sqrt{\varphi - 1}$	−A197762
115					$\sqrt{\varphi - 1}$	+A197762
116			[1, 0, 0, 1, 1]	(+, −)	−1.2207440846...	−A060007
117					+0.7244919590...	+A356032
118				(−, −)	−0.7244919590...	−A356032
119					+1.2207440846...	+A060007
120	7	1	[6, 1]	(+)	$-\frac{1}{6}$	−A020793
121				(−)	$+\frac{1}{6}$	+A020793
122			[1, 6]	(+)	−6	−A001477(6)
123				(−)	+6	+A001477(6)
124			[5, 2]	(+)	$-\frac{2}{5}$	−(A274981 − 1)
125				(−)	$+\frac{2}{5}$	+(A274981 − 1)
tbc	...	...	...	...	...	...

Table 4: Continued: Real algebraic numbers $\{\omega(c)\}_{c=126}^{167}$ of height n and degree k.

c	height n	degree k	Composition	signs	ω	A-number
126	7 cont.	1 cont.	[2, 5]	(+)	$-\frac{5}{2}$	−10 ∗ A020773
127				(−)	$+\frac{5}{2}$	+10 ∗ A020773
128			[4, 3]	(+)	$-\frac{3}{4}$	−A152627
129				(−)	$+\frac{3}{4}$	+A152627
130			[3, 4]	(+)	$-\frac{4}{3}$	−A122553
131				(−)	$+\frac{4}{3}$	+A122553
132		2	[5, 1]	(+)	$-(-1+2\varphi)$	−A002163
133				(−)	$+(-1+2\varphi)$	+A002163
134			[1, 5]	(+)	$-(-1+2\varphi)/5$	−A002163/5
135				(−)	$+(-1+2\varphi)/5$	+A002163/5
136			[4, 1, 1]	(+, −)	$-\frac{1+\sqrt{17}}{8}$	−(A189038 − 1)
137					$\frac{-1+\sqrt{17}}{8}$	+A358945
138				(−, −)	$-\frac{-1+\sqrt{17}}{8}$	−A358945
139					$+\frac{1+\sqrt{17}}{8}$	+A189038 − 1
140			[1, 4, 1]	(+, −)	$-(1+2\varphi)$	−A098317
141					$+(-3+2\varphi)$	+(A134972 − 1)
142				(+, +)	$-(2+\sqrt{3})$	−A019973
143					$-(2-\sqrt{3})$	−A019913
144				(−, −)	$-(-3+2\varphi)$	−(A134972 − 1)
145					$+(1+2\varphi)$	+A098317
146				(−, +)	$+(2-\sqrt{3})$	+A019913
147					$+(2+\sqrt{3})$	−A019973
148			[1, 1, 4]	(+, −)	$-\frac{1+\sqrt{17}}{2}$	−A222132
149					$+\frac{-1+\sqrt{17}}{2}$	+A222133
150				(−, −)	$-\frac{-1+\sqrt{17}}{2}$	−A222133
151					$+\frac{1+\sqrt{17}}{2}$	+A222132
152			[2, 3, 1]	(+, −)	$-\frac{3+\sqrt{17}}{4}$	−A188485
153					$+\frac{-3+\sqrt{17}}{4}$	+A246725
154				(−, −)	$-\frac{-3+\sqrt{17}}{4}$	−A246725
155					$+\frac{3+\sqrt{17}}{4}$	+A188485
156			[1, 3, 2]	(+, −)	$-\frac{3+\sqrt{17}}{2}$	−A178255
157					$+\frac{-3+\sqrt{17}}{2}$	+(A178255 − 3)
158				(−, −)	$-\frac{-3+\sqrt{17}}{2}$	−(A178255 − 3)
159					$+\frac{3+\sqrt{17}}{2}$	+A178255
160		3	[4, 1]	(+)	$-\frac{2^{1/3}}{2}$	−A235362
161				(−)	$+\frac{2^{1/3}}{2}$	+A235362
162			[1, 4]	(+)	$-\frac{2}{2^{1/3}}$	−A005480
163				(−)	$+\frac{2}{2^{1/3}}$	+A005480
164			[3, 2]	(+)	$-\frac{(18)^{1/3}}{3}$	−A358943
165				(−)	$+\frac{(18)^{1/3}}{3}$	+A358943
166			[2, 3]	(+)	$-\frac{(12)^{1/3}}{2}$	−A319034
167				(−)	$+\frac{(12)^{1/3}}{2}$	+A319034
tbc	...	...	...	...	...	...

Table 5: Continued: Real algebraic numbers $\{\omega(c)\}_{c=168}^{211}$ of height n and degree k.

c	height n	degree k	Composition	signs	ω	A-number
168	7 cont.	3 cont.	$[3,1,0,1]$	$(+,+)$	$-0.8241226211...$	$-$A357465
169				$(-,-)$	$+0.8241226211...$	$+$A357465
170				$(-,+)$	$-0.5981934981...$	$-$A357464
171				$(+,-)$	$+0.5981934981...$	$+$A357464
172			$[3,0,1,1]$	$(+,+)$	$-0.5365651646...$	$-$A357467
173				$(+,-)$	$+0.5365651646...$	$+$A357467
174				$(-,+)$	$-0.8513830728...$	$-$A357466
175				$(-,-)$	$+.0.8513830728..$	$+$A357466
176			$[1,3,0,1]$	$(+,+)$	$-3.1038034027...$	$-($A357103 $+ 1)$
177				$(-,-)$	$+3.1038034027...$	$+($A357103 $+ 1)$
178				$(+,-)$	$-2.8793852415...$	$-($A332437 $+ 1)$
179					$-0.6527036446...$	$-$A178959
180					$+0.5320888862...$	$+($A322438 $- 3)$
181				$(-,+)$	$-0.5320888862...$	$-($A322438 $- 3)$
182					$+0.6527036446...$	$+$A178959
183					$+2.8793852415...$	$+($A332437 $+ 1)$
184			$[1,0,3,1]$	$(-,+)$	$-1.8793852415...$	$-$A332437
185					$+0.3472963553...$	$+$A130880
186					$+1.5320888862...$	$+($A332438 $- 2)$
187				$(-,-)$	$-1.5320888862...$	$-($A332438 $- 2)$
188					$-0.3472963553...$	$-$A130880
189					$+1.8793852415...$	$+$A332437
190				$(+,+)$	$-0.3221853546...$	$-$A357104
191				$(+,-)$	$+0.3221853546...$	$+$A357104
192			$[1,1,0,3]$	$(+,+)$	$-1.8637065278...$	$-$A356034
193				$(-,-)$	$+1.8637065278...$	$+$A356034
194				$(-,+)$	$-1.1745594102...$	$-$A357100
195				$(+,-)$	$+1.1745594102...$	$+$A357100
196			$[1,0,1,3]$	$(-,+)$	$-1.6716998816...$	$-$A294644
197				$(-,-)$	$+1.6716998816...$	$+$A294644
198				$(+,+)$	$-1.2134116627...$	$-$A337569
199				$(+,-)$	$+1.2134116627...$	$+$A337569
200			$[2,2,0,1]$	$(+,+)$	$-1.2971565081...$	$-$A357109
201				$(-,-)$	$+1.2971565081...$	$+$A357109
202				$(-,+)$	$-0.5651977173...$	$-$A273065
203				$(+,-)$	$+0.5651977173...$	$+$A273065
204			$[2,0,2,1]$	$(-,+)$	$-1.1914878839...$	$-($A316711 $- 1)$
205				$(-,-)$	$+1.1914878839...$	$+($A316711 $- 1)$
206				$(+,+)$	$-0.4238537990...$	$-$A357463
207				$(+,-)$	$+0.4238537990...$	$+$A357463
208			$[2,1,0,2]$	$(+,+)$	$-1.1974293369...$	$-$A357105
209				$(-,-)$	$+1.1974293369...$	$+$A357105
210				$(-,+)$	$-0.8580943294...$	$-$A357106
211				$(+,-)$	$+0.8580943294...$	$+$A357106
tbc	...	...	...	...	...	...

Table 6: Continued: Real algebraic numbers $\{\omega(c)\}_{c=212}^{257}$ of height n and degree k.

c	height n	degree k	Composition	signs	ω	A-number
212	7 cont.	3 cont.	[2, 0, 1, 2]	(−, +)	−1.1653730430...	−A357107
213				(−, −)	+1.1653730430...	+A357107
214				(+, +)	−0.8351223484...	−A357108
215				(+, −)	+0.8351223484...	+A357108
216			[1, 2, 0, 2]	(+, +)	−2.3593040859...	−A357101
217				(−, −)	+2.3593040859...	+A357101
218				(−, +)	−0.8392867552...	−(A058265 − 1)
219				(+, −)	+0.8392867552...	+(A058265 − 1)
220			[1, 0, 2, 2]	(−, +)	−1.7692923542...	−A273066
221				(−, −)	+1.7692923542...	+A273066
222				(+, +)	−0.7709169970...	−A357102
223				(+, −)	+0.7709169970...	+A357102
224			[2, 1, 1, 1]	(+, −, +)	−1.2337519285...	−A358182
225				(−, −, −)	+1.2337519285...	+A358182
226				(−, −, +)	−0.8294835409...	−A358183
227				(+, −, −)	+0.8294835409...	+A358183
228				(+, +, +)	−0.7389836215...	−A358184
229				(−, +, −)	+0.7389836215...	+A358184
230			[1, 2, 1, 1]	(+, −, +)	−2.5468182768...	−A358181
231				(−, −, −)	+2.5468182768...	+A358181
232				(+, −, −)	−2.2469796037...	−A231187
233					−0.5549581320...	−A255240
234					+0.8019377358...	+(A160389 − 1)
235				(+, +, +)	−1.7548776662...	−A109134
236				(−, +, −)	+1.7548776662...	+A109134
237				(−, −, +)	−0.8019377358...	−(A160389 − 1)
238					+0.5549581320...	+A255240
239					+2.2469796037...	+A231187
240				(−, +, +)	−0.4655712318...	−A088559
241				(+, +, −)	+0.4655712318...	+A088559
242			[1, 1, 2, 1]	(+, −, +)	−2.1478990357...	−A357470
243				(−, −, −)	+2.1478990357...	+A357470
244				(+, −, −)	−1.8019377358...	−A160389
245					−0.4450418679...	−A255241
246					+1.2469796037...	+A225249
247				(−, −, +)	−1.2469796037...	−A225249
248					+0.4450418679...	+A255241
249					+1.8019377358...	+A160389
250				(+, +, +)	−0.5698402909...	−A357471
251				(−, +, −)	+0.5698402909...	+A357471
252				(−, +, +)	−0.3926467817...	−A357472
253				(+, +, −)	+0.3926467817...	+A357472
254			[1, 1, 1, 2]	(+, +, +)	−1.3532099641...	−A357469
255				(−, +, −)	+1.3532099641...	+A357469
256				(−, −, +)	−1.2055694304...	−A137421
257				(+, −, −)	+1.2055694304...	+A137421
tbc	...	...	...	...	...	...

Table 7: Continued: Real algebraic numbers $\{\omega(c)\}_{258}^{291}$ of height n and degree k.

c	height n	degree k	Composition	signs	ω	A-number
258	7 cont.	3 cont.	$[1,1,1,2]$ cont.	$(-,+,+)$	$-0.8105357137...$	−A357468
259				$(+,+,-)$	$+0.8105357137...$	+A357468
260		4	$[3,1]$	$(-)$	$-\frac{27^{\frac{1}{4}}}{3}$	−A358186
261					$+\frac{27^{\frac{1}{4}}}{3}$	+A358186
262			$[1,3]$	$(-)$	$-3^{\frac{1}{4}}$	−A011002
263					$+3^{\frac{1}{4}}$	+A011002
264			$[1,2,0,0,1]$	$(+,-)$	$-2.1069193403...$	−A358188
265					$+0.7166727492...$	+A358187
266				$(-,-)$	$-0.7166727492...$	−A358187
267					$+2.1069193403...$	+A358188
268			$[1,0,2,0,1]$	$(-,-)$	$-1.5537739740...$	−A278928
269					$+1.5537739740...$	+A278928
270				$(+,-)$	$-0.6435942529...$	−A154747
271					$+0.6435942529...$	+A154747
272			$[1,0,0,2,1]$	$(+,-)$	$-1.3953369944...$	−A358190
273					$+0.4746266175...$	+A358189
274				$(-,-)$	$-0.4746266175...$	−A358189
275					$+1.3953369944...$	+A358190
276		5	$[2,1]$	$(+)$	$-\frac{1}{2}^{\frac{1}{5}}$	−A358938
277				$(-)$	$+\frac{1}{2}^{\frac{1}{5}}$	+A358938
278			$[1,2]$	$(+)$	$-2^{\frac{1}{5}}$	−A005531
279				$(-)$	$+2^{\frac{1}{5}}$	+A005531
280			$[1,1,0,0,0,1]$	$(-,+)$	$-0.8566748838...$	−A230152
281				$(+,-)$	$+0.8566748838...$	+A230152
282			$[1,0,1,0,0,1]$	$(-,+)$	$-1.2365057033...$	−A358940
283				$(-,-)$	$+1.2365057033...$	+A358940
284				$(+,+)$	$-0.8376197748...$	−A358939
285				$(+,-)$	$+0.8376197748...$	+A358939
286			$[1,0,0,1,0,1]$	$(+,+)$	$-1.1938591113...$	−A358942
287				$(-,-)$	$+1.1938591113...$	+A358942
288				$(-,+)$	$-0.8087306004...$	−A358941
289				$(+,-)$	$+0.8087306004...$	+A358941
290			$[1,0,0,0,1,1]$	$(-,+)$	$-1.1673039782...$	−A160155
291				$(-,-)$	$+1.1673039782...$	+A160155
...	...	...	...	...	...	...

Table 9: Phi(n, k), the number of real algebraic numbers ω of height n and degree k.

n \ k	**1**	**2**	**3**	**4**	**5**	**6 ...**	**Phi(n)**
1	1						1
2	2						2
3	4	0					4
4	4	8	0				12
5	8	8	12	0			28
6	4	32	20	16	0		72
7	12	28	100	16	16	0	172
...	...	...	...	...	...	...	...