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# On Positive Integer Descartes-Steiner Curvature Quintuplets

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**Abstract**

In Descartes' five circle problem integer curvatures (inverse radii) are considered. The positive integer curvature triple $[c_1,\, c_2,\, c_3]$ (dimensionless), with non-decreasing entries for three given mutually touching circles, leading to integer curvatures $[c_{4,-},\, c_{4,+}]$ for the two circles touching the given ones is called a *Descartes-Steiner* triple. They come in two types: $[c,\, c,\, d]$ (or $[c,\, d,\, d]$) and triples with distinct entries. The first case is related to Pythagorean triples. The distinct curvature case is more involved and needs a combined representations of certain binary quadratic forms of the indefinite and definite type. The degenerate case when a straight line touches the three given touching circles can also be characterized completely.

## 1 Introduction

Motivated by the section on *Descartes* in [21] where his three circle problem is discussed we consider the case of positive integer curvatures (inverse radii or bends) of three given mutually touching circles (we do not consider vanishing curvatures, *i.e.*, the straight line degenerate cases). The touching circular disks are assumed to have as their pairwise intersections only the touching point, *i.e.*, the three touching circles are outside one another, see Figure 1. The question is for which triple $[c_1,\, c_2,\, c_3]$, with non-decreasing positive integer entries (in some inverse length unit), the curvatures for the two circles which touch each of the three given ones are also integer. *Descartes* [4] showed in a letter from 1643 to Princess Elizabeth of Bohemia how to find a quadratic equation for the radius of the inner touching circle, leaving details as an exercise for a laborious calculator, and *Steiner* [17] (not mentioning *Descartes*) found a nice form for this quadratic equation for the curvature $c_4$. For *Descartes*' problem see also [22] , [24] and [7] with the *Beecroft*, *Soddy*, *Pedoe* and *Coxeter* references). This quadratic equation is

$$\left(\sum_{j=1}^{4} c_j\right)^2 - 2\left(\sum_{j=1}^{4} c_j^2\right) = 0\ . \tag{1}$$

The two solutions for $c_4$, called $c_{4,+}$ and $c_{4,-}$, are

$$c_{4,\pm} = c_1 + c_2 + c_3 \pm 2\sqrt{c_1\, c_2 + c_1\, c_3 + c_2\, c_3}. \tag{2}$$

Vanishing $c_{4,-}$ indicates a degenerate case if a straight line touches the three given (non-degenerate) circles (see *Figure 5*). $c_{4,-}$ (which can also be negative) stands for the circle of curvature $|c_{4,-}|$ circumscribing the three given ones. $c_{4,+}$ gives the curvature of the inner circle. See *Fig. 2 (b)* of [7] and the examples in the *Note 1* below after *Proposition 4*. We shall use the abbreviation $q = \sqrt{c_1\, c_2 + c_1\, c_3 + c_2\, c_3} > 0$. A computer program can be written for finding the primitive *Descartes-Steiner* ($DS$)-triples: Select from all in'teger triples $0 < c_1 \le c_2, \le c_3 = N \in \mathbb{N}$ the ones with $\gcd(c_1,\, c_2,\, c_3) = 1$ and integer $q^2$, then compute $c_{4,\pm}$. This produces a list $L_N = [c_1,\, c_2,\, c_3, c_{4,-},\, c_{4,+}, q > 0]$ for all primitive $DS$ triples with curvatures $\le N$. In this way *Tables 2, 3, 4* and *5* will later be found.

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*Section 2* gives the coordinates of the centers of the three touching circles and of the circum- and inscribed ones. A classical problem for certain areas involving these five circles is discussed. Then the definition of primitive $DS$-triples and some of their general properties are treated.

In *Section 3* the primitive $DS$-triples of types $[c,\, c,\, d]$ and $[c,\, c,\, d]$ are studied, and mapped to primitive Pythagorean triples.

In *Section 4* the distinct curvature triples come in three cases $q = c_3$, $q \leq c_3$, and $q > c_3$, named **i**), **ii**), and **iii**), respectively.
The curvatures for case **i**) are obtained from the proper solutions of the indefinite binary form $X^2 - 2,\, Y^2$ of discriminant 8 representing the numbers $-s(n)$, with $s(n) =$ A058529$(n)$, with $n \geq 2$. Such linkable OEIS [15] A-numbers will be given later without quotation. A uniqueness property selects in each pair of conjugate families of infinitely many solutions one with $0 < X < Y$. See *Proposition 6*, following many *Lemmata*.

The curvatures for the cases **ii**) and **iii**) are obtained similarly. There two binary quadratic forms enter, one indefinite like in case **i** but with $Y \to \widehat{Y}$, the other positive definite $t^2 + 2\,k^2$, of discriminant $-8$, with $q = c_3 - k$ and $q = c_3 + k$, with $k > 0$, for the tow cases **ii**) and **iii**), respectively. $\widehat{Y} = Y + k$ and $\widehat{Y} = Y - k$, respectively. These forms have to represent simultaneously a number $-a$ and $a > 2$, respectively. For the indefinite form only one solution for each pair of conjugate solutions will qualify with the respective positive $\widehat{Y}$, and the definite form also has one solution with $k > 0$. The possible values for $a$ are in both cases determined, and come in three types, **a**), **b**)*and***c**), depending on the proper or improper solutions of the two forms. See *Proposition 7* and *Proposition 8* for the two cases.
These solutions have then to be investigated for $0 < X < Y$ and primitivity of $[c_1,\, c_2\, c_3]$. Examples for these investigations are given. There is no general way to find the final curvature solutions for a given candidate $a$.

**Note in passing:** This touching circle problem bears strong resemblance to *G. F. Malfatti*'s circles ([23],[25], *Martin* [10], pp. 92-96) treated in connection with *Malfatti*'s problem from (1803). There the circle problem is to inscribe in a general triangle three mutually (externally) touching circles, each of which touches two of the sides of the triangle. *J. Steiner* [18] generalized this to the present five touching circle problem.

## 2 Touching Circles and DS curvature triples

**Exercise 1: Choice of centers of the three touching circles**

Given the (dimensionless) curvature triple $[c_1,\, c_2,\, c_3]$, with non-decreasing positive entries, we can choose the center of the circle with radius $r_1 = \dfrac{1}{c_1}$ as the origin in the $(x,\, y)-$plane: $M_1 = (0,\, 0)$. The center of the next circle with radius $r_2 = \dfrac{1}{c_2}$ is chosen (if necessary after a rotation) as $M_2 = (r_1 + r_2, 0)$. Then the center of the third circle with radius $r_3 = \dfrac{1}{c_3}$ is chosen (if necessary after a reflection on the $x$-axis) in the upper half-plane and it will be $M_3 = (x_3,\, y_3)$ with $y_3^2 = (r_1 + r_3)^2 - x_3^2 = (r_2 + r_3)^2 - (r_1 + r_2 - x_3)^2$. This means that

$$\begin{aligned} x_3 &= \frac{r_1^2 + r_1\, r_2 - r_2\, r_3 + r_1\, r_3}{r_1 + r_2} = \frac{c_1\, c_2 + c_1\, c_3 + c_2\, c_3 - c_1^2}{c_1\, c_3\, (c_1 + c_2)} \\ &= \frac{q^2 - c_1^2}{c_1\, c_3\, (c_1 + c_2)}\,, \qquad (3) \\ y_3 &= 2\, \frac{\sqrt{r_1\, r_2\, r_3\, (r_1 + r_2 + r_3)}}{r_1 + r_2} = \frac{2\, q}{(c_1 + c_2)\, c_3} > \frac{1}{c_3} = r_3\,. \qquad (4) \end{aligned}$$

The polar representation will be given later in *Exercise 4.* For example, if $[c_1, c_2, c_3] = [1, 2, 3]$, *i.e.*, $[r_1, r_2, r_3] = \left[1, \frac{1}{2}, \frac{1}{3}\right]$ (in some length unit), then $q = \sqrt{11}$ and $M_3 = \left(\frac{10}{9}, \frac{2}{3}\sqrt{11}\right)$. See *Figure 1.*

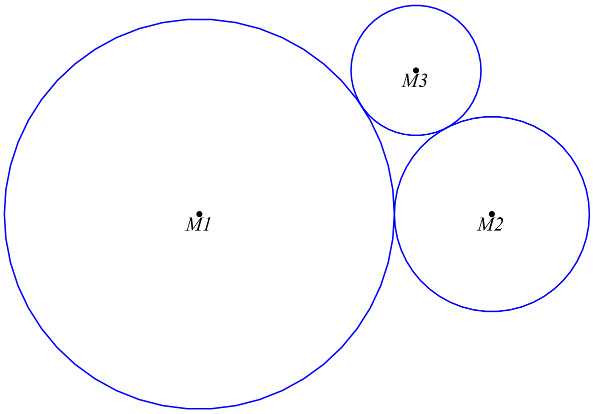


**Fig. 1:** Input circles for the curvature triple [1, 2, 3].

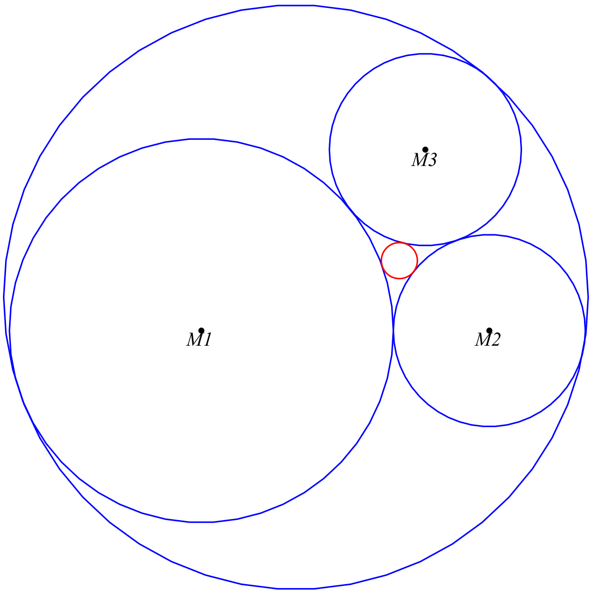


**Fig. 2:** Five circles for the curvature triple $\left[\frac{1}{2}, 1, 1\right]$, and their outer and inner circles.

**Exercise 2: Case $\mathbf{c_{4,-} = 0}$**

A straight line appears as fourth touching 'circle' for $c_{4,-} = 0$. In this case the three given circles with curvatures $[c_1, c_2, c_3]$, where $1 \le c_1 \le c_2 \le c_3$, have to satisfy the original eq. (1) with only three curvatures. Therefore, the third curvature is given in terms of the two other ones by the quadratic equation $c_3^2 - 2(c_1 + c_2)c_3 + (c_1 - c_2)^2 = 0$. The result is

$$c_3 = c_1 + c_2 + 2\sqrt{c_1 c_2}\ . \tag{5}$$

The other solution with a negative square root does not qualify because $c_3 \ge c_2 \ge c_1$, and for this solution $c_3 \ge c_2$ leads to $c_1 \ge c_2$ which is a contradiction.
*I.e.*, if $c_{4,-} = 0$ then $\frac{c_3}{2}$ is the sum of the arithmetic and geometric mean of the two other curvatures.
*E.g.*, [1, 4, 9] has $c_{4,-} = 0$ (see *Table 2*) and indeed $9 = 1 + 4 + 2 \cdot 2$, or $\frac{9}{2} = \frac{5}{2} + 2$.
This case is later analyzed further in *Exercise* 4, **ii**) and *Propositions* 5 and 6.

**Exercise 3: A classical problem**

Given a circle of radius $r_1$ touching two smaller touching circles of the same radius $r_2 < r_1$, but not circumscribing them. See *Figure 2.* Compute the area $F$ which is left from the circumscribing fourth touching circle of curvature $-c_{4,-}$ if the three interior circular disks are taken away. ($c_4 < 0$ because $c_1 = \frac{1}{r_1} < 4\,c_2 = \frac{4}{r_2}$, and $r_1 > r_2 > \frac{r_2}{4}$). Note that in this exercise the curvatures ar not necessary integer.

A similar problem is to find the area $F'$ which remains after the fifth interior disk with curvature $c_{4,+}$ together with the three given circular disks is subtracted from the circumscribing circular disk.

In [21], p. 43, the first version of this problem is related to one of the 15 from 150 problems proposed by *Sawaguchi Kazuyuki* in 1670 (according to [20] in 1671) which he could not solve. *Seki Takakazu* (also known as *Seki Kowa*) [19] gave solutions to these 15 problems in his book *Hatsubi Sampō* (1674), and they have later been reviewed by *Tanaka Yoshizane* in his *Sampō-meikai*.

**Solution:** Take as length unit ($l.u.$) the length of the small radius $r_2$. Then the given big circle has dimensionless radius $\hat{r}_1 := \dfrac{r_1}{r_2}$. The circumscribing fourth circle has dimensionless radius $\hat{r} := \frac{r}{r_2}$ with negative dimensionless curvature $\hat{c}_{4,-} = -\frac{1}{\hat{r}}$. Take $r_1 = r_2 + a\, l.u.$, *i.e.*, $\hat{r}_1 = 1 + a$, with $a > 0$. Thus, $\dfrac{1}{\hat{r}} = -\hat{c}_{4,-} = 2\sqrt{1 + 2\,\hat{c}_1} - (2 + \hat{c}_1)$, with $\hat{c}_1 = r_2/r_1$. The negativity of $\hat{c}_{4,-}$ follows from $\hat{c}_1 < 4$, *i.e.*, $a > -\frac{3}{4}$, which is satisfied because here $a > 0$. The result is $\dfrac{1}{\hat{r}} = -\hat{c}_{4,-} = 2\sqrt{\dfrac{3+a}{1+a}} - \dfrac{3 + 2\,a}{1+a} > 0$. The dimensionless area $\hat{F} := \dfrac{F}{r_2^2}$ satisfies therefore,

$$\frac{\hat{F}}{\pi} = \hat{r}^2 - (2 + \hat{r}_1^2) = \frac{1}{\hat{c}_{4,-}^2} - \left(2 + \frac{1}{\hat{c}_1^2}\right) , \tag{6}$$

which becomes, after some rewriting,

$$\frac{\hat{F}(a)}{\pi} = \frac{4\,(1+a)^2\,(3 + 5\,a + 2\,a^2)\,\sqrt{\frac{a+3}{a+1}} - 2\,(1 + 2\,a)\,(3 + 4\,a + 2\,a^2 + 2\,a^3)}{(3 + 4\,a)^2} . \tag{7}$$

For the original *Sawaguchi Kazuyuki* problem, according to [21], $a = \dfrac{5}{2}$ and the area $F$ was given as 120 in some length unit $L.u.$ squared. Thus, the scaling factor $\lambda = \frac{l.u. = r_2}{L.u.}$ to be applied to the present computation is given by $120 = \frac{F}{L.u.^2} = \lambda^2 \hat{F}\left(\frac{5}{2}\right)$, hence $\lambda = \sqrt{\dfrac{120}{\hat{F}\left(\frac{5}{2}\right)}} \approx 2.493$, due to $\hat{F}\left(\dfrac{5}{2}\right) = \pi\,(\dfrac{1}{169}\,(196\,\sqrt{77} - 681)) \approx 19.312$.

Thus, in the original problem the small radius is $\lambda L.u.$ and the radius of the large one is $\frac{7}{2}\,\lambda\, L.u.$.

If the area of the inner touching circle with curvature $\hat{c}_{4,+} =: \dfrac{r_2}{r_+}$ is also subtracted, the result becomes

$$\frac{\hat{F}'(a)}{\pi} = \frac{8\,(1+a)^3\,(3 + 2\,a)\,\sqrt{\frac{a+3}{a+1}}}{(3 + 4\,a)^2} - (2 + (1+a)^2) . \tag{8}$$

For $a = \dfrac{5}{2}$ this becomes $\hat{F}'\left(\dfrac{5}{2}\right) = \pi\left(\dfrac{392}{169}\sqrt{77} - \dfrac{57}{4}\right) \approx 19.176$. $\square$

Starting with a curvature triple $[c_1, c_2, c_3]$ where $1 \le c_1 \le c_2 \le c_3$ (the non-degenerate circle cases), we ask for the conditions for which $c_1\,c_2 + c_1\,c_3 + c_2\,c_3 =: q^2$ with an integer $q$, and we choose $q > 0$, in fact, $q \ge 3$. This is equivalent to $c_{4,\pm}$ being integers. Such curvature triples will be called **Descartes-Steiner triples** or *DS*-triples. (This should not be confused with the *Steiner* triples, or triple systems, from block design.) Of course, $[c_1', c_2', c_3'] = [g\,c_1, g\,c_2, g\,c_3]$ with positive integer $g$ leads to $q' = g\,q$, and it is sufficient to study **primitive DS-triples**.

**Exercise 4: Centers of the fourth and fifth touching circle**

**i) Case $\mathbf{c_{4,-}} \neq 0$**

The two circles which touch the given three touching circles can be found from the so-called complex *Descartes* equation [7]:

$$\left(\sum_{j=1}^{4} c_j\, z_j\right)^2 - 2\left(\sum_{j=1}^{4} (c_j\, z_j)^2\right) = 0 , \tag{9}$$

where in our convention the complex centers of the three given touching circles are $z_1 = 0$, $z_2 = r_1 + r_2$ and $z_3 = x_3 + y_3\, i$, with $x_3$ and $y_3$ from eqs. (3) and (4). For the polar representation of $z_3 = \rho_3\, e^{\varphi_3\, i}$ one has $\rho_3 = \frac{q^2 + c_1^2}{c_1\, c_3\, (c_1 + c_2)}$ and $\varphi_3 = \arctan\left(\frac{2\, q\, c_1}{q^2 - c_1^2}\right)$. One can replace in eq. (1) $c_j$ by $c_j\, z_j$, and therefore one obtains for the centers of the two touching circles $z_{4,-}$ and $z_{4,+}$

$$z_{4,\pm} = \frac{1}{c_{4,\pm}} \left(c_1\, z_1 + c_2\, z_2 + c_3\, z_3 \pm 2\, \sqrt{c_1\, c_2\, z_1\, z_2 + c_1\, c_3\, z_1\, z_3 + c_2\, c_3\, z_2\, z_3}\,\right), \tag{10}$$

where now the square root of a complex number has to be taken. The square root will now be abbreviated as $Q$. With our choice for $z_j$, $j = 1, 2, 3$, one finds, for the center $M4p = (x_{4,+} = Re(z_{4,+}),\, y_{4,+} = Im(z_{4,+}))$, with the real $z_2 = \frac{1}{c_1} + \frac{1}{c_2}$,

$$\begin{aligned}
x_{4,+} &= \frac{1}{c_{4,+}} \left(c_2\, z_2 + c_3\, x_3 + 2\, \sqrt{c_2\, c_3\, z_2\, \rho_3}\, \cos\left(\frac{\varphi_3}{2}\right)\right) & (11)\\
&= \frac{1}{c_{4,+}} \left(\frac{c_1 + c_2}{c_1} + c_3\, x_3 + \frac{2\, q}{c_1}\right), & (12)\\
y_{4,+} &= \frac{1}{c_{4,+}} \left(c_3\, y_3 + 2\, \sqrt{c_2\, c_3\, z_2\, \rho_3}\, \sin\left(\frac{\varphi_3}{2}\right)\right) & (13)\\
&= \frac{1}{c_{4,+}} \left(c_3\, y_3 + 2\right). & (14)
\end{aligned}$$

The simplification occurs due to the identities $\arctan x = \arccos \frac{1}{\sqrt{1 + x^2}}$ for $x > 0$, and $\cos\left(\frac{x}{2}\right) = \sqrt{\frac{1}{2}\,(1 + \cos\, x)}$, with the formula with a minus sign for $\sin\left(\frac{x}{2}\right)$.
The center $M4m = (x_{4,-},\, y_{4,-})$ is given by

$$\begin{aligned}
x_{4,-} &= \frac{1}{c_{4,-}} \left(c_2\, z_2 + c_3\, x_3 - 2\, \sqrt{c_2\, c_3\, z_2\, \rho_3}\, \cos\left(\frac{\varphi_3}{2}\right)\right), & (15)\\
&= \frac{1}{c_{4,-}} \left(\frac{c_1 + c_2}{c_1} + c_3\, x_3 - \frac{2\, q}{c_1}\right), & (16)\\
y_{4,-} &= \frac{1}{c_{4,-}} \left(c_3\, y_3 - 2\, \sqrt{c_2\, c_3\, z_2\, \rho_3}\, \sin\left(\frac{\varphi_3}{2}\right)\right) & (17)\\
&= \frac{1}{c_{4,-}} \left(c_3\, y_3 - 2\right). & (18)
\end{aligned}$$

The positive square roots are chosen. For example, for $[c_1, c_2, c_3] = [2, 3, 6]$ one finds $q = 6$, $x_3 = \frac{8}{15}$, $y_3 = \frac{2}{5}$, $c_{4,-} = -1$, $c_{4,+} = 23$ (see *Table 2*), $x_{4,+} = \frac{1}{23}\left(\frac{57}{10} + 6\right) = \frac{117}{230} \approx 0.509$ and $y_{4,+} = \frac{1}{23}\left(\frac{12}{5} + 2\right) = \frac{22}{115} \approx 0.191$. $x_{4,-} = -\left(\frac{57}{10} - 6\right) = \frac{3}{10} = 0.3$ and $y_{4,-} = -\left(\frac{12}{5} - 2\right) = -\frac{2}{5} = -0.4$
For another example [8, 9, 17] with $q = 19$ and $[c_{4,-}, c_{4,+}] = [-4, 72]$ see *Figure 4* and *Table 2.*

**ii) Case $\mathbf{c_{4,-} = 0}$**

The center of the inner touching circle $M4p = (x_{4,+},\, y_{4,+})$ is identical with the one given in eqs. (11) and (13). Now a straight line will touch the three given touching circles with centers $M_1$, $M_2$ and $M_3$ chosen as explained above. To find the slope and the touching points $P$ and $Q$ with the first and second circle, respectively, one can employ a geometric construction shown in *Figure 3*, found, *e.g.*, in [11]. This tangent will then automatically be tangent also to the third circle with center $M_3$, and the touching point is $R$. One draws two auxiliary circles: a circle around the origin $M1$ with radius $r_1 - r_2$ and a circle around $MT = \left(\frac{r_1 + r_2}{2}, 0\right)$. The relevant intersection point of these two circles is $C = (x_C, y_C)$. By *Thales* the angle $\angle(M_1, C, M_2) = \pi/2$. The touching point $P$ on the first circle $(M1; r_1)$ is obtained by extending the radial line $\overline{M1, C}$. The tangent point $Q$ on the second circle $(M2; r_2)$ is obtained by the parallel to the line segment $\overline{C, M2}$ through $P$. The slope of the straight line through $P$ and $Q$ (the tangent) is given by $\tan(\pi - \alpha) = -\tan\alpha$ with $\tan\alpha = \frac{y_C}{r_1 + r_2 - x_C}$. The coordinates of $C$ as intersection point of the two auxiliary circles are

$$x_C = \frac{(r_1 - r_2)^2}{r_1 + r_2}, \qquad y_C = 2\sqrt{r_1 r_2}\,\frac{r_1 - r_2}{r_1 + r_2}. \tag{19}$$

Written in terms of curvatures this becomes

$$x_C = \frac{(c_1 - c_2)^2}{c_1\, c_2\, (c_1 + c_2)}, \qquad y_C = 2\,\frac{1}{\sqrt{c_1\, c_2}}\,\frac{c_2 - c_1}{c_2 + c_1}. \tag{20}$$

The coordinates of $P$ satisfy (use an intercept theorem) $x_P = d\sqrt{r_1^2 - x_P^2}$ with $d = \frac{x_C}{y_C}$, or $x_P = \frac{d\, r_1}{\sqrt{1 + d^2}}$ and $y_P = \sqrt{r_1^2 - x_P^2}$, which yields

$$x_P = r_1\,\frac{r_1 - r_2}{r_1 + r_2}, \qquad y_P = 2\sqrt{r_1 r_2}\,\frac{r_1}{r_1 + r_2}. \tag{21}$$

In terms of curvatures this becomes

$$x_P = \frac{c_2 - c_1}{c_1\,(c_2 + c_1)}, \qquad y_P = \frac{2}{\sqrt{c_1\, c_2}}\,\frac{c_2}{c_2 + c_1}. \tag{22}$$

The tangent through $P$ satisfies $y = -\tan(\alpha)\, x + c$ *i.e.*, $c = y_P + \tan(\alpha)\, x_P = \frac{r_1\,(r_1 + r_2)}{2\sqrt{r_1 r_2}} = \frac{1}{2\sqrt{c_1\, c_2}}\,\frac{c_1 + c_2}{c_1}$. With $\tan\alpha = \frac{r_1 - r_2}{2\sqrt{r_1 r_2}} = \frac{1}{2\sqrt{c_1\, c_2}}\,(c_2 - c_1)$ this becomes

$$y = \frac{\sqrt{r_1 r_2}}{2\, r_2}\left(-\frac{r_1 - r_2}{r_1}\, x + r_1 + r_2\right). \tag{23}$$

In terms of curvatures this becomes

$$y = \frac{1}{2\sqrt{c_1\, c_2}}\left(-(c_2 - c_1)\, x + \frac{c_1 + c_2}{c_1}\right). \tag{24}$$

The coordinates of the tangent point $Q$ are then obtained from $\sin\beta = \frac{y_P}{r_1} = \frac{y_Q}{r_2}$ and $x_Q = r_1 + r_2 + \sqrt{r_2^2 - y_Q^2}$

$$x_Q = r_1\,\frac{r_1 + 3\, r_2}{r_1 + r_2}, \qquad y_Q = 2\sqrt{r_1 r_2}\,\frac{r_2}{r_1 + r_2}. \tag{25}$$

In terms of curvatures this becomes

$$x_Q = \frac{c_2 + 3\, c_1}{c_1\,(c_2 + c_1)}, \qquad y_Q = \frac{2}{\sqrt{c_1\, c_2}}\,\frac{c_1}{c_2 + c_1}. \tag{26}$$

The coordinates of the touching point $R$ on the third circle $(M3; r_3)$ satisfy $y_R = y_3 + \frac{r_3}{r_1} y_P$ and $x_R = x_3 + \sqrt{r_3^2 - (y_R - y_3)^2} = x_3 + r_3 \frac{r_1 - r_2}{r_1 + r_2}$, hence, with *eqs.* (3) and (4),

$$x_R = r_1 + 2\,\frac{r_1 - r_2}{r_1 + r_2}\, r_3\,, \qquad y_R = 2\,\frac{\sqrt{r_1\, r_2\, r_3}}{r_1 + r_2}\,(\sqrt{r_1 + r_2 + r_3} + \sqrt{r_3}\,)\,, \tag{27}$$

where $r_3$ is from eq. (5) $r_3 = r_1\, r_2/(r_1 + r_2 + 2\,\sqrt{r_1\, r_2})$.
In terms of the curvatures eq. (27) this becomes

$$x_R = \frac{1}{c_1} + 2\,\frac{c_2 - c_1}{c_2 + c_1}\,\frac{1}{c_3}\,, \qquad y_R = \frac{2}{c_3\,(c_1 + c_2)}\,(q + \sqrt{c_1\, c_2})\,, \tag{28}$$

where $c_3$ is given by eq. (5).
For the example $[c_1, c_2, c_3] = [1, 4, 9]$ we have $q = 7$ with $c_{4,-} = 0$ and $c_{4,+} = 28$ *see Table 2* and *Table 6.* $M3 = \left(\frac{16}{15}, \frac{14}{45}\right) \approx (1.06, 0.31)$, $\tan\alpha = \frac{3}{4}$, $\sin\beta = \frac{4}{5}$, $c = \frac{5}{4} = 1.25$, $C = \left(\frac{9}{20}, \frac{3}{5}\right) = (0.45, 0.6)$, $P = \left(\frac{3}{5}, \frac{4}{5}\right) = (0.6, 0.8)$, $Q = \left(\frac{7}{5}, \frac{1}{5}\right) = (1.4, 0.2)$, $R = \left(\frac{17}{15}, \frac{2}{5}\right) \approx (1.13, 0.4)$.

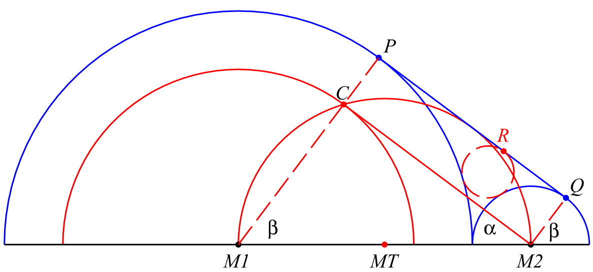


**Fig. 3:** Geometric construction of tangent points $P$, $Q$ and $R$.

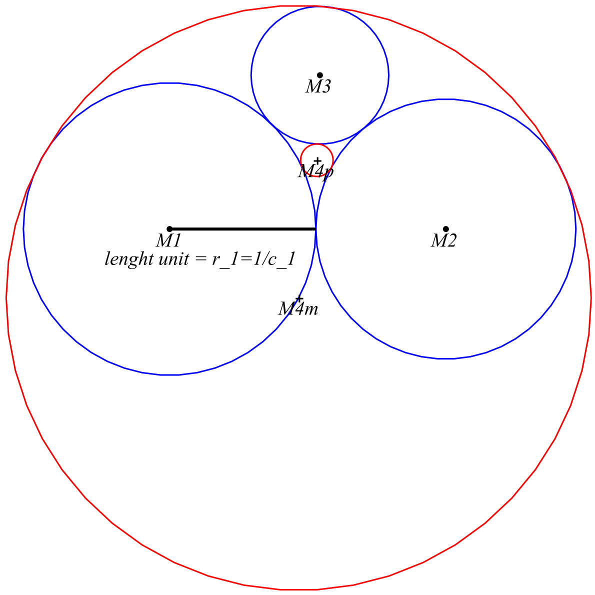


**Fig. 4:** Five touching circles for [8, 9, 17].

**Definition 1: Primitive DS-triples**

*An integer curvature triple* $[c_1, c_2, c_3]$ *with* $0 < c_1 \le c_2 \le c_3$, *and* $\gcd(c_1, c_2, c_3) = 1$ *is called a primitive DS-triple if both curvatures* $c_{4,+}$ *and* $c_{4,-}$ *given in eq. (2) are integers.*

**Proposition 1:** *There is no DS triple of the type* $[c, c, c]$, $c \in \mathbb{N}$.

**Proof**: $\sqrt{3}$ is not an integer number.

Primitive $DS$-triples obey a certain parity rule stated as follows.

**Proposition 2: Parity of primitive DS-triples**

*Two cases are possible for primitive DS-triples* $[c_1, c_2, c_3]$*:*

**i)** *Two of the curvatures are even and the other is odd. Then the sum of the even ones is congruent to* $0\,(mod\,4)$.

**ii)** *Two of the curvatures are odd and the other is even. Then the sum of the odd ones is congruent to* $2\,(mod\,4)$.

Before proceeding with the elementary proof some examples are given.

**Example 1:**

Case **i)**: [2, 3, 6] with $q = 6$ has $2 + 6 = 8 \equiv 0\,(mod\,4)$. [2, 2, 3] with $q = 4$ has $2 + 2 = 4 \equiv 0\,(mod\,4)$.
Case **ii)**: [2, 3, 15] with $q = 9$ has $3 + 15 = 18 \equiv 2\,(mod\,4)$. [25, 25, 48] with $q = 55$ has $25 + 25 = 50 \equiv 2\,(mod\,4)$.

**Proof:** Primitivity forbids that all three curvatures are even. They also cannot be all odd because always $q^2 \equiv 0$ or $1\,(mod\,4)$ ($q$ is even or odd) but then $q^2$ would become $3\,(mod\,4)$. This leaves the two cases **i**) and **ii**). In the first case, with, say, $c_1 = 2\,C_1 + 1$, $c_2 = 2\,C_2$ and $c_3 = 2\,C_3$ one finds $q^2 = 4\,(C_1\,C_2 + C_1\,C_3 + C_2\,C_3) + 2\,(C_2 + C_3)$ but $q^2 \equiv 0$ or $1\,(mod\,4)$, hence $c_2 + c_3 \equiv 0\,(mod\,4)$. In the second case with, say, $c_1 = 2\,C_1$, $c_2 = 2\,C_2 + 1$ and $c_3 = 2\,C_3 + 1$ one has $q^2 = 4\,(C_1\,C_2 + C_1\,C_3 + C_2\,C_3 + C_1) + 2\,(C_2 + C_3) + 1$, which is odd, therefore $q^2 \equiv 1\,(mod\,4)$. Thus, $C_2 + C_3$ has to be even and $c_2 + c_3 \equiv 2(mod\,4)$. □

**Proposition 3:** *For primitive DS-triples one has* $gcd(c_1 + c_2,\, c_1 + c_3,\, c_2 + c_3) = 1$.

**Proof:** Note that this *gcd* certainly has to be odd because otherwise this would require all curvatures to be either even or odd. But assuming that $c_1 + c_2 = g\,X$, $c_1 + c_3 = g\,Y$ and $c_2 + c_3 = g\,Z$, with odd $g \geq 3$, implies $2\,c_1 = g\,(X + Y - Z) =: g\,a_1$, $2\,c_2 = g\,(X - Y + Z) =: g\,a_2$ and $2\,c_3 = g\,(-X + Y + Z) =: g\,a_3$. Hence $a_j = 2\,A_j$, for $j = 1, 2, 3$, and then $c_j = g\,A_j$, but this contradicts the primitivity, hence $g$ has to be 1. □

It is clear that also the converse is true. If for $x := c_1 + c_2$, $y := c_1 + c_3$ and $z := c_2 + c_3$ $gcd(x,\, y,\, z) = 1$ holds, then also $gcd(c_1,\, c_2,\, c_3) = 1$. Assume that $gcd(c_1,\, c_2,\, c_3) = g > 1$. This contradicts $gcd(x,\, y,\, z) = 1$.

## 3 DS-triples with curvatures [c, c, d] or [c, d, d]

For the next *Proposition* we define the following set of ordered pairs $\mathcal{M}$.

**Definition 2: Set of ordered pairs (n, m)**

$$\mathcal{M} = \left\{ (n,\, m) \in \mathbb{N} \times \mathbb{N} \,\middle|\, n > m,\; gcd(n,\, m) = 1, (-1)^{n+m} = -1 \right\}. \tag{29}$$

The last condition means that $n$ and $m$ have opposite parity. For $n \geq 2$ there are $\varphi(2\,n)/2 =$ A055034($n$) values for $m$, where $\varphi$ is *Euler*'s totient function. For the characteristic triangle of $\mathcal{M}$ see A249866.

**Proposition 4: Primitive DS-triples of type [c, c, d] or [c, d, d]**

*Primitive DS-triples of the type* $[c,\, c,\, d]$ *or* $[c,\, d,\, d]$ $(c < d)$ *come in two types, parametrized by* $(n,\, m) \in \mathcal{M}$.

*Type* **I**: *Either* $[(n-m)^2,\, (n-m)^2,\, 2\,n\,m]$ *or* $[2\,n\,m,\, (n-m)^2,\, (n-m)^2]$, *depending on* $n^2 + m^2 < 4\,n\,m$ *or* $n^2 + m^2 > 4\,n\,m$, *respectively. In both cases* $q = n^2 - m^2$. $c_{4,+} = 2\,n\,(2\,n - m)$ *and* $c_{4,-} = 2\,m\,(2\,m - n)$.

*Type* **II**: *Either* $[2\,m^2,\, 2\,m^2,\, n^2 - m^2]$ *or* $[n^2 - m^2,\, 2\,m^2,\, 2\,m^2]$, *depending on* $\sqrt{3}\,m < n$ *or* $\sqrt{3}\,m > n$, *respectively. In both cases* $q = 2\,n\,m$. $c_{4,+} = (n + m)\,(n + 3\,m)$ *and* $c_{4,-} = (n - m)\,(n - 3\,m)$.

**Proof:** This problem reduces to primitive Pythagorean triples as follows. Consider a *DS*-triple of the type $[c,\, c,\, d]$, where $1 \leq c < d$. Now $q^2 = c^2 + 2\,c\,d = (c + d)^2 - d^2$, *i.e.*,

$$(c + d)^2 = q^2 + d^2 =: z^2, \tag{30}$$

where we choose $z = c + d$. This means that we have to consider the Pythagorean triple $(q,\, d,\, z)$. Similarly, a *DS*-triple of the type $[c,\, d,\, d]$, where $1 \leq c < d$, leads to the Pythagorean triple $(q,\, c,\, z)$, with $(c + d)^2 = q^2 + c^2 =: z^2$, with $z = d + c$ like in the $[c,\, c,\, d]$ case.

A Pythagorean triple $(x, y, z)$ is called primitive if the three positive integers $x$, $y$ and $z$ are pairwise relatively prime (see *e.g.*, [14]).

We first prove

**Lemma 1:** *Each primitive DS-triple $[c, c, d]$ leads to a primitive Pythagorean triple $(q, d, z)$. Similarly, each primitive DS-triple $[c, d, d]$ leads to a primitive Pythagorean triple $(q, c, z)$. Here $z := c + d$.*

**Proof:** It is sufficient to prove the $[c, c, d]$ case. The proof of other case runs similarly.
With $\gcd(c, d) = 1$ one finds, by assuming that $\gcd(q, d) = g > 1$, that $c^2 = g\,(g\,Q^2 - 2\,c\,D)$ with $q = g\,Q$ and $d = g\,D$. Thus, $g|c^2$, hence $c = g\,C$, in contradiction to $\gcd(c, d) = 1$. Therefore $g = 1$. It is clear that $\gcd(d, z = c + d) = 1$. Also $\gcd(q, z) = 1$ because $\gcd(q^2, z) = \gcd(c\,(c + 2\,d), z)$ and $\gcd(c, z) = d(c, d) = 1$ as well as $\gcd(c + 2\,d, z) = 1$ due to $\gcd(c + 2\,d, z) = \gcd(c + d, d)$, hence $\gcd(q^2, z) = \gcd(q, z) = 1$. □
Note that it is sufficient to prove $\gcd(q, d) = 1$ because for Pythagorean triples $(q, d, z)$ this is equivalent to pairwise relatively prime $q$, $d$ and $z$. This fact is used in [6], Theorem 225, p. 190.

That each primitive Pythagorean triple leads to two primitive $DS$-triples, will be shown later as a *Corollary 1* to this *Proposition*.

Using a well known theorem on primitive Pythagorean triples with $q^2 + d^2 = z^2$ (see *e.g.*, [14], Theorem 5.5, p. 232, or [6], Theorem 225, p. 190) one proves that the structure of primitive $DS$-triples come in the two claimed types $I$ or $II$.
The primitive Pythagorean triples $(q, d, z)$ are characterized by the elements of the set $\mathcal{M}$ from *Definition* 2 either as case **i**): $q = n^2 - m^2$, $d = 2\,n\,m$ and $z = n^2 + m^2$ ($d$ is even, and then $q$ is odd) or, interchanging $q$ and $d$, as case **ii**): $q = 2\,n\,m$, $d = n^2 - m^2$ and $z = n^2 + m^2$ ($d$ is odd, and then $q$ is even). In both cases $c = z - d$.
For $c < d$, case **i**) needs $(n - m)^2 < 2\,n\,m$ *i.e.*, $n^2 + m^2 < 4\,n\,m$, and the $DS$-triple is $[(n - m)^2, (n - m)^2, 2\,n\,m]$. Case **ii**) needs $2\,m^2 < n^2 - m^2$ *i.e.*, $\sqrt{3}\,m < n$, and the $DS$-triple is $[2\,m^2, 2\,m^2, n^2 - m^2]$.
Similarly, the primitive Pythagorean triples $(q, c, z)$ with case **i**): $q = n^2 - m^2$, $c = 2\,n\,m$ and $z = n^2 + m^2$, leads for $c < d$, *i.e.*, $n^2 + m^2 > 4\,n\,m$, to the $DS$-triple $[2\,n\,m, (n - m)^2, (n - m)^2]$. The case **ii**): $q = 2\,n\,m$, $c = n^2 - m^2$ and $z = n^2 + m^2$, leads for $c < d$, *i.e.*, $n < \sqrt{3}\,m$, to the $DS$-triple $[n^2 - m^2, 2\,m^2, 2\,m^2]$.

What is left is to prove that these $DS$-triples, derived from primitive Pythagorean triples, are also primitive.
For both primitive Pythagorean triples of type **i)** one writes $\gcd((n - m)^2, 2\,n\,m) = \gcd(2\,n\,m, n^2 + m^2) = \gcd(n\,m, n^2 + m^2)$. In the last step we used that $n^2 + m^2$ is odd from *Definition* 2. Now $n^2 + m^2$ cannot be divisible by a factor $f_n > 1$ of $n$ because otherwise $m^2$, hence $m$, would also be divisible by $f_n$ but this contradicts the primitivity of the Pythagorean triple guaranteed by $gcd(n, m) = 1$. Similarly a factor $f_m > 1$ of $m$ cannot divide $n^2 + m^2$. Hence $\gcd((n - m)^2, 2\,n\,m) = 1$.
For both primitive Pythagorean triples of type **ii)** one writes $\gcd(n^2 - m^2, 2\,m^2) = \gcd(n^2 - m^2, m^2)$ (because $n^2 - m^2$ is odd) $= \gcd(m^2, n^2) = 1$ because $\gcd(m, n) = 1$.

Therefore case **I** of the Proposition is given by case **i**) from the primitive Pythagorean triples $(q, d, z)$ and by case **i**) from the primitive Pythagorean triples $(q, c, z)$, and case **II** is given by the cases **ii**) of these two primitive Pythagorean triples.
Note that the parity of the primitive $DS$-triples are in accordance with *Proposition* 2. □

**Corollary 1:** *Each primitive Pythagorean triple leads to two primitive DS-triples, either both of type $[c, c, d]$ or one of type $[c, c, d]$ and the other of type $[c, d, d]$. No pair of triples of the type $[c, d, d]$ is possible.*

This follows from the the conditions for cases **I** and **II**. A pair of primitive $DS$-triples $[c, d, d]$ would require $n^2 + m^2 > 4\,n\,m > 4\,m^2$, *i.e.*, $n^2 > 3\,m^2$, contradicting $3\,m^2 > n^2$.

**Example 2: Primitive DS-triple [c, d, d], and primitive Pythagorean triple (q, c, c + d)**

The primitive $DS$-triple [44, 81, 81] is of type **I** because $n\,m = 22$ and $n - m = 9$, hence $n = 11$ and $m = 2$, and $q = 117 = n^2 - m^2$ with the corresponding primitive Pythagorean triple (117, 44, 125). This satisfies indeed the $[c,\,d,\,d]$ type **I** inequality $n^2 + m^2 = 125 < 88 = 4\cdot 11\cdot 2$. Also $c_{4,-} = -28 = 4\,(4 - 11)$ and $c_{4,+} = 440 = 22\,(22 - 2)$.

Note that this primitive Pythagorean triple $(117, 44, 125)$ leads, because of the symmetry (44, 117, 125), also to the type **II** primitive $DS$-triple of type $[c,\,c,\,d]$ namely $[8,\,8, 117]$ with $q = 2\,n\,m = 44$, $c_{4,-} = 45$ and $c_{4,+} = 221$. The inequality is here $2\,\sqrt{3} < 11$. See *Table 1* for both $DS$-triples with $[n,\,m] = [11,\,2]$.

**Note 1: Special instances of Proposition 4**

For type **I** consider $(n, m = n-1)$, $n \geq 2$, which is indeed an element of $\mathcal{M}$ ($gcd(n, n-1) = 1$ is easy to prove by contradiction). The first inequality applies because $2, n\,(n-1) - 1 > 3 > 0$, hence the primitive $DS$-triple is $[1,\,1,\,4\,T(n-1)]$, where $n = 2$, with the triangular number $T(n-1) =$ A000217$(n-1)$. Here $q = \sqrt{1 + 8\,T(n-1)} = 2\,n - 1 = n^2 - (n-1)^2$. $c_{4,+} = 4\,T(n)$ and $c_{4,-} = 4\,T(n-2)$. These are the triples [1, 1, 4], [1, 1, 12], [1, 1, 24], [1, 1, 40] ... with $q$ values 3, 5, 7, 9 ... and $[c_{4,-},\,c_{4,+}]$ pairs [0, 12], [4, 24], [12, 40], [24, 60], ... .

For type **II** take $m = 1$ , with $\sqrt{3} < n$, leading to primitive triples $[2,\,2,\,4\,k^2 - 1]$, if $n = 2\,k$ and $k \geq 1$. Here $q = 2\,n = 4\,k$, $c_{4,+} = 4\,k\,(k + 2) + 3 = (2\,k + 1)\,(2\,k + 3)$ and $c_{4,-} = (2\,k - 1)\,(2\,k - 3)$. This is [2, 2, 3], [2, 2, 15], [2, 2, 35], ... with $q$ values $4, 8, 12$, ... and $[c_{4,-}, c_{4,+}]$ pairs $[-1,\,15]$, [3, 35], [15, 63], ... .

The touching circles for [1, 1, 4] with $[c_{4,-},\,c_{4,+}] = [0,\,12]$, and for [2, 2, 3] with $[c_{4,-},\,c_{4,+}] = [-1,\,15]$ are shown in *Figure* 5 and 6, respectively.

The possible values for $z = c + d$ give the length of the (dimensionless) hypotenuses of primitive Pythagorean triangles. They are given as a number triangle in A222946, and in ordered form, with multiplicity, in A020882. This is the $z$-sequence 5, 13, 17, 25, 29, 37, 41, 53, 61, 65, 65, 73, 85, 85, 89, 101, .... In *Table 1* one can find for the corresponding $(n,\,m)$ values the primitive $DS$-triples of *Proposition 4*.

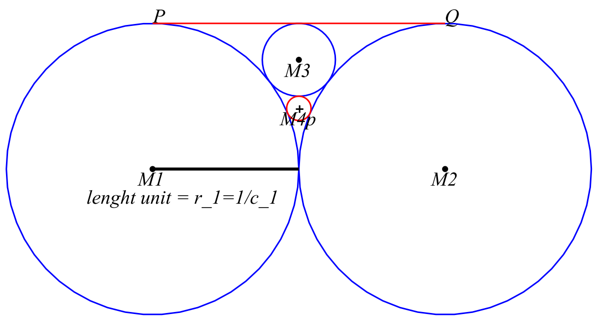


**Fig. 5:** Four touching circles and a straight line for [1, 1, 4].

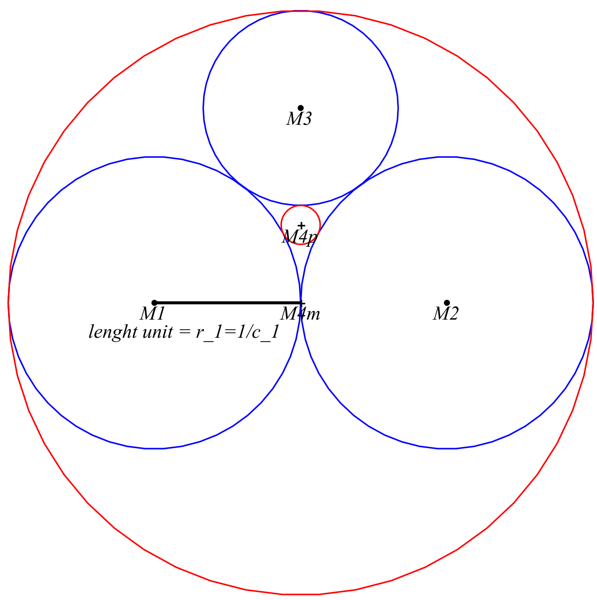


**Fig. 6:** Five touching circles for [2, 2, 3].

The next topic is to find all primitive $DS$-triples which belong to the special degenerated case when $c_{4,-} = 0$, *i.e.*, when a straight line touches the three touching circles. In order for this to happen there is a condition on the curvature of the third given circle in terms of the curvatures of the other two ones.

In *Exercise* 2, eq. (5), it was shown that $c_3 = c_1 + c_2 + 2\sqrt{c_1 c_2}$. This solution for $c_3$ is compatible with $\sum_{j=1}^{3} c_j^2 = 2q^2$, obtained from $c_{4,-} = 0$, with $q$ the square root defined after eq. (2).
Due to the $AGM$ inequality [26] in the case of two variables $4\sqrt{c_1 c_2} \leq c_3$ (of course, $4\sqrt{c_1 c_2} \leq 2(c_1 + c_2) \leq 4c_3$). For the $DS$-triple [1, 4, 9] (see *Tables 2* and *6*) one obtains $8 \leq 9$).
Because of the homogeneity with degree 1 of the $c_3$ equation, a common factor of $c_1$ and $c_2$ will also multiply $c_3$. Therefore one imposes $gcd(c_1, c_2) = 1$. Then the only such $DS$-triple with $c_1 = c_2$ is [1, 1, 4] with $q = 3$ and $c_{4,+} = 12$ (see *Tables 1* and *6*). All other solutions for $1 \leq c_1 < c_2$ with integer $\sqrt{c_1 c_2}$, *i.e.*, $c_1 c_2 = h^2$, with integer $h > 0$, are obtained from $c_1 = M^2$ and $c_2 = N^2$ with positive integers $M$ and $N > M$. This follows from the fact that $c_1$ and $c_2$ have no common factor $> 1$, so each curvature has to be a square by itself. This is summarized in the following *Proposition*.

**Proposition 5: All primitive DS-triples with $\mathbf{c_{4,-} = 0}$**

*The primitive DS-triples which satisfy $c_{4,-} = 0$ are given by*

$$[M^2,\, N^2,\, (M+N)^2],\ \textit{with}\ \gcd(M, N) = 1,\ 1 \leq M \leq N,\ \textit{and}\ 4q = c_{4,+} = 2(M^2 + N^2 + (M+N)^2)\,. \tag{31}$$

**Proof:**
Obviously $\gcd(M, N) = 1$ follows from $gcd(c_1, c_2) = 1$. The case $c_1 = c_2$ needs $M = N = 1$, by primitivity, and $c_3 = 4$, and $c_{4,+} = 4q = 2 \cdot 6 = 12$, from $c_{4,-} = 0$. If $c_1 < c_2$ then $1 \leq M < N$, and $c_3 = (M + N)^2$. $c_{4,+} = 4q = 2(M^2 + N^2 + (M + N)^2) = 4(M^2 + N^2 + MN)$.
The $DS$-triples are primitive because also $\gcd((M + N)^2, M^2) = 1 = \gcd((M+N)^2, N^2)$ due to the same eqs. without the squares: $\gcd(M + N, N) = \gcd(N, M + N) = \gcd(N, M + N - N) = \gcd(N, M) = \gcd(M, N) = 1$. Similarly for the other gcd. □

For the characteristic triangular array $T(N, M)$ with $\gcd(N, M) = 1$ see A054521. There a 0 entry stands for $\gcd(N, M) > 1$.

For the multiplicity $m_3(n)$ of primitive $DS$-triples with $c_{4,-} = 0$ and $c_3 = n^2$, for $n \geq 2$, the following *Proposition* holds.

**Proposition 6: Multiplicity $\mathbf{m_3(n)}$ for $\mathbf{c_3 = n^2}$, if $\mathbf{c_{4,-} = 0}$**

$$m_3(n) = \begin{cases} 1 & \text{if n} = 2\,, \\ \varphi(n)/2 & \text{if n} \geq 3\,. \end{cases} \tag{32}$$

*Here $\varphi$ is Euler's totient function A000010, and $m_3(1) = 0$, of course.*

**Proof:** The multiplicity of $c_3 = (M + N)^2 = n^2$, for $1 \leq M \leq N$, is the sane as the one for $N + M = n$, for $n \geq 2$, which is given by the number of partitions of $n$ with two relative prime parts, given in A023022 = {1, 1, 1, 2, 1, 3, 2, 3, 2, 5, ...}.
This can be proved using the triangle $T(n, M) =$ A054521$(n, M)$, for $n \geq 2$ and $M = 1, 2, ..., n-1$ (the original triangle without its diagonal), because the sum over the anti-diagonals $T(n - M, M)$, for $M = 1, 2, ..., \lfloor \frac{n}{2} \rfloor$, give $m_3(n)$, and due to $\gcd(n - M, M) = \gcd(n, M)$, this is $m_3(n) = \sum_{m=1}^{\lfloor \frac{n}{2} \rfloor} T(n, M)$. Then $m_3(2) = 1$, and for even $n \geq 4$ the entries are symmetric around the middle 0 (due to the gcd property), and for odd $n$ the first $(n-1)/2$ entries are repeated. Therefore, in both cases the sum is one half of the sum over row $n \geq 3$ which is $\varphi(n)$. □

**Example 3:** $n = 5, c_3 = 25$, with the two ($\varphi(5/2) = 2$) primitive $DS$-triples [4, 9, 25] and [1, 16, 25], with $q = 19$ and 21, respectively.

The instances $c_3 = n^2$, for $n = 2, 3, ..., 24$, are shown in *Table 6*.

# 4 DS-triples with distinct curvatures

Now we have to analyze the primitive $DS$-triples with distinct (pairwise different) curvatures $[c_1,\, c_2,\, c_3]$ satisfying $0 < c_1 < c_2 < c_3$ and $gcd(c_1,\, c_2,\, c_3) = 1$. See *Table 2* for these triples with $c_3$ values from 6, 7, ..., 38 (except for $c_3 = 8$). This table was found by testing all distinct primitive increasingly ordered positive integer triples for (positive) integer $q = \sqrt{c_1\, c_2 + c_1\, c_3 + c_2\, c_3}$.
The $c_j$'s are distinct, and $q$ can come in three versions:

$$\textbf{i)}\ q = c_3,\quad \textbf{ii)}\sqrt{2\, c_1^2 + c_2^2} < q < c_3,\quad \text{or}\quad \textbf{iii)}\ q > c_3. \tag{33}$$

In **ii)** the first inequality is trivial. Examples for these three cases are found in *Table 2*: **i)** $c_3 = q = 6$, [2, 3, 6], **ii)** $c_3 = 9 > 7 = q\,( >= 3\sqrt{2}\,)$, [1, 4, 9], and **iii)** $c_3 = 7 < 9 = q$, [3, 6, 7].

**Case i) $\mathbf{q = c_3}$**

This part turns out to become quite lengthy. The result will be given finally in *Proposition 7*.

Put $c_1 = x$, $c_2 = y$ and $c_3 = q$, with $0 < x < y$ and $\gcd(x,\, y,\, q) = 1$. Now $q^2 - (x+y)\, q - x\, y = 0$ or, because integer $q > 0$, $2\, q = x + y + s$ with $s := \sqrt{(x + y)^2 + 4\, x\, y}$, and $s \in \mathbb{N}$. This results in an indefinite binary quadratic form representing $s^2$, *viz*

$$x^2 + 6\, x\, y + y^2 = s^2,\ \text{with}\ s \in \mathbb{N},\ \ 0 < x < y\,. \tag{34}$$

This binary quadratic form is indefinite because the discriminant $D = 6^2 - 4 = 32 > 0$. Therefore, if for given $s \geq 1$ an integer solution $(x,\, y)$ exists (not regarding the inequality) there will be infinitely many such solutions.
Solutions satisfying also the inequality are found within the range $2\sqrt{2}\, x < s < 2\sqrt{2}\, y$. *E.g.*, $(x,\, y) = (2,\, 3)$ with $q = c_3 = 6$ needs $6 \leq s \leq 8$, and there is fact only a solution for $s = 7$ with $(x,\, y) = (2,\, 3)$. Or the $DS$-triple [3, 10, 15] qualifies with $s = 17$ and $9 < 17 < 28$. See *Table 2*.
Note that this quadratic form (not considering the inequality) is symmetric in $x$ and $y$, and under a common sign flip of $x$ and $y$.
We are interested in primitive $DS$-triples, and the following Lemma shows that this means $\gcd(x,\, y) = 1$, hence one is interested only in proper solutions of eq. (34).

**Lemma 2:** $\gcd(x,\, y,\, q) = 1$ *if and only if* $\gcd(x,\, y) = 1$.

**Proof:** $\Rightarrow$) Assume that $\gcd(x,\, y) = g > 1$, then $x = g\, \hat{x}$ and $y = g\, \hat{y}$, with $\gcd(\hat{x},\, \hat{y}) = 1$, From eq. (34) $s^2 = g^2\, \hat{s}^2$, hence $s = g\, \hat{s}$. Therefore $q = g\, \dfrac{\hat{x} + \hat{y} + \hat{s}}{2}$, and because $\hat{x}^2 + 6\, \hat{x}\, \hat{y} + \hat{y}^2 = \hat{s}^2$, one finds, independent of the parity of $\hat{s}$, that $\hat{x} + \hat{y} + \hat{s}$ is always even. Thus, $q = g\, \hat{q}$ with $\tilde{q} \in N$, and $\gcd(x,\, y,\, q) = g > 1$, in contradiction to $\gcd(x,\, y,\, q) = 1$.
$\Leftarrow$) $\gcd(x,\, y,\, q) = \gcd(\gcd(x,\, y),\, q) = \gcd(1,\, q) = 1$. □

It is convenient to diagonalize the binary quadratic form eq. (34) in order to obtain a (generalized) *Pell* equation. This is straightforward because $x^2 + 6\, x\, y + y^2 = s^2$ can be written as $(x - y)^2 - 2\,(x + y)^2 = -s^2$. Defining $X := y - x > 0$ and $Y := x + y > 2\, x > 0$ one has,

$$X^2 - 2\, Y^2 = -s^2\ ,\ \textit{with}\ s \in \mathbb{N},\ \text{and}\ \ 0 < X < Y\,, \tag{35}$$

*i.e.*, $s = \sqrt{2\, Y^2 - X^2}$.
Note that there are no integer solutions of this equation (without the inequality requirement) with vanishing $X$ or $Y$. Every integer solution of eq. (34) with $y > x > 0$ produces an integer solution $(X,\, Y)$ with $Y > X > 0$ of this *Pell* type equation. Conversely, given any integer solution $(X,\, Y)$ of eq. (35) with $Y > X > 0$ leads to $x = \dfrac{Y - X}{2}$ and $y = \dfrac{Y + X}{2}$ with $y > x > 0$. One has to check whether $x$ and $y$ are integers. $X + Y$ (hence $Y - X$) is indeed even due to the following *Lemma*.

**Lemma 3:** *If* $(X, Y)$ *is an integer solution of* $X^2 - 2Y^2 = -s^2$, *for* $s \in \mathbb{N}$, *then* $X + Y$ *is even.*

**Proof:** We show that in fact $X$ and $Y$ have the same parity as $s$, hence $X + Y$ is even. From eq. (36) follows that if $s$ is even then $X$ is even. Then also $Y$ is even, because with $s = 2\,\tilde{s}$ and $X = 2\,\tilde{X}$ one has $Y^2 = 2\,(\tilde{X}^2 + \tilde{s}^2)$. If $s$ is odd then $X$ is odd, and with $s = 2\,\tilde{s} + 1$ and $X = 2\,\tilde{X} + 1$ one has $Y^2 - 1 = 4\,(T(\tilde{X}) + T(\tilde{s}))$ with the triangular numbers $T(n) = n\,(n+1)/2$ (given in A000217). Therefore, $Y^2 \equiv 1 (mod\,4)$, and $Y$ is also odd.

The next question is whether proper solutions $(x, y)$ of eq. (34) correspond to proper solutions $(X, Y)$ of eq. (35). This is indeed the case as the next *Lemma* shows.

**Lemma 4:** $\gcd(x, y) = 1$ *if and only if* $\gcd(X, Y) = 1$.

**Proof:** $\Rightarrow$) Suppose that $\gcd(X, Y) = g > 1$. Hence $y - x = X = g\,\tilde{X}$ and $x + y = Y = g\,\tilde{Y}$ and $\gcd(\tilde{X}, \tilde{Y}) = 1$. Hence $2\,x = g\,\tilde{x}$ and $2\,y = g\,\tilde{y}$ with $\tilde{x} = \tilde{Y} - \tilde{X}$ and $\tilde{y} = \tilde{Y} + \tilde{X}$.
Case $g$ odd, $\geq 3$: $\tilde{x}$ and $\tilde{y}$ are both even and $x = g\,\frac{\tilde{x}}{2}$ and $y = g\,\frac{\tilde{y}}{2}$ and $\gcd(x, y) \geq g > 1$ contradicting $\gcd(x, y) = 1$.
Case $g$ even, $\geq 2$: $g = 2\,\tilde{g}$ with $x = \tilde{g}\,\tilde{x}$ and $y = \tilde{g}\,\tilde{y}$. If $\tilde{g} \geq 2$ then $\gcd(x, y) \geq \tilde{g} \geq 2$, contradicting $\gcd(x, y) = 1$. If $\tilde{g} = 1$, *i.e.*, $\gcd(X, Y) = 2$, hence $X = 2\,\tilde{X}$ and $Y = 2\,\tilde{Y}$, with $\gcd(\tilde{X}, \tilde{Y}) = 1$, satisfying $\tilde{X}^2 - 2\,\tilde{Y}^2 = -\tilde{s}^2$, with $s = 2\,\tilde{s}$. But $\tilde{X} + \tilde{Y}$ has to be even by applying the argument from *Lemma 3*. But both numbers can only be odd because their gcd equals 1, and then $\gcd(x, y) = \gcd(\tilde{Y} - \tilde{X}, \tilde{Y} + \tilde{X}) = 2\gcd\left(\frac{\tilde{Y} - \tilde{X}}{2}, \frac{\tilde{Y} + \tilde{X}}{2}\right) \geq 2 > 1$, (because $\tilde{X}$ and $\tilde{Y}$ are both odd), contradicting $\gcd(x, y) = 1$.
$\Leftarrow$) Suppose that $\gcd(x, y) = g > 1$, then $x = g\,\tilde{x}$ and $y = g\,\tilde{y}$ with $\gcd(\tilde{x}, \tilde{y}) = 1$. Thus, $X = g\,(\tilde{y} - \tilde{x}) > 0$ and $Y = g\,(\tilde{y} + \tilde{x}) > 0$, and $\gcd(X, Y) \geq g > 1$ contradicting $\gcd(X, Y) = 1$. □

For proper solutions $(X, Y)$ of eq. (35) the even $s$ case is excluded because then $X$ and $Y$ would both be even (as shown in the proof of *Lemma 3*). Therefore we only have to consider odd $s$ with odd $X$ and odd $Y$. In this case eq. (35) is equivalent to the following problem involving triangular numbers $T(n) =$ A000217$(n)$.

$$T(\widehat{X}) + T(\widehat{s}) = 2\,T(\widehat{Y})\,, \tag{36}$$

where $s = 2\,\hat{s} + 1$, $X = 2\,\widehat{X} + 1$, $Y = 2\,\widehat{Y} + 1$, and $0 \leq \widehat{X} < \widehat{Y}$.

**Example 4:** A proper integer solution of eq. (35) with $s = 7$ is $(X, Y) = (1, 5)$ (one of the two proper positive fundamental solutions of this equation satisfying also $X < Y$; the other being $(17, 13)$ with $Y < X$). This corresponds to the solution of eq. (36) with $(\hat{X}, \hat{Y}) = (0, 2)$ and $\hat{s} = 3$. This shows that the relevant solutions $(\hat{X}, \hat{Y})$ of eq (36) do not have to be proper even if $\gcd(X, Y) = 1$. (The other solution leads to $T(8) + T(3) = 2\,T(6)$, *i.e.*, $36 + 6 = 2 \cdot 21$. Again, this is not a proper solution of eq. (36), and it does not satisfy the inequality).
The solution $(X, Y) = (1, 5)$ corresponds to one of the proper solutions of eq. (35), *viz* $(x, y) = (2, 3)$ with $0 < x < y$. The other solution $(17, 13)$ leads to $(x, y) = (-2, 15)$ which belongs to the family with the positive fundamental solution $(3, 2)$, which does not satisfy the inequality.

After this digression we give the *Proposition* with the solution of eq. (35) with the restrictions on $X$ and $Y$.

**Proposition 7: Solution of $\mathbf{X^2 - 2\,Y^2 = -s^2}$, with $\mathbf{0 < X < Y}$**

*Proper solutions* ($\gcd(X, Y) = 1$) *of the generalized Pell equation eq. (35) with restriction are possible if and only if* $s = s(n) =$ *A058529*$(n)$, *with* $n \geq 2$, *i.e.,* $s$, $X$ *and* $Y$ *are odd, and there are exactly* $P(n) = 2^{P_1(n) + P_7(n) - 1}$ *solutions, where* $P_1(n)$ *and* $P_7(n)$ *are the numbers of distinct prime divisors of* $s(n)$ *congruent to* $1$ *and* $7$ *modulo* $8$, *respectively.*

*The (here irrelevant) improper solutions arise only from composite* $s$ *with divisors that are primes congruent to* $1$ *or* $7$ *modulo* $8$.

*The case* $s = 1$ *does not qualify because there is only one family of solutions (the ambiguous case) with positive proper fundamental solution* $(X, Y) = (1, 1)$, *and positive solutions have* $X > Y > 0$.

Before giving the proof the following example of such a generalized *Pell* equation, but not representing a negative square, may be useful, and then *Lemmata 5* to *13* will follow.

**Example 5:** $X^2 - 2\,Y^2 = -7 \cdot 17 \cdot 23 = -2737$

With the restriction $0 < Y$ (always obtainable by a sign flip in $X$ and $Y$), but both signs for $X$, one has 8 families of proper solutions (and no improper ones). The *positive proper fundamental solutions* (*ppfs*) $(X, Y)$ are $(1, 37)$ and $(145, 109)$, $(25, 41)$ and $(89, 73)$, $(31, 43)$ and $(79, 67)$, $(41, 47)$ and $(65, 59)$. Each of these pairs generates an infinite conjugate family, consisting of pairs related by sign flip in the $X$ entries. Only the first members of the four pairs are found as the unique solutions satisfying also $X < Y$. There are precisely $4 = 2^{1+2-1}$ solutions with $P_1 = 1$, from 17 and $P_7 = 2$, from 7 and 23.

For the proof of *Proposition 7* the method employing *representative parallel primitive forms* (*rpapfs*) for solving generalized *Pell* equations, adapted to the present case, are first recalled. This method has been described in [8] and [9], where also references are given.

A necessary ingredient is the first *reduced form* obtained from a given unreduced one. For the definition of reduced forms see [8], also for references.

**Lemma 5: Half-reduced right neighbor form**

*The half-reduced right neighbor of a form* $F = [a, b, c]$ *of discriminant* $Disc(F) = b^2 - 4\,a\,c$ *is* $\widetilde{F} = [c, -b + 2\,c\,t, c']$, *with* $c'$ *such that* $Disc(\widetilde{F}) = Disc(F)$, *and a unique* $t = \left\lceil \frac{f(Disc(F)) + b}{2\,c} - 1 \right\rceil$ *if* $c > 0$ *and* $t = \left\lfloor 1 - \frac{f(Disc(F)) + b}{2\,|c|} \right\rfloor$ *if* $c < 0$, *where* $f(Disc(F)) := \left\lceil \sqrt{Disc(F)} \right\rceil$. *Vanishing* $c$ *is excluded because it leads to a factorization of* $F$. *For the present case, eq. (35),* $Disc(F) = 8$ *and* $f(8) = 3$.

**Proof:** For the definition of a half-reduced neighbor form of $F$ see [8], p. 2 (where $\widetilde{F}$ is denoted by $F^R$, and $Disc$ by $D$). Note that the transformation from $F$ to $\widetilde{F}$ is accomplished by the so called $\mathbf{R}(t)$-transformation, with a matrix occurring in the next *Lemma*. Therefore half-reduced right neighbor transformations will simply be called $\mathbf{R}$-transformations in the following. □

**Lemma 6: Pell form, its reduced principal form, cycle, and automorphic matrix**

**a)** *The Pell form* $F_{Pell}(\mathbf{A}, \vec{X}) = \vec{X}^\top \mathbf{A} \vec{X}$ *with* $\vec{X}^\top = (X, Y)$ ($\top$ *for transposed) and* $\mathbf{A} = Matrix([1, 0], [0, -2])$, *i.e.,* $F_{Pell} = X^2 - 2\,Y^2$, *also denoted by* $[1, 0, -2]$, *has first reduced form, called principal (reduced) form* $F_p = [1, 2, -1]$. *This is obtained after two proper (Determinant* $+1$*) equivalence transformation, i.e.,* $\mathbf{A}_p = \mathbf{B}^\top\,\mathbf{A}\,\mathbf{B}$, *where* $\mathbf{B} = \mathbf{R}(0)\,\mathbf{R}(1)$, *where* $\mathbf{R}(t) = Matrix([0, -1], [1, t])$. *Thus,* $\mathbf{B} = -Matrix([1, 1], [0, 1])$. *The discriminant of these two forms is* $Disc = -4\,Det(\mathbf{A}) = 8$, *implying that these are indefinite binary quadratic forms.*

**b)** *The cycle generated by the principal form* $F_p = [1, 2, -1]$ *is a* 2*-cycle with (reduced) partner* $F' = [-1, 2, 1]$, *obtained from* $F_p$ *by an* $\mathbf{R}(-2)$ *transformation. A further* $\mathbf{R}(+2)$ *transformation leads from* $F'$ *back to* $F_p$.

**c)** *The automorphic matrix* **Auto** *related to this* $2-$*cycle is (first* $t = -2$ *then* $t = +2$*)*

$$\begin{aligned} \mathbf{Auto} &= \mathbf{R}(-2)\,\mathbf{R}(2) \;=\; -Matrix([1,2],\,[2,5])\,,\\ \mathbf{Auto}^{-1} &= -Matrix([5,-2],\,[-2,1])\,. \end{aligned} \tag{37}$$

**d)** *Later becomes important:*

$$\begin{aligned} \mathbf{Auto}' &:= \mathbf{B}\,(-\mathbf{Auto})\,\mathbf{B}^{-1} \;=\; Matrix([3,4],\,[2,3])\,,\\ (\mathbf{Auto}')^{-1} &= Matrix([3,-4],\,[-2,3])\,. \end{aligned} \tag{38}$$

**Proof:**

**a)** Two half-reduced right neighbor transformations are needed to obtain the first reduced form $F_p$ from $F_{Pell}$, The $t$ values are 0 and 1. This leads to the transformation matrix $\mathbf{B}$ with the two $\mathbf{R}-$transformations.

**b)** $\mathbf{A}_p = Matrix([1,1],\,[1,-1]$. The matrix for $F'$ is $\mathbf{A}' = \mathbf{R}(-2)^\top\,\mathbf{A}_p\,\mathbf{R}(-2) = Matrix([-1,1],\,[1,1])$. Then $\mathbf{A}_p = \mathbf{R}(2)^\top\,\mathbf{A}'\,\mathbf{R}(2)$, This leads to the automorphic matrix **Auto** given in part **c)**.

The other parts **c)** and **d)** are obvious. □

For some application of automorphic matrices for ordinary *Pell* forms with $D(n)$ = A000037$(n)$ representing $k = +1$ see [9], section 2.

**Lemma 7: Representative parallel primitive forms, and solutions of generalized Pell equations**

**a)** *For the solution of the generalized Pell equation* $X^2 - D\,Y^2 = k$*, with non-vanishing integer* $k$*,* $D \in \mathbb{N}$*, not a square, so-called representative parallel primitive forms (rpapfs) are used. They are determined by Disc and* $k$*. For even discriminant* $Disc = 4\,D$*, (later* $D = 2$*), the rpapfs* $F_{pa}(D,\,k;\,j)$ *are obtained from solving the congruence* $j^2 - D \equiv 0\,(mod\,k)$*, for* $j \in \{0,\,1,\,...,\,|\,k\,|-1\}$*, and restricting to primitive forms* $F_{pa}(D,\,k;\,j) = [k,\,2\,j,\,c(D,\,k;\,j)]$*, with* $c(D,\,k;\,j) = \dfrac{j^2 - D}{k}$*. The case of odd Disc is here not relevant.*

**b)** *All families of infinitely many proper solutions of* $X^2 - D\,Y^2 = k$ *are found from the rpapfs that reach the principal form* $F_p(D)$ *by half-reduced right neighbor transformations. For each such rpapf the infinitude of proper solutions, indexed by* $i$*, is*

$$\begin{pmatrix} X_i(j(D,\,k)) \\ Y_i(j(D,\,k)) \end{pmatrix} = \mathbf{B}(D)\,(\mathbf{Auto}(D))^i\,\mathbf{Rpa}(\vec{t}(j(D,\,k)))^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{for } i \in \mathbb{Z}, \tag{39}$$

*with* $j(D,\,k)$ *a quadratic residue* $j$ *modulo* $k$ *from part* **a)** *that gives a primitive form, and the definition* $\mathbf{Rpa}(\vec{t}(j(D,\,k))) := \mathbf{R}(t_1)\cdots\mathbf{R}(t_L)$ *, with the L-tuple* $\vec{t} = (t_1,\,...,\,t_L)$ *determined from reaching for the first time the reduced principal form* $F_p(D)$ *from the rpapf* $F_{pa}(j(D,\,k))$ *via half-reduced right neighbor* $\mathbf{R}(t)$*-transformations. The number of such families is denoted by* $f = f(D,\,k)$ *(not to be confused with* $f = \left\lceil \sqrt{Disc} \right\rceil$ *in the definition of reduced forms in [8]).*

*Note that if eq. (39) produces a solutions with negative* $Y_i$ *then one applies a common sign change in* $X_i$ *and* $Y_i$*. Only solutions with non-negative* $Y$ *will then be considered. Therefore one could as well use* $-$**Auto** *or* $-$**B** *in eq. (39).*

This method using *rpapfs* for $D$ and $k$ to find all proper solutions of a generalized *Pell* equation of discriminant $4\,D$, representing a non-vanishing $k$, produces automatically some fundamental solution for $i = 0$. But $X$ may be negative. Here we define fundamental solution as positive ones with smallest $X$ (but sometimes the $i = 0$ solutions with negative $X$ are also called fundamental, like in the later *Example 7*). This method differs from *Nagell* [13] where inequalities are used in *Theorem 108* and in

*Theorem 108a*, pp. 205-207, for positive and negative $k$ (there called $N$), respectively, to find fundamental solutions with $|X| > 0$, $Y \geq 0$ and $|X| \geq 0$, $Y > 0$, respectively. Here one has to find first all *rpapfs* and discard those not related to the fundamental form $F_p(D)$.

**Example 6:** For $D = 2$ and $k = -7 \cdot 17 \cdot 23 = -2737$ the *rpapfs* come from the $f = 8$ solutions of $j(2, 2737)$, *i.e.*, 74, 465, 1145, 1201, 1536, 1592, 2272, 2663, with corresponding $c(2, 2737; j)$ values 2, 79, 479, 527, 862, 926, 1886, 2591. All 8 *rpapfs* reach the $F_p(2) = [1, 2, -1]$. See the later proof of *Lemma 8 b)*.

**Proof:**

**a)** The notion of representative parallel primitive forms (*rpapfs*), here for even discriminants, is found in [9], *section 3*, with references to *Scholz-Schöneberg* [16] and *Buell* [2].

**b)** Observe first that the primitivity of forms is respected by each $\mathbf{R}(t)$ transformation. With $\gcd(a, b, c) = 1$ and $a' = c$, $b' = -b + 2ct$ and $c' = a - bt + ct^2$ (from the Disc invariance, with $c \neq 0$) assume that $x' = g\,\tilde{x}'$, for $x \in \{a, b, c\}$, where $g \geq 2$ and $\gcd(\tilde{a}', \tilde{b}', \tilde{c}') = 1$. Then the definition of $a'$ implies $c = g\,\tilde{a}'$. From $b'$ follows then that $b = g\,\tilde{b}$ with $\tilde{b} = 2\,\tilde{a}'\,t - \tilde{b}'$. From $c'$ follows that $a = g\,\tilde{a}$ with $\tilde{a} = \tilde{c}' + (\tilde{b} - \tilde{a}'\,t-)\,t$. Thus, the contradiction $1 = gcd(a, b, c) \geq g \geq 2$ follows.

Also a proper solution $\vec{X} = (X, Y)^\top$, *i.e.*, $\gcd(X, Y) = 1$, of a form $F$ representing some $k$ is transformed to $\vec{X}' = \mathbf{R}^{-1}(t)\,\vec{X} = (t\,X + Y, -X)^\top$ (see the next paragraph). And this implies that $\gcd(t\,X + Y, -X) = \gcd(-X, (t\,X + Y) - t\,X) = \gcd(X, Y) = 1$. Thus, a proper solution transforms into a proper solution.

A parallel form $[k, 2\,j, c(D, k; j)]$ has the special trivial non-negative proper fundamental solution $(X, Y) = (1, 0)$. If forms are related by $\mathbf{A}' = \mathbf{R}^\top\,\mathbf{A}\,\mathbf{R}$, where $F = F(\mathbf{A}, \vec{X})$ and $F' = F'(\mathbf{A}', \vec{X}')$, then the solutions of each representation of $k$ of these two forms are related by $\vec{X}' = \mathbf{R}^{-1}\,\vec{X}$. This explains the product of inverse $\mathbf{R}-$transformations needed to reach a fundamental solution of the principal form $F_p$ representing $k$. Then one has to reach a fundamental solution of $F_{Pell}$ representing $k$ by application of $\mathbf{B}$, because $\mathbf{A}_{Pell} = \mathbf{B}^{-1,\top}\,\mathbf{A}_p\,\mathbf{B}^{-1}$, with $F_p = F_p(\mathbf{A}_p, \vec{X}_p)$. Looping around the reduced cycle determined by $F_p$ in both directions leads to all solutions of the family corresponding to the chosen *rpapf* reaching $F_p$. (Note that what is called here family is called class by *Nagell* [13], p. 204. The notion class is here reserved for the number of reduced cycles for given discriminant, called $h(Disc)$).

That every unreduced primitive form reaches a reduced form after repeated application of $\mathbf{R}$-transformations is the content of of [16], Satz 70 on p. 113. □

**Lemma 8: Conjugated rpapfs for $\mathbf{X^2 - 2\,Y^2 = -s^2}$.**

**a)** *Each representative parallel primitive form (rpapf)* $F_{pa}(D = 2, k = -s^2, j)$, $s > 1$, *of* $X^2 - 2\,Y^2 = -s^2$, *for* $j \in \{j_1, ..., j_{f(2,-s^2)}\}$, *is not reduced.*

**b)** *Each* $F_{pa}(2, -s^2, j)$ *reaches the cycle of reduced forms containing the principal form* $F_p = [1, 2, -1]$ *after applying* $\mathbf{R}$- *transformations. Hence* $f(2, -s^2)$, *the number of families of proper solutions with* $Y > 0$, *equals the number of rpapfs.*

**c)** *If $s$ is an odd positive integer number $\neq 1$ then the rpapfs come in conjugate pairs* $\left[-s^2, 2\,j, -\frac{j^2 - 2}{s^2}\right]$ *and* $\left[-s^2, 2\,(s^2 - j), -\frac{(s^2 - j)^2 - 2}{s^2}\right]$ *for each solution $j$ of the congruence* $j^2 - 2 \equiv 0\,(mod\,s^2)$, *with* $j \in \{0, 1, ..., s^2 - 1\}$. *Hence* $f(2, -s^2)$ *is a positive even integer number.*

**Proof:**

**a)** For the definition of reduced indefinite binary forms $F = [a, b, c]$ of discriminant $Disc$ see the two equivalent definitions in [8] eqs. (1) and (2) (where the discriminant is named $D$). This implies (see eq. (3) there) that $4|a\,c| < Disc = 4\,D$, *i.e.*, $|a\,c| < 2$.

But from *Lemma 7* each rpapf $F_{pa}$ satisfies $a\,c\,=\,j^2\,-\,2$ (where $a\,=\,k$). Hence $|a\,c|\,=\,|j^2\,-\,2|$, for $j\,\in\,\{0,\,1,\,...,\,|k|\,-\,1\}$ (here $|k|\,-\,1\,=\,s^2-1\,>\,3$). Thus, $|j^2\,-\,2|\,>\,2$ for $j\,\geq\,2$, and only $j\,=\,1$ satisfies $|1\,-\,2|\,<\,2$, but $1\,-\,2\,\not\equiv\,0(mod\,-\,s^2)$, for $s\,>\,1$.

**b)** Because the number of reduced cycles of discriminant $Disc\,=\,4\,D\,=\,8$, called the class number $h(Disc)$, that is here $h(8)\,=\,1$ (*e.g.*, [2], Table 2B, p. 241), there is only the reduced 2-cycle from *Lemma 6 b)*, namely the principal form $F_p\,=\,[1,\,2,\,-1]$, and its $\mathbf{R}(-2)$-transform $F'\,=\,[-1,\,2,\,1]$. These two reduced forms can also be found directly by using the definition II of reduced forms given in [8], eq. (3) for $Disc\,=\,8$ (there called $D$).
As mentioned above in the proof of *Lemma 7* b) every indefinite primitive form of discriminant $Disc$ which is not reduced reaches a reduced form after a chain of half-reduced neighbors, by applying $\mathbf{R}(t)$-transformations. This is the content of [16], Satz 70 on p. 113.

**c)** Remember from the proof of *Lemma 3* that for proper solutions $s$ has to be odd. That there are two different solutions for each $j$ follows from the degree 2 of the solvable congruence. These two given solutions are different because $s^2\,=\,2\,j$ is impossible because $s$ is odd. If $j^2\,-\,2\,\equiv\,0\,(mod\,s^2)$, for $s\,>\,1$ and $j\,\in\,\{0,1,\,...,s^2-1\}$ then also $(s-j)^2-2\,\equiv\,0\,(mod\,s^2)$. Most $j$s do not qualify, and not all odd $s\,>\,1$ are possible, as will be discussed later. These two forms are indeed primitive because $j\,=\,0$ is not a solution of the congruence for $s\,>\,1$, and with odd $s$ one has $\gcd(-s,\,2\,j)\,=\,\gcd(s,\,j)\,=\,1$ because the congruence shows that $j$ cannot have any odd prime divisor of odd $s\,\geq\,3$, otherwise this divisor would also divide 2. Finally $\gcd(a,\,b,\,c) = \gcd(\gcd(a,\,b),\,c)$ is used. □

**Example 7: Two conjugate families for $\mathbf{k = -119^2\,=\,-14161}$, $\mathbf{s\,=\,119\,=\,7\cdot 17}$.**

The two pairs of *rpapfs* are $[-14161,\,4136,\,-302]$, $[-14161,\,24186,\,-10327]$ and $[-14161,\,6448,\,-734]$, $[-14161,\,21874,\,-8447]$. With $-\mathbf{B}(D) = -\mathbf{B}(2)$ the corresponding fundamental solutions for $i\,=\,0$ of eq. (39) are $[41,\,89]$, $[-41,\,89]$ and $[79,\,101]$, $[-79,\,101]$. The corresponding $\vec{t}$ values are $(-6,\,2,\,2,\,2,\,2,\,2,\,3,\,1)$, $(-1, 6, 7, 2, 2)$ and $(-4,\,3,\,3,\,2,\,2,\,2,\,1)$, $(-1,\,4,\,2,\,3,\,5,\,2)$.

**Lemma 9: Recurrence for solutions of generalized Pell equations**

*Eq.* (39)*, using* -$\mathbf{Auto}$*(D) instead of* $\mathbf{Auto}$*(D), implies a recurrence for non-negative powers and non-positive powers i with input* $\begin{pmatrix} X_0(j(D,\,k)) \\ Y_0(j(D,\,k)) \end{pmatrix}$*, with* $Y_0(j(D,\,k))\,\geq\,0$*, which involves* $\mathbf{Auto}'(D)$ *like in eq. (38) , but with arguments D (not the instance* $D\,=\,2$*).*

$$\begin{pmatrix} X_i(j(D,k)) \\ Y_i(j(D,k)) \end{pmatrix} \;=\; \mathbf{Auto}'(D) \begin{pmatrix} X_{i-1}(j(D,\,k)) \\ Y_{i-1}(j(D,\,k)) \end{pmatrix}, \quad for\;\; i\,\in\,\mathbb{N}, \tag{40}$$

$$\begin{pmatrix} X_{-|i|}(j(D,\,k)) \\ Y_{-|i|}(j(D,\,k)) \end{pmatrix} \;=\; (\mathbf{Auto}'(D))^{-1} \begin{pmatrix} X_{-(|i|-1)}(j(D,k)) \\ Y_{-(|i|-1)}(j(D,k)) \end{pmatrix}, \quad for\;\; |i|\,\in\,\mathbb{N}, \tag{41}$$

*for every solution* $j\,=\,j(D,\,k)$ *satisfying the congruence given in Lemma 7, a).*

**Proof:** This is obvious. Again, if the result does not produce a non-negative $Y_i$, for $i\,\in\,\mathbb{Z}$, then a common sign flip in $(X_i,\,Y_i)$ will be applied. □

**Lemma 10: Powers of $(-\mathbf{Auto})$ and $\mathbf{Auto}'$ for $\mathbf{D\,=\,2}$**

*Integer powers of the special* $2\,\times\,2$ *matrices* $-\mathbf{Auto}$*, from Lemma 6, c), eq. (37) , and* $\mathbf{Auto}'$*, from Lemma 6, d), eq. (38) , are given by*

$$(-\mathbf{Auto})^n \;=\; \begin{pmatrix} S(n-1,\,6)\,-\,S(n-2,\,6) & 2\,S(n-1,\,6) \\ 2\,S(n-1,\,6) & S(n,\,6)\,-\,S(n-1,\,6) \end{pmatrix}, \quad for\;\; n\,\in\,\mathbb{Z}\,, \tag{42}$$

$$(\mathbf{Auto}')^n \;=\; \begin{pmatrix} S(n,\,6)\,-\,3\,S(n-1,\,6) & 4\,S(n,\,6) \\ 2\,S(n,\,6) & S(n,\,6)\,-\,3\,S(n-1,\,6) \end{pmatrix}, \quad for\;\; n\,\in\,\mathbb{Z}\,, \tag{43}$$

*where the Chebyshev polynomial system* $\{S(n,\,x\,=\,6)\}_{n=-\infty}^{+\infty}$, *with recurrence* $S(n,\,6)\,=\,6\,S(n-1,\,6)\,-\,S(n-1,\,6)$ *entered, with inputs* $S(-1,\,6)\,=\,0$ *and* $S(0,\,6)\,=\,1$.

**Proof:** This is a standard application of the *Cayley-Hamilton* theorem (see *e.g.*, [3]). The relevant recurrence is the one of the *Chebyshev* $S-$polynomial system $\{\mathrm{S}(\mathrm{n},\ 6)\}$ given by A001109$(n+1)$, for integer numbers $n\,\geq\,-1$. For other integer indices this is given *via* the recurrence by $S(-|n|,6)\,=\,-S(|n|-2,\,6)$, for $|n|\,\geq\,2$. For $S(n,\,6)\,-\,S(n-1,\,6)$ see A001653$(n+1)$, for $n\,\in\,\mathbb{N}_0$, and for $S(n,\,6)\,-\,3\,S(n-1,\,6)$ see A001541$(n)$, for $n\,\in\,\mathbb{N}_0$. □

**Corollary 2:**

$$(-\mathbf{B})\,(-\mathbf{Auto})^n\,=\,\begin{pmatrix} 3\,S(n-1,\,6)\,-\,S(n-2,\,6) & S(n-1,\,6)\,+\,S(n,\,6) \\ 2\,S(n-1,\,6) & S(n,\,6)\,-\,S(n-1,\,6)\end{pmatrix},\quad \text{for}\ \ n\,\in\,\mathbb{Z}\,. \tag{44}$$

Note that $3\,S(n-1,\,6)\,-\,S(n-2,\,6)\,=$ A001541$(n)$, for $n\,\geq\,0$.

**Lemma 11: $|\mathbf{X_i}|/\mathbf{Y_i}$ bound, and proper positive fundamental solution**

**a)** *If there is a solution* $(X_i,\,Y_i)$ *of the generalized Pell equation* $X^2\,-\,2\,Y^2\,=\,-s^2$, *for* $s\,\in\,\mathbb{N}$, *with* $Y_i\,>\,0$, *then*

$$\frac{|X_i|}{Y_i}\,=\,\sqrt{2}\,\sqrt{1\,-\,\frac{1}{2}\left(\frac{s}{Y_i}\right)^2}\,=\,\sqrt{2}\Big/\sqrt{1\,+\,\left(\frac{s}{X_i}\right)^2}\,<\,\sqrt{2}\,. \tag{45}$$

**b)** *The positive proper fundamental solution (ppfs)* $(\widehat{X},\,\widehat{Y})$ *for one of the members of each pair of conjugate families of solutions of this Pell equation representing* $s\,>\,1$ *satisfies*

$$0\,<\,\widehat{X}\,<\,\widehat{Y}\,<\,s\,. \tag{46}$$

**Proof:**

**a)** This follows immediately from this *Pell* equation, and the fact that neither a solution with $Y_i\,=\,0$ nor with $X_i\,=\,0$ is possible.

**b)** Here *Nagell*'s [13] inequalities from *Theorem 108a, pp. 206-7* are useful. The positive proper fundamental solution of $X^2\,-\,2\,Y^2\,=\,+1$, with only one family of solutions with $Y\,>\,0$, called an ambiguous family, is $(\widetilde{X},\,\widetilde{Y})\,=\,(3,\,2)$. Hence, for $s\,>\,1$, with $\widehat{Y}\,>\,0$ and the member of a conjugate pair of (proper or improper) solutions (related by $X\mapsto -X,\,Y\mapsto Y$), with $\widehat{X}\,>\,0$ ($\widehat{X}\,=\,0$ is impossible), one has

$$\begin{aligned} 0\,<\,\widehat{Y}\ &\leq\ \frac{2\,s}{\sqrt{2\,(3\,-\,1))}}\,=\,s, \\ 0\,<\,\widehat{X}\ &\leq\ \sqrt{\frac{1}{2}\,(3\,-\,1)\,s}\,=\,s\,. \end{aligned} \tag{47}$$

Therefore, if there is a proper solution to $X^2\,-\,2\,Y^2\,=\,-s^2$, where $s\,\in\,\mathbb{N}\setminus\{1\}$, then from part **a)**,

$$0\,<\,\frac{\widehat{X}}{\widehat{Y}}\,=\,\sqrt{2}\Big/\sqrt{1\,+\,\left(\frac{s}{\widehat{X}}\right)^2}\,<\,\sqrt{2}/\sqrt{2}\,=\,1, \tag{48}$$

because if $\widehat{X}\,=\,s$ then also $\widehat{Y}\,=\,s$, and the solution is not proper, hence $\widehat{X}\,<\,s$.
For $s\,=\,1$ the proper positive fundamental solution of the ambiguous family is $(1,\,1)$. Therefore this case has been excluded. □

**Definition 3: Conjugated family pairs I and II**

*For each conjugate family pair of proper solutions of* $X^2 - 2Y^2 = -s^2$*, for solvable odd* $s \geq 3$ *(later shown in* Proposition 7 *to be* $s = s(n) =$ A058529$(n)$*, for* $n \geq 2$*) family I is defined as the one with the positive proper fundamental solution (ppfs)* $X_{i'} = \widehat{X}$ *and* $Y_{i'} = \widehat{Y}$ *of Lemma 11 b) for a certain index* $i = i'$ *in eq. (39), with* $D = 2$*,* $k = -s^2$*, after a certain sign choice for* $\mathbf{B}(2) = \mathbf{B}$*.*
*The conjugate family II has then, for a certain index* $i = i''$*,* $X_{i''} = -X_{i'} = -\widehat{X} < 0$ *and* $Y_{i''} = Y_{i'} = \widehat{Y} > 0$*.*

**Example 8: Proper fundamental solutions for families I and II**

For $D = 2$, $s = 17$, family $I$ has the *ppfs* $(\widehat{X}, \widehat{Y}) = (7, 13)$ with $i = i' = 0$ in eq. (39) with $\mathbf{B}(2) = -\mathbf{B} = Matrix([1, 1], [0, 1])$ and $\mathbf{Auto}(2) = -\mathbf{Auto} = Matrix([1, 2], [2, 5])$. For family $II$ $(-\widehat{X}, \widehat{Y}) = (-7, 13)$ for $i = i'' = 0$. For $i = 1$ this becomes the fundamental positive solution (31, 25) (later not qualifying because $X > Y$). The corresponding conjugate *rpapfs* are $[-289,\ 488,\ -206]$ and $[-289,\ 90,\ -7]$ with $\vec{t}$ tuples $(-1,\ 6,\ 2,\ 1)$ and $(-6,\ 2,\ 2)$, respectively.

**Lemma 12: Relations between solutions of conjugate families**

i) The solution $(X_{i''}, Y_{i''})^{II} = (-X_{i'}, Y_{i'})^{I} = (-\widehat{X}, \widehat{Y})$,

ii) for $(X_{i''\pm k}, Y_{i''\pm k})^{II} = (-X_{i'\mp k}, Y_{i'\mp k})^{I}$, for $k \in \mathbb{N}$, with correlated signs.

**Proof**

**i)** From *Definition 3.*

**i)** Induction over $k$ is used, with the help of the recurrences from *Lemma 9*, with eq. (40) or eq. (41) (adapted with $D \to 2$, $k \to -s^2$) for positive or negative signs, respectively. The base case $k = 0$ is given in part **i)**. Assuming that for positive signs the claim holds for $k = 1, 2, ..., j$ then (we omit the superscripts) $X_{i''+j+1} = 3X_{i''+j} + 4Y_{i''+j} = -(3X_{i'-j} - 4Y_{i'-j})$ and $Y_{i''+j+1} = 2X_{i''+j} + 3Y_{i''+j} = -2X_{i'-j} + 3Y_{i'-j}$ Using the recurrence eq. (41) in family $I$ this implies $X_{i''+j+1} = -X_{i'-(j+1)}$. Similarly for $Y_{i''+j+1} = Y_{i'-(j+1)}$.
The proof for negative signs runs along the same line. □

**Lemma 13: Relations between $\mathbf{X_i}$ and $\mathbf{Y_i}$, signs of $\mathbf{X_i}$, and uniqueness**

*Consider only proper solutions of* $X^2 - 2Y^2 = -s^2$*, with odd* $s \geq 3$*, and a fixed family of conjugate pairs.*

**a)** *Family I*

**i)** *There is only one solution satisfying* $0 < X < Y$*, namely* $(\widehat{X}, \widehat{Y})$ *from Lemma 11 b), with* $\widehat{X} = X_{i'}$ *and* $\widehat{Y} = Y_{i'}$*, for a certain* $i = i'$ *in eq. (39).*

**ii)** *The solutions* $(X_i, Y_i)$ *for* $i > i'$ *satisfy* $0 < Y_i < X_i$*.*

**iii)** *The solutions* $(X_i, Y_i)$ *for* $i < i'$ *satisfy* $X_i < 0$ *and* $|X_i| > Y_i > 0$*.*

**b)** *Family II*

**i)** *The solution* $(X_i, Y_i)$ *for* $i = i''$ *satisfies* $X_{i''} < 0$ *and* $0 < |X_{i''}| < Y_{i''}$*.*

**ii)** *The solutions* $(X_i, Y_i)$ *for* $i > i''$ *satisfy* $0 < Y_i < X_i$*.*

**iii)** *The solutions* $(X_i, Y_i)$ *for* $i < i''$ *satisfy* $X_i < 0$ *and* $0 < Y_i < |X_i|$*.*

**Note 2:** As a necessary condition for a final solution $(X_i, Y_i)$ one needs later $0 < X_i$, Therefore if $X_i < 0$ a common sign flip in $(X_i, Y_i)$ can be used, which is a proper equivalence transformation.

**Proof:** This uses the recurrences eqs. 40 and 41 for $D = 2$, $k = -s^2$ (only certain odd $s \geq 3$ will later be shown to qualify) and $\mathbf{Auto'}(2) = \mathbf{Auto'} = Matrix([3, 4], ], [2, 3])$ from eq. (38) together with its inverse. In these recurrences the input $(X_0, Y_0)$ is then taken as $(X_{i'}, Y_{i'})$ or $(X_{i''}, Y_{i''})$ from *Definition 3.*

The proofs will be done by induction starting with $i' + 1$, respectively with $i'' + 1$, using $\mathbf{Auto}'$, and starting with $i' - 1$, respectively with $i'' - 1$, using $(\mathbf{Auto}')^{-1}$.

**a) ii)** Base case of the induction: $X_{i'+1} = 3\,X_{i'} + 4\,Y_{i'} = 3\,\widehat{X} + 4\,\widehat{Y}$ and $Y_{i'+1} = 2\,X_{i'} + 3\,Y_{i'} = 2\,\widehat{X} + 3\,\widehat{Y}$ by the recurrence Eq(40) for $i = i' + 1$. Hence $0 < Y_{i'+1} < X_{i'+1}$ because $\widehat{X} + \widehat{Y} > 2\,\widehat{X} > 0$ from *Lemma 11* b).

Assuming that $0 < Y_i < X_i$ for $i = i' + k$, for $k = 1, 2, ..., j$, it follows that $0 < Y_{j+1} = 2\,X_j + 3\,Y_j < 3\,X_j + 4\,Y_j = X_{j+1}$, because from the assumption $X_j + Y_j > 2\,Y_j > 0$.

**a) iii)** Base case of the induction: $X_{i'-1} = 3\,X_{i'} - 4\,Y_{i'} = 3\,\widehat{X} - 4\,\widehat{Y} < -\widehat{Y} < 0$, and $|X_{i'-1}| = -3\,\widehat{X} + 4\,\widehat{Y} > -2\,\widehat{X} + 3\,\widehat{Y} = Y_{i'-1}$ because $-\widehat{X} + \widehat{Y} > -\widehat{X} + \widehat{X} = 0$.

Assuming that $X_i < 0$ and $|X_i| > Y_i > 0$ for $i = i' - k$, for $k = 1, 2, ..., j$, it follows that $X_{j+1} = 3\,X_j - 4\,Y_j < -4\,Y_j < 0$ and $|X_{j+1}| = -3\,X_j + 4\,Y_j > -2\,X_j + 3\,Y_j = Y_{j+1}$ because $-X_j + Y_j = |X_j| + Y_j > Y_j > 0$.

**a) i)** For solvable odd $s \geq 3$ there exists the proper solution $(X_{i'}, Y_{i'})$ from *Definition 3* satisfying $0 < X_{i'} < Y_{i'}$. This is the only solution in family I as shown in parts **ii)** and **iii)**.

**b) ii)** Base case of the induction: $X_{i''+1} = 3\,X_{i''} + 4\,Y_{i''} = -3\,\widehat{X} + 4\,\widehat{Y}$ and $Y_{i''+1} = 2\,X_{i''} + 3\,Y_{i''} = -2\,\widehat{X} + 3\,\widehat{Y}$. Hence $0 < Y_{i''+1} < X_{i''+1}$ because $\widehat{X} - \widehat{Y} < \widehat{Y} - \widehat{Y} = 0$.

Assuming that $0 < Y_i < X_i$ for $i = i'' + k$, for $k = 1, 2, ..., j$, it follows that $0 < Y_{j+1} = 2\,X_j + 3\,Y_j < 3\,X_j + 4\,Y_j$, because from the assumption $X_j + Y_j > 2\,Y_j > 0$.

**b) iii)** Base case of the induction: $X_{i''-1} = 3\,X_{i''} - 4\,Y_{i''} = -3\,\widehat{X} - 4\,\widehat{Y} < -7\widehat{X} < 0$, and $|X_{i''-1}| = 3\,\widehat{X} + 4\,\widehat{Y}$. $Y_{i''-1} = -2\,X_{i''} + 3\,Y_{i''} = 2\,\widehat{X} + 3\,\widehat{Y} > 0$. Hence $|X_{i''-1}| > Y_{i''-1}$ because $\widehat{X} + \widehat{Y} > 0$.

Assuming that $X_i < 0$ and $|X_i| > Y_i > 0$ for $i = i'' - k$, for $k = 1, 2, ..., j$, it follows that $X_{j+1} = 3\,X_j - 4\,Y_j < -4\,Y_j < 0$ and $|X_{j+1}| = -3\,X_j + 4\,Y_j > Y_{j+1} = -2\,X_j + 3Y_j$ because $X_j - Y_j < -Y_j < 0$.

**b) i)** This follows from the *Definition 3*. □

**Corollary 3: Lower bound for $\mathbf{Y_i}$ for both families**

$$Y_i \geq \left\lceil \frac{s}{\sqrt{2}} \right\rceil , \tag{49}$$

From $Y_i = \frac{1}{\sqrt{2}} \sqrt{X_i^2 + s^2} > s/sqrt2$, because $|\,X_i\,| > 0$

For some $s = s(n) =$ A058529$(n)$ this bound is attained; *e.g.*, ., $n = (1), 7, 17, 23, 41, 103, 137, 193, ....$

**Lemma 14: $\mathbf{X_i}$ decreases with decreasing i**

*In both members of each pair of conjugate proper solutions of the generalized Pell equation $X^2 - 2\,Y^2 = -s^2$, representing positive odd $s \geq 3$, and positive $Y$ values, the values $X_i$ decrease with decreasing index $i$.*

**Proof:**

For each family $I$,

1) if $i > i'$, with $i'$ from *Definition 3*, $\triangle X_i := X_i - X_{i-1} = X_i - (3\,X_i - 4\,Y_i)$, because by inversion of eq. (40), with $(\mathbf{Auto}')^{-1}$ from eq. (38). Hence $\triangle X_i = 2\,Y_i\left(2 - \frac{X_i}{Y_i}\right) > 0$ because $X_i > Y_i > 0$ from *Lemma 13 a) ii)*, and $\frac{X_i}{Y_i} < \sqrt{2} < 2$. from eq. (45).

2) If $i < i'$ then $\triangle X_i := X_i - X_{i-1} = X_i - (3\,X_i - 4\,Y_i) = 2\,(-X_i + 2\,Y_i)$ like above, but now $X_i < 0$ from *Lemma 13 a) iii)*. Hence $\triangle X_i > 0$.

3) The case $i = i'$, with $\triangle X_{i'} := X_{i'} - X_{i'-1}$ works like in part 1) with $\frac{X_{i'}}{Y_{i'}} = \frac{\widehat{X}}{\widehat{Y}} < 1 < 2$ from *Lemma 13 a) i)*.

For each family $II$, the claim follows then with $X_i^{II} = -X_{-i}^{I}$ and $Y_i^{II} = +Y_{-i}^{I}$, for $i > i''$ as well as for $i < i''$, and $X_{i''}^{II} = -X_{i'}^{I}$.

Now the existence and uniqueness proof for each family of *Proposition 7* follows.

**Proof of Proposition 7:**

If there exist proper solutions of $X^2 - 2\,Y^2 = -s^2$, for odd $s > 1$ then *Lemma 13* shows that for each conjugate pair of families of solutions there is exactly one such solution with $0 < X < Y <$, namely the positive proper fundamental solution of family I. The family pairs I and II are given in *Definition 3*. The it w.l.g.) positive $s$ has to be odd $\geq 3$ as proved in *Lemma 3* because even $s$ leads only to improper solutions. The case $s = 1$ where only one family with $Y > 0$ exists does not qualify because all positive proper solutions, obtained from $(X\,Y) = (1,\,1)$, by applying powers of $\mathbf{Auto}'$ from eq. (38), satisfy $X > Y > 0$, as is seen by induction.

An aside: One can use the *rpapf* method eq. (39) to find this trivial $(1,\,1)$ solution for $s = 1$. From the parallel form $F_{pa}(j = 0) = [-1,\,0,\,2]$ one needs three $\mathbf{R}$-transformations with $\vec{t} = (0,\,-1,\,2)$ to reach $F_p = [1, 2,\,-1]$, leading with $i = 1$ and $-\mathbf{B}$ and $-\mathbf{Auto}$, to the positive *Pell* solution $(1,\,1)$ representing $-1$.

What is left is to investigate for which odd $s \geq 3$ values proper solutions of $X^2 - 2\,Y^2 = -s^2$ exist. For this all solutions of the *rpapf*s of *Lemma 7* for the present $D = 2$ case have to be found, which means to solve the congruence $j^2 - 2 \equiv 0 \pmod{-s^2}$, *i.e.*, $j^2 - 2 \equiv 0 \pmod{s^2}$, for $s = \prod_{i=1}^{k} p_i^{\beta_i}$, with odd primes $p_i$ and $\beta_i \geq 1$.

This quadratic congruence is reduced to the congruence with odd prime power moduli $p_i^{\alpha_i}$, with $\alpha_i = 2\,\beta_i$. The number of *rpapf*s is then $\nu(s^2) = \prod_{i=1}^{k} \nu(p_i^{\alpha_i})$ with the number of solutions to the congruence $j^2 - 2 \equiv 0 \pmod{p_i^{\alpha_i}}$. See *e.g.*, [1], *Theorem 5.28*, pp. 118, 119. In the second step this congruence is treated by the lifting theorem [1], *Theorem 5.30*, pp. 121 - 122.

One starts with finding the odd primes for which 2 is a quadratic residue modulo $p$. This is easy because for odd $p$ the *Legendre* symbol $L(2,\,p) = (-1)^{\frac{p^2-1}{8}}$. See, *e.g.*, [1], *Theorem 9.5*, p. 181. Thus, $L(2,\,p) = +1$ for $p$ congruent to $-1 \pmod 8$ or $+1 \pmod 8$ A001132. From the degree of the congruence for each of these primes there are two solutions $j = r$ and $j = p - r$, with $r \in \{0,\,1, ...,\,p-1\}$.

Now the lifting of the congruence to a power $p^k$ is obtained inductively from $p^{k-1}$ to $p^k$, for k ¿ 2. In our case the lifting is unique for each solution; part a) of the lifting theorem. This is because if there is a solution $r$ of $j^2 - 2 \equiv 0 \pmod{p^{k-1}}$ for $0 \leq r < p^{k-1}$ then $2\,r \not\equiv 0 \pmod p$ because otherwise $(2\,r)^2 = (K\,p)^2$, for $K \in \mathbb{Z}$, but then $8 = p\,(K^2\,p - 4\,L\,p^{k-2})$, with $L \in \mathbb{Z}$, *i.e.*, $8 \equiv 0 \pmod p$; a contradiction for odd $p$. Thus, the two solutions for each $p \equiv 1 \pmod 8$ or $p \equiv 7 \pmod 8$ are lifted to two solutions for any modulus $p^k$.

In our case, for $s^2$, only even powers $\alpha = 2\,\beta$ appear.

In the proof of *Lemma 8* part b) it is shown that all *rpapf*s reach the (only) fundamental 2-cycle. Therefore the number of solutions corresponds to the number of *rpapf*s for the generalized *Pell* equation for $Disc = 8$ and $k = -s^2$. This number is $2^{P_1 + P_7}$, where $P_1$ and $P_7$ are the number of prime divisors of $s$ congruent to 1 and 7 modulo 8, respectively. The sequence of these solvable $s$ is A058529$(n)$, for $n \geq 2$.

Each *rpapf* leads to a solutions by eq. (39) for $D = 2$, where only solutions with $Y > 0$ are taken into account. Of course, there are also the obvious solutions with $Y = -0$, not considered here.

For general quadratic congruences see [1] chapter 9.1, pp. 178 ff.

Because the *rpapf*s come in conjugate pairs, leading to families of solutions (*Lemma 8* and *Definition 3*), and only family I contains the unique solution with $0 < X < Y$ there are only $2^{P_1 + P_7 - 1}$ solutions to eq. (35). $\square$

For these $s = s(n) =$ A058529$(n)$, with $n \geq 2$, see *Table 3*, for the $s-$ values $7,\,17, ...,\,313$, also for $(x = c_1,\,y = c_2)$ solutions , together with $q = c_3$, and the $t$-tuples for the *rpapf*s needed in eq. (39). The $t$-tuples come in conjugate pairs, but only one of them leads to family I solutions with the required unique $0 < X < Y$ solution. The other $t$-tuple is given within solid brackets. Composite $s$ values (with also improper solutions) are underlined.

**Case ii) $\mathbf{q} < \mathbf{c_3}$**

We use $q = c_3 - k$, with $1 \leq k < \sqrt{2\,x^2 + y^2}$, where $x = c_1 < c_2 = y < q + k$, and $gcd(x,\, y,\, q+k) = 1$.

This case will lead to two binary quadratic fields, one indefinite with $Disc = 8$, the other a positive definite form with $Disc = -8$, representing a negative integer $-a$ and the positive number $a$, respectively. The proper solutions will be found by the method of the representative parallel primitive forms (*rpapfs*) like above. Improper cases are reduced to proper ones.

**Lemma 15: Quadratic equation for $(\mathbf{x},\, \mathbf{y})$**

$$x^2 + y^2 + 6\,x\,y + 4\,(x + y)\,k = t^2, \tag{50}$$

*w.l.g.* $t \geq 0$.
Only solutions with $0 < x < y$ are of interest.

**Proof:** From the definitions

$$q^2 = x\,y + (q + k)\,(x + y), \tag{51}$$

leading to the positive solution (the negative one is excluded because $q > 0$)

$$2\,q = x + y + \sqrt{x^2 + y^2 + 6\,x\,y + 4\,k\,(x + y)} =: x + y + t, \tag{52}$$

Integer solutions of $q$ need the claimed formula. $\square$

Diagonalization of eq. (50) leads, with $X = y - x$, $Y = y + x$ and $\widehat{Y} = Y + k$, to

**Lemma 16: Quadratic form for $(\mathbf{X},\, \widehat{\mathbf{Y}})$**

$$X^2 - 2\,\widehat{Y}^2 = -a(t,\, k) := -(t^2 + 2\,k^2)\,. \tag{53}$$

Only solutions satisfying $0 < X < \widehat{Y} - k$ are of interest, because $2\,x = Y - X$, $2\,y = Y + X$ and $Y = \widehat{Y} - k$.

**Proof:** Elementary. $\square$

**Corollary 4:**

$$2\,q = \widehat{Y} + t - k, \text{ and } 2\,c_3 = 2\,(q + k) = \widehat{Y} + t + k. \tag{54}$$

The left-hand side of eq. (53) shows an indefinite binary quadratic form of discriminant $Disc = 8$ representing a negative integer $-a$, and the right-hand side is a positive definite form of $Disc = -8$ representing a positive integer $a$.

From the use of $t$ in eq. (52) it is clear that $t > 0$. One can show that $t = 0$ is indeed excluded in eq. (53) .

**Lemma 17: Exclusion of instance $\mathbf{t} = \mathbf{0}$.**

*There is no solution for* $(X,\, \widehat{Y})$ *and* $0 < X < Y$ *if* $t = 0$.

**Proof:** If $t = 0$ then $X$ is even, $X = 2\,\widetilde{X}$, *i.e.*, $\widetilde{Y}^2 - k^2 = 2\,\tilde{X}^2$, $2\,(\widehat{Y} + k)\,(\widehat{Y} - k) = X^2$, *i.e.*, $\sqrt{2\,(Y + 2\,k)\,Y} = X$, hence $X > \sqrt{2\,Y\,Y} > Y > 0$, contradicting the required inequality. $\square$

**Note 3: Symmetry between X and t**

In the equation $X^2 - 2\,\widehat{Y}^2 + t^2 + 2\,k^2 = 0$ there is a symmetry between $X$ and $t$. However, if a solution satisfies $0 < X < Y = \widehat{Y} - k$ then $0 < t < Y$ cannot be satisfied for $k > 0$. Because then $0 = X^2 - 2\,\widehat{Y}^2 + t^2 + 2\,k^2 < (\widehat{Y} - k)^2 - 2\,\widehat{Y}^2 + (\widehat{Y} - k)^2 + 2\,k^2 = -4\,\widehat{Y}\,k + 4\,k^2 = -4\,k\,Y < 0$, a contradiction.

The possible values $a$ for the two quadratic forms are now determined one after the other, and then the intersection of these $a$ values are of interest as candidates for testing the required $(X, Y)$ inequalities.

**A) The indefinite binary quadratic form**

**Lemma 18: All values a for proper solutions of $\mathbf{X^2 - 2\,\widehat{Y}^2 = -a}$**

**a)** *The indefinite quadratic form $X^2 - 2\,\widehat{Y}^2$ represents $-a$, for $a \geq 2$, properly if and only if,*

$$a = 2^{e_2} \prod_{i=1}^{P_1} p_{1,i}^{e_{1,i}} \prod_{i=1}^{P_7} p_{7,i}^{e_{7,i}}, \tag{55}$$

*with $e_2 \in \{0, 1\}$, $e_{1,i} \geq 0$, $e_{7,i} \geq 0$, $P_1 \geq 0$ and $P_7 \geq 0$, but not all exponents vanish. The products are over powers of odd primes with $p_{1,i}$ congruent to $1 \pmod 8$, A007519, and $p_{7,i}$ are the primes congruent to $7 \pmod 8$, A007522. The combination of these odd primes is shown in A001132, together with prime 2 in A038873.*

*$-a = -1$ is also represented by one family (the ambiguous case) with $\widehat{Y} > 0$ with the fundamental solution $(1, 1)$. This will not be of interest because $0 < X < Y = \widehat{Y} - k$ is required later.*

**b)** *The number of rpapfs, i.e., the number of families of solutions with positive integer $\widehat{Y}$, is $f(-a) = 2^{P_1 + P_7}$.*

**c)** *The sequence of these numbers $a$ is given in A057126.*

**Proof:**

**a)** It is sufficient to find the $a$ values for proper representations $(X, \widehat{Y})$. The improper solutions reduce to these solutions. Improper solutions are later also of interest because $\gcd(X, \widehat{Y} = Y + k)$ is not necessarily 1.

All proper solutions are found using the representative parallel primitive forms (*rpapfs*), See *Lemma 7* for $D = 2$ and $k = -a$. They are given by solving the congruence $j^2 - 2 \equiv 0 \pmod{-a} \equiv 0 \pmod a$ for $j \in \{0, 1, ..., a-1\}$ as $F_{pa}(-a, j) = [-a, 2j, (2 - j^2)/a]$. Each solution $j$ of this congruence leads by $\mathbf{R}$-transformations always to the principal form $F_p = [1, 2, -1]$ (see the proof of *Lemma 8* b)). Therefore each such $F_{pa}(-a\, j)$ produces a family of proper representations of $-a$ by $X^2 - 2\,\widehat{Y}^2$. These infinite $j$-families are obtained by eq. (39) (with $D = 2$ and $k = -a$), restricting to positive $\widehat{Y}$ by a possible overall sign flip in $(X, \widehat{Y})$, The number of such families is called f(-a) (there are then also solutions with $\widehat{Y} \to -\widehat{Y}$, of course).

The congruence for $j$ is solved by using the prime number decomposition $a = 2^{e_2} \prod_{i=1}^{K} p_i^{e_i}$, with odd primes $p_i$. The solution for each prime number is then determined.

If $a$ is even the solution of $j^2 - 2 \equiv 0 \pmod{p = 2}$ is $j = \pm 0$. A lifting to powers of prime 2 is not possible by the lifting theorem [1], *Theorem 5.30*, pp. 121 - 122, because there case $(b), (b_2)$ applies inductively for each power $\alpha > 2$, since $2 \cdot 0 \equiv 0 \pmod 2$. Therefore $e_2 \in \{0, 1\}$.

For each odd prime $p$ the congruence $j^2 - 2 \equiv 0 \pmod p$ has two solutions $j = r$ and $j = p - r$, for a certain $r \in \{0, 1, ..., p-1\}$, only if $p \equiv \pm 1 \pmod 8$ from the *Legendre* symbol $(2, p) = +1$. In this case there is an inductive proof of a unique lifting of each of the two solutions to any power $k \geq 2$ of such a $p$. Adapt this proof to the one given in the proof of *Proposition 7*.

**b)** The number $f(-a)$ of solutions of $j$, *i.e.*, the number of *rpapfs*, is
$f(-a) = \nu(2) \prod_{i=1}^{P_1} \nu(p_1, i) \prod_{i=1}^{P_7} \nu(p_7, i) = 2^{0 + P_1 + P_7}$, where $\nu(p)$ is the number of solutions of the congruence for $j$ for prime $p$, a factor of $a$, viz $\nu(2) = 1$ and $\nu(p) = 2$ for each odd $p$.

**c)** This follows from the definition of A057126. See the first comment there, and the following conjecture which is true because $X^2 - 2\,\widehat{Y} = -a$ has the same numbers $a$ as proper representations of the form $2\,x^2 - y^2$ (see a Nov 09 2009 comment in A035251 where also the $a$ values with only uninteresting improper representations are listed). □

**Example 9: Proper solutions for $\mathbf{a = 2 \cdot 7 \cdot 17 = 238}$**

The number of *rpapfs* $F_{pa}(-238,\, j)$ is $f(-238) = 2^{1+1} = 4$, viz $[-238,\, 148,\, -23]$, $[-238,\, 216,\, -49]$ $[-238,\, 260,\, -71]$, $[-238,\, 328,\, -113]$, for $j = 74,\, 108,\, 130,\, 164$, respectively. The corresponding $\vec{t}$ values are $(-3,\, 4,\, 2)$, $(-2,\, 5,\, 1)$, $(-1,\, 2,\, 2,\, 2,\, 2,\, 2,\, 2)$ and $(-1,\, 3,\, 2,\, 2,\, 2,\, 1)$. The corresponding fundamental solutions $(X_0,\, \widehat{Y}_0)$ are from eq. (39), with $Y = \widehat{Y}$, $D = 2$, $k = -238$ and $\mathbf{B(2)} = -\mathbf{B} = Matrix([1,1],\, [0,1])$ and $\mathbf{Auto}(2) = -\mathbf{Auto}$ from eq. (37), $(-10, 13)$, $(2,\, 11)$, $(-2,\, 11)$, $(10,\, 13)$. The only solutions satisfying $0 < X < \widehat{Y}$ in these four families are $(2,\, 11)$ and $(10,\, 13)$.
However, these solutions will not survive as final solution of eq. (53) because the solutions for $(t,\, k)$ will have no solution for $a = 238$. See the later *Lemma 18*, which shows that there is no solution with prime $p \equiv 7 \pmod 8$.

*Lemma 11* can be rewritten with $Y \to \widehat{Y}$, and $s = \sqrt{a}$, with the positive proper fundamental solution *(ppfs)* $(X',\, \widehat{Y}')$ *as*

**Corollary 5:**

**a)** *If there is a solution* $(X_i,\, \widehat{Y}_i)$ *for the generalized Pell equation* $X^2 - 2\,\widehat{Y}^2 = -a$*, for* $a \in \mathbb{N}$*, with* $\widehat{Y}_i > 0$*, then*

$$\frac{|X_i|}{\widehat{Y}_i} = \sqrt{2}\sqrt{1 - \frac{1}{2}\left(\frac{\sqrt{a}}{\widehat{Y}_i}\right)^2} = \sqrt{2}\Big/\sqrt{1 + \left(\frac{\sqrt{a}}{X_i}\right)^2} < \sqrt{2}\,. \tag{56}$$

**b)** *The positive proper fundamental solution (ppfs)* $(X',\, \widehat{Y}')$ *for one of the members of a pair of conjugate families of solutions of this Pell equation representing* $-a$ *satisfies*

$$0 < X' < \widehat{Y}' < \sqrt{a}\,. \tag{57}$$

The definition of conjugate family pairs $I$ and $II$ is taken over from *Definition 3* with $s \to \sqrt{a}$. Hence family $I$ is the one which has the *ppfs* solution $(X',\, \widehat{Y}')$. The conjugate family has then the solution $(-X',\, \widehat{Y}')$.
In *Example 9* there are two conjugate family pairs. The first pair has in family $I$ the *ppfs* $(2,\, 11)$, and $(-2,\, 11)$ in family $II$. The other pair has *ppfs* $(10,\, 13)$ in family $I$, and $(-10,\, 13)$ in family $II$.

Adapting *Definition 3*, *Lemma 13* and *Lemma 14* to the present case, with $s \to \sqrt{a}$ and $Y \to \widehat{Y}$, $\widehat{X} \to X'$ and $\widehat{Y} \to \widehat{Y}'$,shows that there is in each conjugate pair of solutions precisely one solution, namely $(X',\, \widehat{Y}')$ in family $I$, for each represented $a$ value of eq. (55), satisfying the necessary $0 < X < \widehat{Y}$ requirement.
Therefore the number of solutions of $X^2 - 2\,\widehat{Y}^2 = -a$, $a$ from eq. (55), also satisfying $0 < X < \widehat{Y}$ has precisely $2^{P_1 + P_7 - 1}$ solutions.

But in order to obtain the final result for eq. (53) one has also to solve the definite quadratic form for $(t,\, k)$ of eq. (53) representing certain positive integers $a$ (not the one of eq. (55)).

**B) The positive binary quadratic form**

Before treating the special $(t,\, k)$ case some general remarks on definite binary quadratic forms are collected. Compare with [16], §29 and §30, 104 - 112.
A definite quadratic form $F = [a,\, b,\, c; x,\, y]$, with discriminant $Disc = -(4\,a\,c - b^2) < 0$ is positive or negative definite, *i.e.*, $F > 0$ or $F < 0$ for all integers $(x,\, y) \neq (0,\, 0)$, if $a > 0$ or $a < 0$. This implies that $c > 0$ or $c < 0$, respectively. This follows from the identity (see [16], p. 102, eq. (126)). $4\,a\,F = (2\,a, x + b\,y)^2 + |\,Disc\,|\,y^2)$.
An aside: Note that $Disc = -q^2 = (i\,q)^2$, for $q \geq 0$, is not considered because $a\,F$ factorizes into a complex conjugate pair: $a\,F = K\,\overline{K}$, with $K = (a\,x + \frac{b}{2}) + \frac{q}{2}\,y\,i := A + B\,i$. (Compare [16] eq. (127), p. 103, for the indefinite $Disc = q^2 > 0$ case). For the positive definite $F$ representing $k \in \mathbb{N}$ this leads to the representations of two squares $A^2 + B^2 = a\,k \in \mathbb{N}$. For this see [5], §3, Theorem 3, p. 15, or [12], Theorem 2.5, p. 27.

A positive definite Form is called *reduced* if $0 \le |b| \le a \le c$. This implies that $0 < a \le \left\lfloor \sqrt{\frac{-Disc}{3}} \right\rfloor$, hence there is only finite number of solutions for $a$ hence for $b$ ($-a \le b \le a$) with given $Disc$ (determining $c$). Thus, the number of these reduced forms for given $Disc < 0$ is finite, see [2], *Theorem 2.2.*, p. 13.

We call a positive definite quadratic form $F$ *half-reduced* if $0 = |b| \ge a$. Hence such an $F$ is then not reduced if $c > a$.

A $t$-family of *parallel forms*, each improperly (with a $Det = -1$ transformation) equivalent to $F = [a,\, b,\, c]$ is defined by $F'(t) = [a,\, b'(t),\, c'(t)]$, with $b'(t) = -b + 2\,a\,t$ and $c' = c - b\,t + a\,t^2$, for $t \in \mathbb{Z}$. This corresponds to $\mathbf{A}' = (\mathbf{N}\,\mathbf{R}(\mathbf{t}))^\top\,\mathbf{A}\,(\mathbf{N}\,\mathbf{R}(\mathbf{t}))$, with $\mathbf{N} = Matrix([0,1],\, [1,0])$, $Det\,\mathbf{N} = -1$, and $\mathbf{R}(t) = Matrix([1,-1],\, [0,t])$, $Det\,\mathbf{R} = +1$, as used earlier in *Lemma 6*.

Each primitive positive definite form $[a,\, b,\, c]$ representing properly, with $\vec{x} = (x,\, y)^\top$, an integer $k > 0$ (for the positive definite case) is improperly equivalent to a $t$-family of parallel forms $F''(t)$ obtained after two steps.
Start with $F = F_1$ and $\vec{x}_1$, then $F' = [k,\, b',\, c']$, with $Disc\,F = Disc\,F'$ and $\mathbf{A}' = \mathbf{M}^\top\mathbf{A}\,\mathbf{M}$, where $\mathbf{M} = Matrix([x_1, v_1],\, [y_1, w_1])$, with $Det\,\mathbf{M} = +1 = x_1\,w_1 - y_1\,v_1$. $b'$ and $c'$ look complicated, not needed explicitly here. Thus, $\vec{x}' = \mathbf{M}^{-1}\vec{x}_1 = (1,\, 0)^\top$.
In a second step a $t$-family of parallel forms $F''(t) = [k,\, b'',\, c'']$ is obtained improperly *via*
$\mathbf{A}'' = \mathbf{T}^\top(t)\,\mathbf{A}'\,\mathbf{T}(t)$, where from above $\mathbf{T}(t) = \mathbf{N}\,\mathbf{R}(t) = Matrix([1,t],\, [0,-1])$. Thus, $\vec{x}'' = \mathbf{T}^{-1}\,\vec{x}' = \mathbf{T}\,\vec{x}' = (1,\, 0)^\top$. Now $b''(t) = -b' + 2\,k\,t$ and $c''(t) = c' - b'\,t + k\,t^2$.
Because $b'' \equiv = -b' \pmod{2\,k}$, a representative parallel form $F'' = F''(t_0)$ with $b''(t_0) \in [0,\, 2\,k)$ can be chosen.
This leads to a program for all representative parallel positive definite forms forms (*rpaf*)s $F$ (forget the superscripts $''$) of given $Disc < 0$ representing an integer $k > 0$. We are interested later only on even $Disc$, *i.e.*, $Disc = -4\,D$. This becomes then a $j-$family of *rpaf*s, *i.e.*, $F(j) = \left[k,\, 2\,j,\, \frac{j^2 + D}{k}\right]$, for $j \in [0,\, k-1]$. This is because from the Disc formula $b$ is even, $b = 2\,j$ and $b \in [0, 2\,k)$ ($b''(t_0)$ from above).
A representative parallel form $F''$ that is improperly equivalent to $F$ may not be primitive, even if $F$ was primitive. Therefore one discards in the $j$-family of the *rpaf*s the imprimitive forms and stays with a set of primitive ones the *rpapf*s.

**Example 10: Representative parallel forms for some Disc and k**

1) The representable $k$ values for $Disc = -32$, $D = 8$ are, including also imprimitive forms, 1, 2, 3, 4, 6, 8, 9, 11, 1
Those with primitive forms (*rpapf*s ) are 1, 3, 4, 8, 9, 11, 12, 17, 19, 24, 27, 33, 36, ... .
The number of $j$-values for these *rpapf*s are 1, 2, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 4, 2, ... .
For $k = 2$ there is only the imprimitive $j = 0$ *rpaf* $[2, 0, 4]$.
For $k = 4$ the two *rpaf*s with $j = 0$ and 2 are $[4, 0, 2]$ and $[4, 4, 3]$, but only $[4, 4, 3]$ is primitive, the *rpapf* for $k = 4$.

2) The representable $k$ values for $Disc = -20$, $D = 5$ are given in A343238: 1, 2, 3, 5, 6, 7, 9, 10, 14, 15, 18, ... . The number of $j$-solutions is given in A343240. It seems that for each such $k$ value all $j$ solutions lead to primitive parallel forms (tested for the first 70 of these $k$-values).
For $k = 18$ the two $j$-values are 7 and 11 with corresponding *rpapf*s [18, 14, 3] and [18, 22, 7].

A $t$-family of *right neighbor forms* is defined by $\widetilde{F}(t) = [c,\, -b + 2\,c\,t,\, a - b\,t + c\,t^2]$ this is obtained like above in the parallel case but omitting $\mathbf{N}$, hence exchanging $a$ and $c$. This is a proper equivalence transformation $\mathbf{R}(\mathbf{t})$.

To obtain a unique *half-reduced right neighbor form* (*hredrnf*) one chooses $t = \left\lceil \frac{b - c}{2\,c} \right\rceil$, from the half-reduced definition.

Now we treat the reduced positive definite quadratic form $t^2 + 2\,k^2$, with $Disc = -8$, representing an integer $a \ge 1$. This is the reduced form $[1,\, 0,\, 2]$ which is the only one for $Disc = -8$. See the Example

table in [2], p. 18 (where $\triangle = Disc$; class number $h(-8) = 1$). Therefore there is only this 1-cycle of reduced forms.
The *rpapf*s are determined from the congruence $j^2 + 2 \equiv 0 \pmod{a}$ for $j \in \{0, 1, ..., a-1\}$ by $F_{pa}(a, j) = [a, 2j, (j^2 + 2)/a]$. Every *rpapf* leads *via* **R**-transformations to this primitive form [1, 0, 2]. That every definte form leads via a chain of right neighbor forms to a reduced one is proven in [16] Satz 75, §38, pp. 108 - 109. As shown above in the proof of *Lemma 7*, part **b)**, **R**-transformations respect primitivity of forms. The formula for the $(t, k)$ solutions for given $a \geq 1$, obtained from the *rpapfs* with the $\vec{t}$ values for these **R**-transformations, is here (compare this with eq. (39) )

$$\begin{pmatrix} t(j(a)) \\ k(j(a)) \end{pmatrix} = \pm \mathbf{Rpa}(\vec{t}(j(a)))^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{58}$$

with the solutions of the $j-$congruence determining the *rpapfs*. A sign is chosen to obtain $k > 0$.

**Example 11: Chain of R-transformations**

For $a = 17$ the two *rpapf*s are [17, 14, 3] and [17, 20, 6], for $j = 7$ and 10. The $\vec{t}$-tuples are $(2, -1)$ and $(2, 2)$ with the chains $\{[17, 14, 3], [3, -2, 1], [1, 0, 2]\}$ and $\{[17, 20, 6], [6, 4, 1], [1, 0, 2]\}$.

**Lemma 19: All values $\mathbf{a > 0}$ for proper solutions of $\mathbf{t^2 + 2\,k^2 = a}$**

**a)** *The positive definite quadratic form $t^2 + 2k^2$, with $Disc = -8$, represents an integer $a$, for $a \geq 2$, properly if and only if,*

$$a = 2^{e_2} \prod_{i=1}^{P_1} p_{1,i}^{e_{1,i}} \prod_{i=1}^{P_3} p_{3,i}^{e_{3,i}}, \tag{59}$$

*with $e_2 \in \{0, 1\}$, $e_{1,i} \geq 0$, $e_{3,i} \geq 0$, $P_1 \geq 0$ and $P_3 \geq 0$, but not all exponents vanish. The products are over powers of odd primes with $p_{1,i}$ congruent to 1 (mod 8), A007519, and $p_{3,i}$ are the primes congruent to 3 (mod 8), A007520. The combination of these odd primes is shown in A033200, together with prime 2 in A033203.*
*The case $a = 1$ is also represented properly but the solutions $(\pm 1, 0)$ is not of interest because $k$ should be positive.*

**b)** *The number of rpapfs, i.e., the number of families of solutions with positive integer $k$, is $f(a) = 2^{P_1 + P_3}$ (for $a \geq 2$). These rpapfs appear as conjugate pairs for values $j$ and $a - j$, differing in the sign of $t > 0$.*

**Proof:**

**a)** One has to solve the congruence $j^2 + 2 \equiv 0 \pmod{a}$. See the proof of *Lemma 17 a)* where now, besides for prime 2 treated there and applying here too, one has to solve for odd primes $p$ the congruence $j^2 + 2 \equiv 0 \pmod{p}$, *i.e.*, to compute the *Legendre* symbol $L(-2, p) = +1$. By multiplicativity of this symbol this means $L(-1, p)\, L(2, p) = +1$. The solution for signs ++ is $p \equiv 1 \pmod{4}$ and $p \equiv \pm 1 \pmod{8}$, *i.e.*, $p \equiv +1 \pmod{8}$, because the other case means $p = 1 + 4K = 7 + 8L$, with integers $K$ and $L$, leading to the contradiction $3 = 2(K - 2L)$. The other solution with signs $--$ is $p \equiv 3 \pmod{4}$ and $p \equiv \pm 3 \pmod{8}$, *i.e.*, $p \equiv +3 \pmod{8}$, because the other case means $p = 3 + 4K = 5 + 8L$, with integers $K$ and $L$, leading to the contradiction $1 = 2(K - 2L)$. Therefore one find $L(-2, p) = +1$ for $p \equiv 1 \pmod{8}$ or $p \equiv 3 \pmod{8}$.
The unique lifting to powers of these primes for each of the two solutions of the $j-$congruence is proved inductively like in the proof of Proposition 7.

**b)** Follows like in *Lemma 18 b)* where also the *rpapf*s appeared in conjugate pairs.

**Example 12: Proper solutions for $a = 3^2\, 17 = 153$**

The number of *rpapfs* $F_{pa}(153, j)$ is $f(153) = 2^{1+1} = 4$, viz [153, 82, 11], [153, 116, 22] [153, 190, 59] [153, 224, 82], for $j = 41, 58, 95, 112$, respectively. Observe the two conjugate $j$-pairs (41, 112) and (58 95), and the corresponding pairs of *rpapf*s . The $\vec{t}$ values are (4, 3), (3, 3, 1), (2, 3, 2, 0), (1, −3, −3), respectively. The solutions are $(t, k > 0)$ are $(-11, 4)$, $(-5, 8)$, $(5, 8)$, $(11, 4)$, respectively. The pairs of conjugate solutions differ by the sign of $t$, Later only the solution (11, 4) will survive if also the $(X, Y)$ values for the indefinite form representing $-a$ with $0 < X < Y$ are determined.

### C) Combining solutions of sections A) and B)

The proper representations of $X^2 - 2\,\widehat{Y}^2$ of certain $-a$ of section **A)** (*Lemma* 17) and those of $t^2 + 2\,k^2$ for certain $a$ of section **B)** for the case **ii)** with $q = c_3 - k$, are combined to find the representations of eq. (53) (*Lemma* 15), and then the requirement $0 < X < Y = \widehat{Y} - k$, for $1 \leq k < \sqrt{X^2 + Y^2}$ is enforced. Because $Y = \widehat{Y} - k$ this uses solutions of the form for $(t, k)$. Finally, $c_1 = x = \frac{Y - X}{2}$. $c_2 = y = \frac{Y + X}{2}$, and $c_3 = \frac{\widehat{Y} + t + k}{2}$ have to be checked for $\gcd(c_1, c_2, c_3) = 1$.

**Proposition 8: Combined solutions**

*The solutions of $X^2 - 2\,\widehat{Y}^2 = -a$ and $t^2 + 2\,k^2 = a$, for $a \geq 2$, qualifying for the final solutions come in three versions. The trivial case, if both congruences are solved by improper solutions (obtainable from proper ones) is not considered because the curvature triples will not be primitive.*

**Type a)**

*If both binary quadratic forms are solved properly then*

$$a = 2^{e_2} \prod_{i=1}^{P_1} p_{1,i}^{e_{1,i}}, \tag{60}$$

*with $e_2 \in \{0, 1\}$, $e_{1,i} \geq 0$ and $P_1 \geq 0$, but not all exponents vanish. The products are over powers of distinct odd primes with $p_{1,i}$ congruent to $1 \pmod 8$, A007519. For the sequence $a$ see A192453 without member 1.*

*It is guaranteed (by Corollary 5 b)) that there is exactly one solution satisfying the necessary condition $0 < X < \widehat{Y}$, later to be tested with $Y = \widehat{Y} - k$ for $X < Y$.*

*The first instance for a final solution appears for $a = 89$. See Table 4 for $c_3 = 9$.*

**Type b)**

*If the solutions $(X, \widehat{Y})$ are improper and those of $(t, k)$ are proper, then*

$$\begin{aligned} a &= A^2\,\tilde{a}, \\ A &= \prod_{i=1}^{P_1'} p_{1,i}^{e_{1,i}} \prod_{i=1}^{P_3} p_{3,i}^{e_{3,i}}, \text{ and} \\ \tilde{a} &= 2^{e_2} \prod_{i=1}^{P_1''} p_{1,i}^{e_{1,i}}, \end{aligned} \tag{61}$$

*where $P_1' + P_1'' = P_1$, the number of distinct prime factors of $a$ congruent to $1 \pmod 8$, A007519, and $P_3$ is the number of distinct primes congruent to $3 \pmod 8$, A033200. Also $P_1' + P_3 > 0$ in order that $A \geq 2$, i.e., the exponents $e_{1,i}$ and $e_{3,i}$ are nonnegative and do not both vanish. $e_2 \in 0, 1$, and $\tilde{a} > 2$. $X = A\,\tilde{X}$, $\widehat{Y} = A\,\tilde{\widehat{Y}}$, and there is exactly one proper solution for $(\tilde{X}, \tilde{\widehat{Y}})$ representing $-\tilde{a}$ satisfying the necessary condition $0 < \tilde{X} < \tilde{\widehat{Y}}$ for each pair of conjugate families of solutions (therefore $\tilde{a} = 2$ is excluded). Later $Y = \widehat{Y} - k$ has to be tested for $X < Y$ for each unique solution from each family pair, with each of the $2^{P_1 + P_3 - 1}$ solutions of $k$ from the $(t, k)$ equation for positive $t$ and $k$. Finally the curvature triples have to be tested for primitivity.*

*For the sequence $A$ see A225771 without 1. For the sequence $\tilde{a}$ see A192453 without 1 and 2.*

*The first instance for a final solution appears for $a = 153 = 3^2 \cdot 17$. See Table 4 for $c_3 = 12$.*

**Type c)**

*If the solutions* $(X, \widehat{Y})$ *are proper and those of* $(t,\,k)$ *are improper, then*

$$\begin{aligned} a &= A^2\,\tilde{a}, \ \ \textit{with} \\ A &= \prod_{i=1}^{P_1'} p_{1,i}^{e_{1,i}} \prod_{i=1}^{P_7} p_{7,i}^{e_{7,i}}, \ \ \textit{and} \\ \tilde{a} &= 2^{e_2} \prod_{i=1}^{P_1''} p_{1,i}^{e_{1,i}}, \end{aligned} \tag{62}$$

*where* $e_2 \in \{0,\,1\}$*,* $P_1' + P_1'' = P_1$*, the number of distinct prime factors of* $a$ *congruent to* $1\,(\mathrm{mod}\,8)$*, A007519, and* $P_7$ *the number of distinct prime factors of* $a$ *congruent to* $7\,(\mathrm{mod}\,8)$*, A007522. For the primes congruent to* $\{1,7\}(\mathrm{mod}\,8)$ *see A001132, or A038873 with the member* $2$*. Also* $P_1' + P_7 > 0$ *in order that* $A \geq 2$*;* $e_2 = 1$ *with* $P_1'' = 0$ *is not allowed because the forbidden* $t = 0$ *would follow. The case* $e_2 = 0$ *and* $P_1'' = 0$ *is also not allowed because this leads to* $k = 0$*.*

*For the sequence* $A$ *see A058529 without member* $1$*. For the sequence* $\tilde{a}$ *see A192453 without* $1$ *and* $2$*.*

$t = A\,\tilde{t}$*,* $k = A\,\tilde{k}$*, with the* $2^{P_1''-1}$ *proper solutions* $(\tilde{t},\,\tilde{k})$ *representing* $\tilde{a}$*, with positive* $t$ *and* $k$*. Then* $Y = \widehat{Y} - k$ *has to be tested for* $X < Y$ *for each of the* $2^{P_1+P_7-1}$ *solutions with* $Y > 0$ *and* $0 < X < \widehat{Y}$ *for each of the* $2^{P_1''-1}$ *solutions for* $\tilde{k}$*. Finally the primitivity of all* $(c_1,\,c_2,\,c_3)$ *has to be checked.*

*The first instance for a final solution appears for* $a = 7^2\cdot 2\cdot 17 sspeq 1666$ *(larger than the values shown in Table 4). with* $[X,\,\widehat{Y}] = [4,\,29]$*,* $[t,\,k] = [28,\,21]$*,* $[X,\,\widehat{Y}] = [4,\,8]$*,* $[c_1,\,c_2,\,c_3] = [2,\,6,\,39]$*) and* $[c_{4,-},\,c_{4,-}] = [11,\,83]$*.*

Before giving the proof several examples will illustrate this *Proposition*.

**Example 13**

Necessary conditions for the final solution are $0 < X < Y$ and $c_2 < c_3$, *i.e.*, $X < t + 2\,k$.

**Type a)**

**i)**: $a = 2\cdot 17^2 = 578$, $(X,\,\widehat{Y}) = (12,\,19)$, $(t,\,k) = (24,\,1)$. $Y = \widehat{Y} - k = 18$. Hence $0 < X < Y$. $c_3 = (\widehat{Y} + t + k)/2 = 44/2 = 22$. $q = (\widehat{Y} + t - k)/2 = 21$. $c_1 = x = (Y - X)/2 = 3$ and $c_2 = x = (Y + X)/2 = 15$. Thus, there is the primitive $DS$-triple $(3,\,15,\,22)$, with $c_{4,\pm} = c_1 + c_2 + c_3 \pm 2\,q$ given by $c_{4,-} = -2$ and $c_{4,+} = 82$. See *Table 4*.

**ii)**: $a = 17\cdot 41 = 697$. Two solutions $(X,\,\widehat{Y})_1 = (5,\,19)$ and $(X,\,\widehat{Y})_2 = (19,\,23)$. Two solutions $(t,\,k)_1 = (25,\,6)$ and $(t,\,k)_2 = (7,\,18)$. Four possibilities: $Y_{1,1} = \widehat{Y}_1 - k_1 = 13$, $Y_{1,2} = \widehat{Y}_1 - k_2 = 1$, and $Y_{2,1} = \widehat{Y}_2 - k_1 = 17$, $Y_{2,2} = \widehat{Y}_2 - k_2 = 5$. Only $(X_1,\,Y_{1,1}) = (5,\,13)$ qualifies with $X < Y$. $c_3 = (19 + 25 + 6)/2 = 25$, $q = 19$, $c_1 = x = (13 - 5)/2 = 4$ and $c_2 = y = (13 + 5)/2 = 9$. The $DS$-triple $(4,\,9,\,25)$ is primitiv. $c_{4,-} = 0$ and $c_{4,+} = 76$. This is a degenerate case. See *Table 4* and *Table 6* under $n = 5$.

**Type b)**

**i)** $a = (3^2\cdot 17^2)\,17 = 44217$. Two pairs of conjugate families. $A = 3\cdot 17$, $\tilde{a} = 17$. $X = A\,\tilde{X}$, $\widehat{Y} = A\,\widetilde{\widehat{Y}}$. Positive proper solutions representing $3^2\,17^3$: $(t,\,k)_1 = (107,\,128)$ and $(t,\,k)_2 = (197,\,52)$. Proper positive solution representing $-17$: $(\tilde{X},\,\widetilde{\widehat{Y}}) = (1,\,3)$, $(X,\,\widehat{Y}) = (51,\,153)$. $Y_1 = \widehat{Y} - k_1 = 153 - 128 = 25$, $Y_2 = \widehat{Y} - k_2 = 153 - 52 = 101$, $(X,\,Y_1) = (51,\,25)$ not qualifying, and the solution is $(X,\,Y_2) = (51, 101)$. $c_3 = (\widehat{Y} + t_2 + k_2)/2 = 201$ and $q = 149$. $x = (101 - 51)/2 = 25$ and $y = (101 + 51)/2 = 76$, The final primitive $DS$-triple is $(25,\,76,\,201)$ with $[c_{4,-},\,c_{4,+}] = [4, 600]$.

**ii)** $a = 3^2\cdot 17^3 = 44217$. Two pairs of conjugate families. $A = 3$, $\tilde{a} = 17^3$. $X = A\,\tilde{X}$, $\widehat{Y} = A\,\widetilde{\widehat{Y}}$. Proper solutions representing $3^2\cdot 17^3$ like above: $(t,\,k)_1 = (107,\,128)$ and $(t,\,k)_2 = (197,\,52)$. Proper solution representing $-17^3$: $(\tilde{X},\,\widetilde{\widehat{Y}}) = (55,\,63)$, $(X,\,\widehat{Y}) = (165,\,189)$. $Y_1 = \widehat{Y} - k_1 = 189 - 128 = 61$,

$Y_2 = \widehat{Y} - k_2 = 189 - 52 = 137$, $(X, Y_1) = (165, 61)$, not qualifying, and $(X, Y_2) = (165, 137)$, also not qualifying.
Conclusion: different splittings of $a$ with improper solutions for $(X, \widehat{Y})$ lead to different answers. Here only the final solution of part **i)** is possible.

**iii)** A similar situation appears for $a = 3^2 \cdot 17^2 \cdot 41 = 106641$. The splitting $a = A^2 \tilde{a}$ with $A = 3 \cdot 17$ and $\tilde{a} = 41$ does not lead to a final solution, whereas $A = 3$ and $\tilde{a} = 17^2 \cdot 41 = 11849$ leads to only two (not eight) final solutions with $(X, Y)_1 = (9, 85)$ and $(X, Y)_2 = (9, 65)$. $c_{3,1} = 315$ and $c_{3,2} = 312$. $q_1 = 169$ and $q_2 = 146$. $[c_1, c_2, c_3]_1 = [38, 47, 315]$ and $(c_1, c_2, c_3)_2 = (28, 37, 312)$. $[c_{4,-}, c_{4,+}]_1 = [62, 738]$ and $[c_{4,-}, c_{4,+}]_2 = [85, 669]$.
A splitting with $A = 17$ and $\tilde{a} = 3^2 \cdot 17$ is not possible because $\tilde{a} \not\equiv 1 \pmod 8$.
In all these examples neither type **a)** nor type **c)** is possible because there is no proper solution for $(X, \widehat{Y})$ due to the factor 3 of $a$.

**Type c)**
**i)** $a = 17^2\,(2 \cdot 17) = 9826$. $A = 17$, $\tilde{a} = 2 \cdot 17$. Positive proper solution representing $-2 \cdot 17^3$: $(X, \widehat{Y}) = (16, 71)$. Positive proper solution representing $2 \cdot 17 = 34$: $(\tilde{t}, \tilde{k}) = (4, 3)$. $(t, k) = (68, 51)$. $Y = \widehat{Y} - k = 20$. $(X, Y) = (16, 20)$. $c_3 = (71 + 68 + 51)/2 = 95$, $q = 44$. Final $DS$-triple $(c_1, c_2, c_3) = (2, 18, 95)$, and $[c_{4.-}, c_{4,+}] = [27, 203]$.
No other splitting of $a$ is possible.
Note that there is no final solution of type **a)** for $a = 2 \cdot 17^3$ because $(t, k) = (76, 45)$, hence $(X, Y) = (16, 71 - 45) = 2 \cdot (8, 13)$ is improper.
There is also no final solution of type **b)** with splitting $A = 17$ and $\tilde{a} = 2 \cdot 17$ because with $(X, \widehat{Y}) = 17 \cdot (4, 5) = (66, 85)$ and the just given $(t, k) = (76, 45)$ one obtains the improper solution $(X, Y) = 2 \cdot (33, 20)$. Thus, for $a = 2 \cdot 17^3$ there is only the one type **c)** solution.

**ii, a)** $a = (7 \cdot 17)^2\, 17 = 240737$. $A = 7 \cdot 17$, $\tilde{a} = 17$. Two positive proper solution representing $-240737$: $(X, \widehat{Y})_1 = (9, 347)$ and $(X, \widehat{Y})_2 = (115, 363)$. Positive proper solution representing 17: $(\tilde{t}, \tilde{k}) = (3, 2)$, $(t, k) = (357, 238)$. $Y_1 = \widehat{Y}_1 - k = 109$, and $Y_2 = \widehat{Y}_2 - k = 98$ is not qualifying. The solution $(X, Y) = (9, 109)$ leads to $c_3 = (347 + 357 + 238)/2 = 471$ and $q = 233$. The final primitive $DS$-triple is $(c_1, c_2, c_3) = (50, 59, 471)$, with $[c_{4.-}, c_{4,+}] = [114, 1046]$.

**ii, b)** $a = 7^2\, 17^3 = 34391$. $A = 7$, $\tilde{a} = 17^3$. Like in the previous example the two proper solutions $(X, \widehat{Y})_1 = (9, 347)$ and $(X, \widehat{Y})_2 = (115, 363)$. Positive proper solution representing $17^3 = 4913$: $(\tilde{t}, \tilde{k}) = (45, 38)$. $(t, k) = (315, 266)$. $Y_1 = \widehat{Y}_1 - k = 81$, and $Y_2 = \widehat{Y}_2 - k = 97 < 115$ does not qualify. The solution $(X, Y) = (9, 81)$ leads to $c_3 = (347 + 315 + 266)/2 = 464$ and $q = 198$. The only final primitive $DS$-triple is $(c_1, c_2, c_3) = (36, 45, 464)$, with $[c_{4.-}, c_{4,+}] = [149, 941]$.
In these examples **ii**) neither type **a)** nor type **b)** applies because of the factor 7 of $a$.

**Proof of Proposition 8**:

**a)** This follows from the intersection of *Lemma 18, a)* eq. (55) and *Lemma 19, a)* eq. (59).

**b)** The proper solution $(t, k)$ of $t^2 + 2\,k^2 = A^2\,\tilde{a}$ allows distinct primes 2, $p_{1,i}$, $p_{3,i}$ by *Lemma 19* a). The improper solution $(X, \widehat{Y}) = (A\,\tilde{X}, A\,\widetilde{\widehat{Y}})$, with $\gcd(\tilde{X}, \widetilde{\widehat{Y}}) = 1$, of $X^2 - 2\,\widehat{Y}^2 = -A^2\,\tilde{a}$, allows for $(\widetilde{X}, \widetilde{\widehat{Y}})$ only distinct primes 2, $p_{1,i}$ because the previous restriction for $(t, k)$ implies $P_7 = 0$. $A$ allows distinct $p_{1,i}$ and $p_{3,i}$ but not 2, because $e_2 \in \{0, 1\}$.
For the fact that for $a \geq 2$ pairs of conjugate families, of types $I$ and $II$, appear for proper solution the of $(X, \widehat{Y})$ see *Corollary 5 b)*. This also proves the existence of a unique positive solution $0 < X' < \widetilde{Y}'$ in each family $I$. Therefore the number of solutions is $2^{P_1 + P_7 - 1}$. The proper $(t, k)$ solutions come for $a > 2$ in a pair of conjugate solutions with $k > 0$, and there is one solution with also $t > 0$ ($t = 0$ is forbidden by *Lemma 17*). Thus, the number of positive proper solutions $(t, k)$ is $2^{P_1 + P_3 - 1}$.

**c)** Only primes 2, $p_{1,i}$, and $p_{i,7}$ can appear because of the proper solutions of $(X, \widehat{Y})$ by *Lemma 18*. Hence the improper solutions $(t, k) = (A\,\tilde{t}, A\,\tilde{k})$ can have $A$ with primes of type $p_{1,i}$ and $p_{7,i}$, and $\tilde{a}$ can

only have primes of type 2 and $p_{1,i}$ for the solutions of $(\tilde{t},\, \tilde{k})$. The splitting of the primes $p_{1,i}$ in $A$ and $\tilde{a}$ is $P_1' + P_1'' = P_1$.
The case $\tilde{a} = 2$ $(P_1''\!=\!0)$ is forbidden because the $(\tilde{t},\, \tilde{k})$ solution with $k > 0$ would be $(0,\, 1)$, with forbidden $t = 0$. Also $\tilde{a} = 1$ is excluded because the solutions $(\tilde{t},\, \tilde{k}) = (\pm 1,\, 0)$ lead to $k = 0$. $\square$

**Case iii) q > c₃**

The case $0 < q = c_3 + k$, with $k \geq 1$, is treated by solving eq. (50) of the previous case *ii)* after substituting $k$ by $-k$. The quadratic equations eq. (53) are the same but the variable $\widehat{Y}$ is now $Y - k$, thus $0 < X < Y = \widehat{Y} + k$ has to be satisfied. The inequalities for the curvatures are

$$c_1 = \frac{\widehat{Y} + k - X}{2} < c_2 = \frac{\widehat{Y} + k + X}{2} < c_3 = q - k\,. \tag{63}$$

Here two cases follow from eq. (51) with $k \to -k$, and $2\,\mathrm{x} = \mathrm{Y} - \mathrm{X}$ and $2\,\mathrm{y} = \mathrm{Y} + \mathrm{X}$, namely:
Case a) $0 < 2\,q = Y + t$, with $t \geq 0$ and $t := \sqrt{2\,Y^2 - X^2 - 4\,Y\,k} = \sqrt{2\,\widehat{Y}^2 - X^2 - 2\,k^2}$.
Case b) $0 < 2\,q = Y - t$, with $t \geq 0$ defined in case a).

**Lemma 20: Upper bound for k**
$1 \leq k \leq \left\lfloor \sqrt{\frac{a}{2}} \right\rfloor$, with $a = 2\,Y^2 - X^2 = t^2 + 2\,k^2$.

Proof: In order that $t$ is real one needs $a = 2\,\widehat{Y}^2 - X^2 \geq 2\,k^2$. Thus, $k \leq \sqrt{\frac{a}{2}}$. $\square$

The next Lemma shows that for both cases the instance $t = 0$ does not qualify.

**Lemma 21:** *There is no solution for* $(X,\, \widehat{Y})$ *and* $c_2 < c_3$ *if* $t = 0$.

**Proof:** If $t = 0$ then $0 < X^2 = 2\,\widehat{Y}^2 - 2\,k^2 = 2\,(\widehat{Y} - k)\,(\widehat{Y} + k)$. In both cases $2\,q = \widehat{Y} + k$, hence $0 < \widehat{Y} - k = 2\,(q - k) = 2\,c_3$. One shows that, in fact, $c_2 > c_3$. $2\,c_2 = \widehat{Y} + k + X > 2\,c_3 = \widehat{Y} - k$, meaning that $2\,k + X > 0$, which is true because $X$ and $k$ are positive. $\square$

From $t^2 > 0$ the upper bound becomes $k < \sqrt{\frac{a-1}{2}}$.

**Lemma 22:** *Case b) does not qualify for a final solution because* $c_2 > c_3$.

Proof: For this case $2\,q = \widehat{Y} + k - t$, with $t > 0$. $2\,c_3 = 2\,(q - k) = \widehat{Y} - k - t$. Hence $2\,c_2 = \widehat{Y} + k + X > 2\,c_3$ because $2\,k + X + t > 0$, which is satisfied because each term is positive. $\square$

Therefore only case a) has to be treated.

Concerning the representable values $a \geq 2$ of the two quadratic forms (the case $a = 1$ is not relevant for nonvanishing $k$; see also the remark in *Lemma 18 a)*) the results from *Proposition 8* for case ii) apply. For $X^2 - 2\,\widehat{Y}^2 = -a$ there is *Lemma 18*, eq. (55), and for $t^2 + 2\,k^2 = a$ see *Lemma 19*, eq. (59). The combined solutions come also in the three cases *a*), *b*), and *c*) of *Proposition 8*, eqs. (60), (61), and (62), respectively.

The uniqueness of the solution $(X',\, \widehat{Y}')$ with $a \geq 2$ and $0 < X' < \widehat{Y}' < \sqrt{a}$ for one of the members of family I (with a certain index $i = i'$) of a conjugate pairs of solutions is also guaranteed, like in *Lemma 11 b)*, eq. (46), but with a change of variables $s \to \sqrt{a}$, $Y \to \widehat{Y}$, $\widehat{Y} \to \widehat{Y}'$ and $\widehat{X} \to X'$.

For final solutions one has to check whether $c_2 < c_3$ and the primitivity of the curvature triple holds for those solutions corresponding to the unique one in each family $I$ which satisfies $0 < X' < \widehat{Y}' < Y$.

But because now $Y = \widehat{Y} + k$, in contrast to case ii) with $q = c_3 - k$ and $Y = \widehat{Y} - k$, further candidates may qualify. It could happen in both families that also solutions with $\widehat{Y}_i < X_i$ and $X_i > 0$ (after a possible common sign flip) satisfy the necessary condition $Y_i = \widehat{Y}_i + k > X_i$ provided $k$ is large enough. Fortunately, this will be proved not to happen, except for one member in each family $II$ for which a proof is missing. This member is $(X_{i''+1}), \widehat{Y}_{i''+1}$. For these instances the conjecture is that they cannot lead to a final solution because the additional condition $c_3 > c_2$ fails. This is the content of the next *Lemma*.

**Lemma 23: Excluding certain candidates for final solutions.**

*There is no qualifying solution* $(X, \widehat{Y})$ *with* $X > 0$ *and* $X > \widehat{Y}$*, with* $Y = \widehat{Y} + k > X$*. This means that* $X - \widehat{Y} \geq k$*, except for possibly one solution of each family* $II$*.*
*If such a solution of family* $II$*, viz* $(X_i, \widehat{Y}_i)$*, where* $i = i'' + 1$*, (i.e., the first positive solution) satisfies* $0 < X_i < Y_i$ *then the* **conjecture** *is that it cannot be a final solution because the necessary condition* $c_3 > c_2$ *is not satisfied, i.e.,* $X_i > 0$ *and* $X_i \geq t - 2\,k$*,*

**Proof:**
a) Family $I$ (see *Definition 3* and *Lemma 13* with the above mentioned adaption of $s$, $Y$, $\widehat{Y}$ and $\widehat{X}$)
Case i) For $i = i'$ there is no such solution because $0 < X' < \widehat{Y}'$.

Case ii) With $i > i'$, one has solutions $X_i > \widehat{Y}_i > 0$. An inductive proof, using the recurrence eq. (40) and the upper bound $k < \sqrt{a/2}$, shows that a qualifying solutions with $0 < X_i < Y_i = \widehat{Y}_i + k$, *i.e.,* $X_i - \widehat{Y}_i < k$, does not exist.
Start with $i = i' + 1$, and the recurrence $X_{i'+1} = 4\,X' + 3\,\widehat{Y}'$ and $\widehat{Y}_{i'+1} = 4\,X' + 3\,\widehat{Y}'$, with $0 < X' < \widehat{Y}'$. Hence $X_i - \widehat{Y}_i = X' + \widehat{Y}'$. But $X' + \widehat{Y}' < k$ implies, with the upper bound of $k$, that this is also $< \sqrt{\widehat{Y}'^{\,2} - X'^2/2}$. Squaring leads to $3\,X' + 4\,\widehat{Y}' < 0$, because $X' > 0$; a contradiction, because also $\widehat{Y}' > 0$.
The same contradiction appears for any solution $(X_i, \widehat{Y}_i)$ for $i \geq i' + 2$ with $X_i > \widehat{Y}_i > 0$. Assuming that $X_i - \widehat{Y}_i = X_{i-1} + \widehat{Y}_{i-1} < k$. then the upper bound of $k$ yields $3\,X_{i-1} + 4\widehat{Y}_{i-1} < 0$; a contradiction because both terms are positive.

Case iii) We use a common sign flip in $X_i$ and $\widehat{Y}_i$ (a proper equivalence transformation) for $i < i'$ of the adapted *Lemma 13 a) iii)* such that $Y_i < 0$ and $0 < |\widehat{Y}_i| < X_i$. In this case no recurrence is needed. The proof that $0 < X_i < Y_i = \widehat{Y}_i + k$ is not possible runs as follows. Assume that $X_i - \widehat{Y}_i = X_i + |\widehat{Y}_i| < k$. But using $k < \sqrt{\widehat{Y}_i^2 - X_i^2/2}$ leads, after squaring and using $X_i > 0$, to the contradiction $3\,X_i + 4\,|\widehat{Y}_i| < 0$.
Note that this direct proof, not using the recurrence equation, does not work for the above case ii). It would lead for $i > i'$ only to $3\,X_i - 4\,\widehat{Y}_i$, which cannot be proved to be $\geq 0$; it only follows that $> -X_i$ or $> -\widehat{Y}_i$.

b) Family $II$

Case i) After adaption of *Lemma 13 b) i)* with $i = i''$ and a common sign flip $0 < X_i < |Y_i|$. Assume $0 < X_i < \widehat{Y}_i = \widehat{Y}_i$, *i.e.,* $X_i - \widehat{Y}_i = X_i + |\widehat{Y}_i| < k$. Then the upper bound on $k$ leads, after squaring and division by $X_i > 0$, to the contradiction $X_i + |\widehat{Y}_i| < 0$.

Case ii) for $i > i''$ has $0 < \widehat{Y}_i < X_i$. One has to use the recurrence eq. (40) like above in case a) ii). Assume $0 < X_i < Y_i = \widehat{Y}_i + k$, then by the recurrence $X_i - \widehat{Y}_i = X_{i-1} + \widehat{Y}_{i-1} < k$. For all $i \geq i'' + 2$ the upper bound on $k$ leads, after squaring and dividing by $X_{i-1}$, to $3\,X_{i-1} + 4\,\widehat{Y}_{i-1} < 0$, a contradiction because both terms are positive, because $i - 1 > i''$.
However, here the mentioned gap in the proof appears for $i = i'' + 1$.
With $0 < \widehat{Y}_i < X_i$ assume that $X_i - \widehat{Y}_i < k$. With Recurrence eq. (40) this becomes $X_{i''} + \widehat{Y}_{i''} = -X' + \widehat{Y}' < k$. Using the upper bound for $k$ like above will not lead to a contradiction because it leads to $3\,X' - 4\widehat{Y}' < 0$. *E.g.,* $a = 89$. In family $I$ $(X', \widehat{Y}') = (3, 7)$ and $(t, k) = (9, 2)$. The upper bound for $k$ is 6, hence $3 \cdot 3 - 7 = 4 < 6$, and the assumption seems to be valid but with the proper $k$ value it leads to a contradiction. The used upper bound is too large.

**Conjecture:**
*For the proper solution* $(X_i, \widehat{Y}_i)$ *with* $i = i'' + 1$ *in family* $II$ *one checks first if the proper positive* $(t, k)$ *solution of* $t^2 - 2\,k^2 = a$ *satisfies the* $c_3 > c_2$ *condition* $X_i < t - 2\,k$*. If it fails no final solution is possible, even if* $0 < X_i < Y_i$ *may hold. Otherwise one has* $X_i - \widehat{Y}_i \geq k$*. Using the recurrence this means that* $-X' + \widehat{Y}' \geq k$*, where* $(X', \widehat{Y}')$ *is the unique positive fundamental solution of* $X^2 - 2\,\widehat{Y}^2 = -a$ *in the corresponding family* $I$*, satisfying* $0 < X' < \widehat{Y}'$*.*

*Using the correct $k$ value the conjecture means that $X'(3X' - 4\hat{Y}') + t^2 \geq 0$, where $(t, k)$ is the unique solution of $t^2 + 2k^2 = a$ with positive $k$ and $t$, where the $a$ value is from A192453 without members* 1 *and* 2.

Case iii) For $i < i''$ we use,as above in case a), a common sign flip, to get $Y_i < 0$ and $0 < |\hat{Y}_i| < X_i$. Like in case a) above assume that $0 < X_i < Y_i = \hat{Y}_i + k$, *i.e.*, $X_i + |\hat{Y}_i| < k$. Using the upper bound of $k$, squaring and dividing by $X_i > 0$, leads to the contradiction $3X_i + 4|Y_i| < 0$. □

Thus, *Proposition 8* can be taken over as **Proposition 9**, just replacing there $k \to -k$ in the remarks at the end of each of the three cases.

Examples for the three types follow.

**Example 14:**

Necessary conditions for the final solution are $0 < X < Y$ and $c_2 < c_3$, *i.e.*, $X < t - 2k$.
In the following examples the prime superscript on $X$ and $\hat{Y}$ for the unique family $I$ solution with $O < X' < \hat{Y}'$ will be omitted.

**Type a) Both binary quadratic forms are solved properly.**

**i)** $a = 2 \cdot 17^2 = 578 =$A192453(30). $(X, \hat{Y}) = (12, 19)$, $(t, k) = (24, 1)$ (see *Example 11, a)i)*). $Y = \hat{Y} + k = 20$. Hence $0 < X < Y$. $c_3 = (\hat{Y} + t - k)/2 = 21$. $q = (\hat{Y} + t + k)/2 = 22$. $c_1 = x = (Y - X)/2 = 4$ and $c_2 = x = (Y + X)/2 = 16$. Thus, there is the primitive $DS$-triple $[4, 16, 21]$ with $c_{4,\pm} = c_1 + c_2 + c_3 \pm 2q$, hence $c_{4,-} = -3$ and $c_{4,+} = 85$. See *Table 5*.
The conjecture in *Lemma 23* for the family $II$ element $(X_{i''+1}, \hat{Y}_{i''+1}) = (40, 33)$ is true because already the necessary condition $X_{i''+1} < t - 2k$ fails, viz $40 > 24 - 1$. Indeed, $c_3 = 23 < 26 = c_2$. In this case also $40 - 33 \geq 1$.

**ii)**: $a = 17 \cdot 41 = 697 =$A192453(47). Two solutions $(X, \hat{Y})_1 = (5, 19)$ and $(X, \hat{Y})_2 = (19, 23)$. Two solutions $(t, k)_1 = (25, 6)$ and $(t, k)_2 = (7, 18)$. The last solution has a negative $t - 2k$ value, hence no positive $X$ solution qualifies. The other solution has $t - 2k = 13$, hence the second $(X, \hat{Y})$ solution fails. Thus, only $(X, \hat{Y}) = (5, 19)$ with $(X, Y) = (5, 25)$ qualifies, and $c_3 = 19$, $2q = 50$, and $[c_1, c_2] = [10, 15]$. Thus, $[c_1, c_2, c_3] = [10, 15, 19]$ qualifies, is primitive, and $[c_{4,-}, c_{4,+}] = [-6, 94]$. This is the only final solution for $a = 697$, provided the conjecture is true. See *Table 50*.
The conjecture in *Lemma 23* for the family $II$ element is true because for $(X_{i''+1}, \hat{Y}_{i''+1}) = (61, 47)$ the necessary condition $5 < t - 2k = 13$ fails. Indeed, $c_2 = 57 > 33 = c_3$. In this case also $47 + 6 < 61$.

**iii)**: $a = 17^3 = 4913 =$A192453(268).
The positive proper solution $(t, k)$ representing $17^3$ is $(45, 38)$, with $t - 2k$ negative. Therefore the family $I$ qualifying solution $(X, \hat{Y}) = (55, 63)$, with $(X, Y) = (55, 101)$ will not lead to a final solution. The conjecture for the first positive family $II$ element $(X_{i''+1}, \hat{Y}_{i''+1}) = (87, 79)$ is true because $t - 2k$ is negative. This is an example where one has for $(X_i, \hat{Y}_i)$, for $i = i'' + 1$, the promising looking condition $0 < X_i < Y_i = \hat{Y}_i + k$, because $0 < 87 < 79 + 38$.

**iv)**: $a = 17^2 \cdot 41 = 11849 =$A192453(598).
The two positive proper $(t, k)$ solutions representing 11849 are (93, 40) and (99, 32) with $t - 2k$ values 13 and 35. The two positive proper fundamental solutions $(X, \hat{Y})$ from family $I$ representing $-17^2 \cdot 41$ are (3, 77) and (101, 105). Only the first solution survives the test $X < t - 2k$ for both $(t, k)$ solutions. The two $(X, Y)$ solutions are then (3, 117) and (3, 109). The corresponding $DS$-triples are [57, 60, 65] and [53, 56, 72] with $[c_{4,-}, c_{4,+}]$ values $[-28, 392]$ and $[-27, 389]$.
The conjecture for the first positive family $II$ element $(X_i, \hat{Y}_i)$, for $i = i'' + 1$, is true, because the two solutions are (299, 225) and (117, 113), and in both cases $X_i \geq t - 2k$, for each of the two $t - 2k$ values 13 and 35. In this case both $(X_i, \hat{Y}_i)$ solutions look promising for both $k$ values 40 and 32, but fail as final solutions.

**Type b) $(\mathbf{X},\, \widehat{\mathbf{Y}})$ improper and $(\mathbf{t},\, \mathbf{k})$ proper**

**i)** $a = 3^2{\cdot}73 = 657$. $3 =$ A225771(2) and $73 =$ A192453(6). There is one qualifying solution from family $I$ *viz* $(X,\, \widehat{Y}) = 3\cdot(5,\, 7) = (15,\, 21)$, and two positive solutions $(t,\, k)_1 = (25,\, 4)$ and $(t,\, k)_2 = (23,\, 8)$. The second solution fails because $X > t - 2\,k$, *i.e.*, $15 > 7$. The surviving solution has $Y = \widehat{Y} + k_1 = 21 + 4 = 25$, $c_3 = (\widehat{Y} + t_1 - k_1)/2 = 21$ and $q = 25$. $c_1 = (25-15)/2 = 5$ and $c_2 = (25+15)/2 = 20$. Thus, the final primitive curvature solution is $(5,\, 20,\, 21)$ with $[c_{4,-},\, c_{4,+}] = [-4,\, 96]$. See *Table 5*.
The first instance of type *b*) appears for $a = 3^2{\cdot}41 = 369$ with the $DS$-triple $[4,\, 13,\, 16]$ with $[c_{4,-},\, c_{4,+}] = [-3,\, 69]$. See *Table 5*.

The conjecture in *Lemma 23* for the family $II$ proper representation for $-\tilde{a} = -73$, *viz* $(\tilde{X}_i,\, \tilde{\widehat{Y}}_i) = (13,\, 11)$, for $i = i'' + 1$, is true because $(t,\, k) = (1,\, 6)$, hence $t - 2\,k$ is negative. Here also $X_i = 3{\cdot}13 > 17 = t_1 - 2\,k_1$, and also $> 7 = t_2 - 2\,k_2$, confirming the conjecture. $\tilde{X}_i - \tilde{\widehat{Y}}_i = 2 < 6$ looks promising, but it fails as final solution representing $-73$. In this example also $X_i - \widehat{Y}_i = 39 - 33 = 6 > 4 = k_1$ fails, but looks promising for $k_2 = 8$.

**ii)** $a = (3^2\cdot 17^2)\,17 = 44217$. $A = 3\cdot 17 = 51 =$ A225771(11) and $\tilde{a} = 17 =$ A192453(3). The two positive proper solutions representing $a = 3^2\,17^3$ are $(t,\, k)_1 = (107,\, 128)$ and $(t,\, k)_2 = (197,\, 52)$. The first solutions fails because $t - 2\,k$ is negative. The positive fundamental proper representation of $-17$ from family $I$ is $(\tilde{X},\, \tilde{\widehat{Y}}) = (1,\, 3)$ This corresponds to the improper solution $(X,\, \widehat{Y}) = (51,\, 153)$, leading to the proper solution $(X,\, Y) = (51,\, 153 + 52) = (51,\, 205)$. $c_3 = (153 + 197 - 52)/2 = 149$ and $q = 201$. $c_1 = 77$ and $c_2 = 128$. The primitive $DS$-trip;e is $[77,\, 128,\, 149]$, with $[c_{4,-},\, c_{4,+}] = [-48,\, 756]$.
The conjecture for the first positive family $II$ element representing $-17$, *viz* $(\tilde{X}_i,\, \tilde{\widehat{Y}}_i) = (9,\, 7)$, for $i = i'' + 1$, is true,because $(t,\, k) = (3,\, 2)$ representing 17 has negative value $t - 2\,k$. Here with $\tilde{X}_i - \tilde{\widehat{Y}} = 9 - 7 < 2 = k$ looks promising, but it fails as final solution The improper solution $(X_i,\, \widehat{Y}_i) = (459,\, 357)$ fails because $t_2 - 2\,k_2 = 93 < X_i$. Also $X_i - \widehat{Y}_i = 102 > 52 = k_2$, hence fails, but it looks promising for $k_1 = 104$, but fails as final solution.
Therefore the only final $DS$-triple for this decomposition of $a = 44217$ is the found one, *viz* $[77,\, 128,\, 149]$. It will be shown next that the other possible decomposition will not survive.

**iii)** $a = 3^2\cdot 17^3 = 44217$. $A = 3 =$ A225771(1), $\tilde{a} = 17^3 = 4913 =$ A192453(269). From the two positive proper solutions $(t,\, k)$ given in the previous section **ii)**, again only $(197,\, 52)$ survives. The unique positive proper solution representing $-17^3$, have been given above in type a)iii) $(\tilde{X},\, \tilde{\widehat{Y}}) = (55,\, 63)$. Now $(X,\, Y) = (3\cdot 55,\, 3\cdot 63 + 52) = (165,\, 241)$ looks promising, but $165 > 83 = t - 2\,k$, thus fails as final solution.
The conjecture for the first positive family $II$ solution $(\tilde{X}_i,\, \tilde{\widehat{Y}}_i) = (87,\, 79)$ for $-\tilde{a} = -17^3$, from above type **a), iii)**, is true, because $t - 2\,k = 45 - 2{\cdot}38$ is negative. Here $\tilde{X}_i - \tilde{\widehat{Y}}_i = 8 < 38$ looks promising. The improper solution $(X_i,\, \widehat{Y}_i) = (261,\, 237)$ fails also because $X_i > 93 = t_2 - 2\,k_2$. Here $X_i - \widehat{Y}_i = 24$ looks promising for $k_2 = 52$ as well as for $k_1 = 128$. Therefore the conjecture is true for this improper as well as proper case.
Thus, with this splitting of $a$ no final solution exists.

Conclusion: different splittings of $a$ with improper solutions for $(X,\, \widehat{Y})$ may lead to different answers. For $a = 44217$ only the final solution of part **ii)** is possible.

**iv)** A similar situation appears for $a = 3^2\cdot 17^2\cdot 41 = 106641$.

**$\alpha$)** The splitting $a = A^2\,\tilde{a}$ with $A = 3\cdot 17 =$ A225771(11) and $\tilde{a} = 41 =$ A192453(5) does not lead to a final solution, because all four positive proper solutions $(t,\, k)$ of $a$ have negative $t - 2\,k$ values. These solutions are $(29,\, 230)$, $(67, 226)$, $(227,\, 166)$ and $(253,\, 146)$. Therefore no final solution exists.
The only qualifying improper positive fundamental solutions from family $I$ and for this splitting is $(X,\, \widehat{Y}) = (153,\, 255)$, but it will not lead to a final result.

The conjecture for the first positive family $II$ proper solution representing $-41$, *viz* $(\tilde{X}_i,\, \tilde{\widehat{Y}}_i) = (11,\, 9)$, where $i = i'' + 1$, representing $-41$, is true because $(t,\, k) = (3,\, 4)$. with a negative $t - 2\,k$ value.Here

$(X_i,\, Y_i) \,=\, (11,\, 9\,+\,4)$ looks promising but does not lead to a final solution for $-41$ because $c_2 \,>\, c_3$. Also the improper solution $X_i,\, \widehat{Y}_i \,=\, (561,\, 459)$ representing $-106641$ does not lead to a final solution, because all four $t \,-\, 2\,k$ values are negative, even though it looks promising for all four $k$ values given above.

$\boldsymbol{\beta}$) The splitting with $A \,=\, 3 \,=$A225771(2) and $\tilde{a} \,=\, 17^2 \cdot 41 \,=\, 11849 \,=$A192453(598) has the four positive proper $(t,\, k)$ solutions given in part $\boldsymbol{\alpha}$), and $t \,-\, 2,\, k$ is negative for all four solutions. Therefore the two improper positive $(X, \widehat{Y})$ solutions $(9,\, 231)$ and $(303,\, 315)$ will not lead to a final solution. Together with $(153,\, 255)$ from the $\boldsymbol{\alpha}$) part they give all three positive improper solutions of $a \,=\, 3^2{\cdot}17^2{\cdot}41$. For the proper solutions representing $-17^2 \cdot 41$ see type **a) iv)**.
For the truth of the the conjecture for the two family $II$ proper solutions representing $-17^2 \cdot 41$, *viz* $(\tilde{X}_i,\, \tilde{\widehat{Y}}_i)$, with $i \,=\, i'' \,+\, 1$, see type **a)iv)** above.
The corresponding two first positive improper solutions representing $a$, *viz* $(X_i,\, \widehat{Y}_i)$, for $i \,=\, i'' \,+\, 1$, are $(897,\, 675$ and $(351,\, 339)$. They satisfy the conjecture because all four above given $t \,-\, 2\,k)$ values are negative.
Therefore there is no $DS$-triple for $a \,=\, 3^2 \cdot 17^2 \cdot 41 \,=\, 106641$ of this type **b**).

**Type c)** $(\mathbf{X},\, \widehat{\mathbf{Y}})$ **proper and** $(\mathbf{t},\, \mathbf{k})$ **improper**.

**i)** $a \,=\, 7^2 \cdot 2 \cdot 17 \,=\, 1666$. $A \,=\, 7 \,=$A058529(2), $\tilde{a} \,=\, 2 \cdot 17 \,=$A192453(4). $(\tilde{t},\, \tilde{k}) \,=\, (4,\, 3)$, $(t,\, k) \,=\, (28,\, 21)$. Because $t - 2\,k$ is negative there will be no final solution, whatever the two proper fundamental family $I$ solutions $(X,\, \widehat{Y})$ representing $a \,=\, 1666$ are (they are $(X,\, \widehat{Y})_1 \,=\, (4,\, 29)$ and $(X,\, \widehat{Y})_2 \,=\, (16,\, 31)$ with $(X,\, Y)_1 \,=\, (4,\, 50)$ and $(X,\, Y)_2 \,=\, (16,\, 52)$, but $c_2 \,>\, c_3$ in both cases).
The conjecture for the two first proper positive family $II$ solutions $(X_i,\, \widehat{Y}_i)$, for $i \,=\, i'' \,+\, 1$, representing 1666 (they are $(104,\, 79)$ and $(76,\, 61)$) is true because $t \,-\, 2\,k$ is negative. Only the second solution satisfies $0 \,<\, X_i \,<\, Y_i$, with $k \,=\, 21$, but it fails the $c_2 \,<\, c_3$ test.

**ii)** $a \,=\, 7^2 \cdot 89 \,=\, 4361$. $A \,=\, 7$, $\tilde{a} \,=\, 89 \,=$A192453(8). $(\tilde{t},\, \tilde{k}) \,=\, (9,\, 2)$, $(t,\, k) \,=\, (63,\, 14)$. The two family $I$ positive proper fundamental solutions are $(X,\, \widehat{Y})_1 \,=\, (29,\, 51)$ and $(X,\, \widehat{Y})_2 \,=\, (51,\, 59)$. Because $t \,-\, 2\,k \,=\, 35$ only the first solution has to be tested. We skip the subscript 1: $(X,\, Y) \,=\, (29,\, 65)$, $c_1 \,=\, (65 \,-\, 29)/2 \,=\, 18$, $c_2 \,=\, 47$, $c_3 \,=\, (\widehat{Y} \,+\, t \,-\, k)/2 \,=\, 50$, and $2\,q \,=\, Y \,+\, t \,=\, 128$. Therefore a final $DS$-triple for $a \,=\, 4361$ is $[c_1,\, c_2,\, c_3] \,=\, [18,\, 47,\, 50)]$, with $(c_{4.-},\, c_{4.+}) \,=\, (-13,\, 243)$.
The conjecture for the two proper family $II$ members $(X_i,\, \widehat{Y}_i)$, for $i \,=\, i'' \,+\, 1$, representing 4361, *viz* $(117,\, 95)$ and $(83,\, 75)$, is true because with $k \,=\, 14$ only the second solution looks promising, but because $83 \,>\, 35$ it fails the $c_2 \,<\, c_3$ test.
Therefore the found $DS$-triple is the only one for $a \,=\, 4361$.
In fact $a \,=\, 7^2 \cdot 89 \,=\, 4361$ is the smallest instance of type c) with a $DS$-triple. This is because $7^2 \cdot 17$, $7^2 \cdot 34$, $7^2 \cdot 41$ and $7^2 \cdot 82$ all do not pass the $X \,<\, t \,-\, 2\,k$ test. The instance $17^2 \cdot 17 \,=\, 4913$ (a candidate to be tested) is already larger. The type **a**) solution for $a \,=\, 17^3$ does also not pass this test because $t \,-\, 2\,k \,=\, 45 \,-\, 2 \cdot 38 \,<\, 0$.

**iii)** $a \,=\, 7^2 \cdot 17^2 \cdot 2 \,=\, 28322$.
$\boldsymbol{\alpha}$) $A \,=\, 7 \cdot 17 \,=\, 119 \,=$A058529(16), $\tilde{a} \,=\, 2$. There is no final solution because $(\tilde{t},\, \tilde{k}) \,=\, (0,\, 1)$, hence $t \,=\, 0$. See *Lemma 17*.

$\boldsymbol{\beta}$) $A \,=\, 7$, $\tilde{a} \,=\, 17^2 \cdot 2 \,=\, 578 \,=$A192453(39). $(\tilde{t},\, \tilde{k}) \,=\, (24,\, 1)$, $(t,\, k) \,=\, (168,\, 7)$, with $t \,-\, 2\,k \,=\, 154$. $(X,\, \widehat{Y})_1 \,=\, (44,\, 123)$ with $(X,\, Y)_1 \,=\, (44,\, 130)$. $c_{3,1} \,=\, (123 + 161)/2 \,=\, 142$, $2\,q_1 \,=\, 130 + 168 \,=\, 298$. $c_{1,1} \,=\, (130 \,-\, 44)/2 \,=\, 43$, $c_{2,1} \,=\, 87$, Hence $(c_1,\, c_2,\, c_3)_1 \,=\, (43,\, 87,\, 142)$ is a $DS$-triple for $a \,=\, 28322$ with $(c_{4.-},\, c_{4.+})_1 \,=\, (-26,\, 570)$.
$(X,\, \widehat{Y})_2 \,=\, (96,\, 137)$ with $(X,\, Y)_2 \,=\, (96,\, 144)$. $c_{3,2} \,=\, (137 + 161)/2 \,=\, 149$, $2\,q_2 \,=\, 144 + 168 \,=\, 312$. $c_{1,2} \,=\, (144 \,-\, 96)/2 \,=\, 24$, $c_{2,2} \,=\, 120$, Hence $(c_1,\, c_2,\, c_3)_2 \,=\, (24,\, 120,\, 149)$ is a second $DS$-triple for $a \,=\, 28322$ with $(c_{4.-},\, c_{4.+})_2 \,=\, (-19,\, 605)$.
There is no other splitting $A \,=\, 17$ and $\tilde{a} \,=\, 7^2 \cdot 2$ because $7 \,(\mathrm{mod}\, 8) \,\neq\, 1$.

The critical first proper positive family $II$ members for the two solutions $(X_i, \widehat{Y}_i)$, corresponding to the two family $I$ solutions, are $(624, 457)$ and $(220, 219)$. Because $k = 7$ only the second solution looks promising because $0 < 220 < 226$, but $X_i = 220 \geq 154$, failing the $c_2 < c_3$ test. Also the first. already discarded, solution fails this test. Therefore the conjecture is true for these two examples.

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Table 1: Primitive Cartesian triples of type $[\mathbf{c}, \mathbf{c}, \mathbf{d}]$ and $[\mathbf{c}, \mathbf{d}, \mathbf{d}]$

| [n, m] | [c, c, d] | q | $[\mathbf{c_{4,-}}, \mathbf{c_{4,+}}]$ | [n, m] | [c, d, d] | q | $[\mathbf{c_{4,-}}, \mathbf{c_{4,+}}]$ |
|---|---|---|---|---|---|---|---|
| [2, 1] | [1, 1, 4] | 3 | [0, 12] | [3, 2] | [5, 8, 8] | 12 | [−3, 45] |
| [2, 1] | [2, 2, 3] | 4 | [−1, 15] | [4, 1] | [8, 9, 9] | 15 | [−4, 56] |
| [3, 2] | [1, 1, 12] | 5 | [4, 24] | [4, 3] | [7, 18, 18] | 24 | [−5, 91] |
| [4, 1] | [2, 2, 15] | 8 | [3, 35] | [5, 4] | [9, 32, 32] | 40 | [−7, 153] |
| [4, 3] | [1, 1, 24] | 7 | [12, 40] | [6, 1] | [12, 25, 25] | 35 | [−8, 132] |
| [5, 2] | [9, 9, 20] | 21 | [−4, 80] | [6, 5] | [11, 50, 50] | 60 | [−9, 231] |
| [5, 2] | [8, 8, 21] | 20 | [−3, 77] | [7, 6] | [13, 72, 72] | 84 | [−11, 325] |
| [5, 4] | [1, 1, 40] | 9 | [24, 60] | [8, 1] | [16, 49, 49] | 63 | [−12, 240] |
| [6, 1] | [2, 2, 35] | 12 | [15, 63] | [8, 5] | [39, 50, 50] | 80 | [−21, 299] |
| [6, 5] | [1, 1, 60] | 11 | [40, 84] | [8, 7] | [15, 98, 98] | 112 | [−13, 435] |
| [7, 2] | [25, 25, 28] | 45 | [−12, 168] | [9, 2] | [36, 49, 49] | 77 | [−20, 288] |
| [7, 2] | [8, 8, 45] | 28 | [5, 117] | [9, 8] | [17, 128, 128] | 144 | [−15, 561] |
| [7, 4] | [9, 9, 56] | 33 | [8, 140] | [10, 1] | [20, 81, 81] | 99 | [−16, 380] |
| [7, 4] | [32, 32, 33] | 56 | [−15, 209] | [10, 7] | [51, 98, 98] | 140 | [−33, 527] |
| [7, 6] | [1, 1, 84] | 13 | [60, 112] | [10, 9] | [19, 162, 162] | 180 | [−17, 703] |
| [8, 1] | [2, 2, 63] | 16 | [35, 99] | [11, 2] | [44, 81, 81] | 117 | [−28, 440] |
| [8, 3] | [25, 25, 48] | 55 | [−12, 208] | [11, 8] | [57, 128, 128] | 176 | [−39, 665] |
| [8, 3] | [18, 18, 55] | 48 | [−5, 187] | [11, 10] | [21, 200, 200] | 220 | [−19, 861] |
| [8, 5] | [9, 9, 80] | 39 | [20, 176] | [12, 1] | [24, 121, 121] | 143 | [−20, 552] |
| [8, 7] | [1, 1, 112] | 15 | [84, 144] | [12, 7] | [95, 98, 98] | 168 | [−45, 627] |
| [9, 2] | [8, 8, 77] | 36 | [21, 165] | [12, 11] | [23, 242, 242] | 264 | [−21, 1035] |
| [9, 4] | [25, 25, 72] | 65 | [−8, 252] | [13, 2] | [52, 121, 121] | 165 | [−36, 624] |
| [9, 4] | [32, 32, 65] | 72 | [−15, 273] | [13, 8] | [105, 128, 128] | 208 | [−55, 777] |
| [9, 8] | [1, 1, 144] | 17 | [112, 180] | [13, 10] | [69, 200, 200] | 260 | [−51, 989] |
| [10, 1] | [2, 2, 99] | 20 | [63, 143] | [13, 12] | [25, 288, 288] | 312 | [−23, 1225] |
| [10, 3] | [49, 49, 60] | 91 | [−24, 340] | [14, 1] | [28, 169, 169] | 195 | [−24, 756] |
| [10, 3] | [18, 18, 91] | 60 | [7, 247] | [14, 3] | [84, 121, 121] | 187 | [−48, 700] |
| [10, 7] | [9, 9, 140] | 51 | [56, 260] | [14, 9] | [115, 162, 162] | 252 | [−65, 943] |
| [10, 9] | [1, 1, 180] | 19 | [144, 220] | [14, 11] | [75, 242, 242] | 308 | [−57, 1175] |
| [11, 2] | [8, 8, 117] | 44 | [45, 221] | [14, 13] | [27, 338, 338] | 364 | [−25, 1431] |
| [11, 4] | [49, 49, 88] | 105 | [−24, 396] | [15, 2] | [60, 169, 169] | 221 | [−44, 840] |
| [11, 4] | [32, 32, 105] | 88 | [−7, 345] | [15, 4] | [120, 121, 121] | 209 | [−56, 780] |
| [11, 6] | [25, 25, 132] | 85 | [12, 352] | [15, 14] | [29, 392, 392] | 420 | [−27, 1653] |
| [11, 6] | [72, 72, 85] | 132 | [−35, 493] | [16, 1] | [32, 225, 225] | 255 | [−28, 992] |
| [11, 8] | [9, 9, 176] | 57 | [80, 308] | [16, 3] | [96, 169, 169] | 247 | [−60, 928] |
| [11, 10] | [1, 1, 220] | 21 | [180, 264] | [16, 11] | [135, 242, 242] | 352 | [−85, 1323] |
| [12, 1] | [2, 2, 143] | 24 | [99, 195] | [16, 13] | [87, 338, 338] | 416 | [−69, 1595] |
| [12, 5] | [49, 49, 120] | 119 | [−20, 456] | [16, 15] | [31, 450, 450] | 480 | [−29, 1891] |
| [12, 5] | [50, 50, 119] | 120 | [−21, 459] | [17, 2] | [68, 225, 225] | 285 | [−52, 1088] |
| [12, 7] | [25, 25, 168] | 95 | [28, 408] | [17, 4] | [136, 169, 169] | 273 | [−72, 1020] |
| [12, 11] | [1, 1, 264] | 23 | [220, 312] | [17, 10] | [189, 200, 200]] | 340 | [−91, 1269] |
| [13, 2] | [8, 8, 165] | 52 | [77, 285] | [17, 12] | [145, 288, 288] | 408 | [−95, 1537] |
| [13, 4] | [81, 81, 104] | 153 | [−40, 572] | [[17, 14] | [93, 392, 392] | 476 | [−75, 1829] |
| [13, 4] | [32, 32, 153] | 104 | [9, 425] | [17, 16] | [33, 512, 512] | 544 | [−31, 2145] |
| cont′d | ... | ... | ... | ... | ... | ... | ... |

**Table 1 continued: Primitive Cartesian triples of type $[\mathbf{c}, \mathbf{c}, \mathbf{d}]$, $n \leq 17$**

| $[\mathbf{n}, \mathbf{m}]$ | $[\mathbf{c},\ \mathbf{c},\ \mathbf{d}]$ | $\mathbf{q}$ | $[\mathbf{c_{4,-}},\ \mathbf{c_{4,+}}]$ | $[\mathbf{n}, \mathbf{m}]$ | $[\mathbf{c},\ \mathbf{d},\ \mathbf{d}]$ | $\mathbf{q}$ | $[\mathbf{c_{4,-}},\ \mathbf{c_{4,+}}]$ |
|---|---|---|---|---|---|---|---|
| $[\mathbf{13}, \mathbf{6}]$ | $[49, 49, 156]$ | 133 | $[-12, 520]$ | ... | ... | ... | ... |
| $[\mathbf{13}, \mathbf{6}]$ | $[72, 72, 133]$ | 156 | $[-35, 589]$ | | | | |
| $[\mathbf{13}, \mathbf{8}]$ | $[25, 25, 208]$ | 105 | $[48, 468]$ | | | | |
| $[\mathbf{13}, \mathbf{10}]$ | $[9, 9, 260]$ | 69 | $[140, 416]$ | | | | |
| $[\mathbf{13}, \mathbf{12}]$ | $[1, 1, 312]$ | 25 | $[264, 364]$ | | | | |
| $[\mathbf{14}, \mathbf{1}]$ | $[2, 2, 195]$ | 28 | $[143, 255]$ | | | | |
| $[\mathbf{14}, \mathbf{3}]$ | $[18, 18, 187]$ | 84 | $[55, 391]$ | | | | |
| $[\mathbf{14}, \mathbf{5}]$ | $[81, 81, 140]$ | 171 | $[-40, 644]$ | | | | |
| $[\mathbf{14}, \mathbf{5}]$ | $[50, 50, 171]$ | 140 | $[-9, 551]$ | | | | |
| $[\mathbf{14}, \mathbf{9}]$ | $[25, 25, 252]$ | 115 | $[72, 532]$ | | | | |
| $[\mathbf{14}, \mathbf{11}]$ | $[9, 9, 308]$ | 75 | $[176, 476]$ | | | | |
| $[\mathbf{14}, \mathbf{13}]$ | $[1, 1, 364]$ | 27 | $[312, 420]$ | | | | |
| $[\mathbf{15}, \mathbf{2}]$ | $[8, 8, 221]$ | 60 | $[117, 357]$ | | | | |
| $[\mathbf{15}, \mathbf{4}]$ | $[32, 32, 209]$ | $120, 6$ | $[33, 513]$ | | | | |
| $[\mathbf{15}, \mathbf{8}]$ | $[49, 49, 240]$ | 161 | $[16, 660]$ | | | | |
| $[\mathbf{15}, \mathbf{8}]$ | $[128, 128, 161]$ | 240 | $[-63, 897]$ | | | | |
| $[\mathbf{15}, \mathbf{14}]$ | $[1, 1, 420]$ | 29 | $[364, 480]$ | | | | |
| $[\mathbf{16}, \mathbf{1}]$ | $[2, 2, 255]$ | 32 | $[195, 323]$ | | | | |
| $[\mathbf{16}, \mathbf{3}]$ | $[18, 18, 247]$ | 96 | $[91, 475]$ | | | | |
| $[\mathbf{16}, \mathbf{5}]$ | $[121, 121, 160]$ | 231 | $[-60, 864]$ | | | | |
| $[\mathbf{16}, \mathbf{5}]$ | $[50, 50, 231]$ | 160 | $[11, 651]$ | | | | |
| $[\mathbf{16}, \mathbf{7}]$ | $[81, 81, 224]$ | 207 | $[-28, 800]$ | | | | |
| $[\mathbf{16}, \mathbf{7}]$ | $[98, 98, 207]$ | 224 | $[-45, 851]$ | | | | |
| $[\mathbf{16}, \mathbf{9}]$ | $[49, 49, 288]$ | 175 | $[36, 736]$ | | | | |
| $[\mathbf{16}, \mathbf{9}]$ | $[162, 162, 175]$ | 288 | $[-77, 1075]$ | | | | |
| $[\mathbf{16}, \mathbf{11}]$ | $[25, 25, 352]$ | 135 | $[132, 672]$ | | | | |
| $[\mathbf{16}, \mathbf{13}]$ | $[9, 9, 416]$ | 87 | $[260, 608]$ | | | | |
| $[\mathbf{16}, \mathbf{15}]$ | $[1, 1, 480]$ | 31 | $[420, 544]$ | | | | |
| $[\mathbf{17}, \mathbf{2}]$ | $[8, 8, 285]$ | 68 | $[165, 437]$ | | | | |
| $[\mathbf{17}, \mathbf{4}]$ | $[32, 32, 273]$ | 136 | $[65, 609]$ | | | | |
| $[\mathbf{17}, \mathbf{6}]$ | $[121, 121, 204]$ | 253 | $[-60, 952]$ | | | | |
| $[\mathbf{17}, \mathbf{6}]$ | $[[72, 72, 253]$ | 204 | $[-11, 805]$ | | | | |
| $[\mathbf{17}, \mathbf{8}]$ | $[81, 81, 272]$ | $225,$ | $[-16, 884]$ | | | | |
| $[\mathbf{17}, \mathbf{8}]$ | $[128, 128, 225]$ | 272 | $[-63, 1025]$ | | | | |
| $[\mathbf{17}, \mathbf{10}]$ | $[49, 49, 340]$ | $189,$ | $[60, 816]$ | | | | |
| $[\mathbf{17}, \mathbf{12}]$ | $[25, 25, 408]$ | $145,$ | $[168, 748]$ | | | | |
| $[\mathbf{17}, \mathbf{14}]$ | $[9, 9, 476]$ | 93 | $[308, 680]$ | | | | |
| $[\mathbf{17}, \mathbf{16}]$ | $[1, 1, 544]$ | 33 | $[480, 612]$ | | | | |
| ... | ... | ... | ... | | | | |

Table 2: **Distinct primitive Cartesian triples with curvatures $c_3 = 1..38$**

| $c_3$ | $[c_1, c_2, c_3]$ | q | $[c_{4,-}, c_{4,+}]$ | $c_3$ | $[c_1, c_2, c_3]$ | q | $[c_{4,-}, c_{4,+}]$ |
|---|---|---|---|---|---|---|---|
| **6** | [2, 3, 6] | 6 | [−1, 23] | | [4, 9, 28] | 20 | [1, 81] |
| **7** | [3, 6, 7] | 9 | [−2, 34] | | [4, 21, 28] | 28 | [−3, 109] |
| **8** | ∅ | ∅ | ∅ | | [13, 21, 28] | 35 | [−8, 132] |
| **9** | [1, 4, 9] | 7 | [0, 28] | | [21, 24, 28] | 42 | [−11, 157] |
| **10** | [3, 7, 10] | 11 | [−2, 42] | **29** | [5, 24, 29] | 31 | [−4, 120] |
| **11** | [2, 6, 11] | 10 | [−1, 39] | | [8, 12, 29] | 26 | [−3, 101] |
| **12** | [1, 4, 12] | 8 | [1, 33] | **30** | [6, 11, 30] | 24 | [−1, 95] |
| | [5, 8, 12] | 14 | [−3, 53] | | [7, 22, 30] | 32 | [−5, 123] |
| **13** | [4, 12, 13] | 16 | [−3, 61] | **31** | [3, 22, 31] | 29 | [−2, 114] |
| **14** | [3, 6, 14] | 12 | [−1, 47] | | [6, 30, 31] | 36 | [−5, 139] |
| **15** | [2, 3, 15] | 9 | [2, 38] | | [10, 19, 31] | 33 | [−6, 126] |
| | [3, 10, 15] | 15 | [−2, 58] | **32** | [5, 20, 32] | 30 | [−3, 117] |
| | [11, 14, 15] | 23 | [−6, 86] | | [21, 29, 32] | 47 | [−12, 176] |
| **16** | [1, 9, 16] | 13 | [0, 52] | **33** | [1, 4, 33] | 13 | [12, 64] |
| | [4, 13, 16] | 18 | [−3, 69] | | [1, 12, 33] | 21 | [4, 88] |
| **17** | [8, 9, 17] | 19 | [−4, 72] | | [4, 12, 33] | 24 | [1, 97] |
| **18** | [2, 11, 18] | 16 | [−1, 63] | | [8, 17, 33] | 31 | [−4, 120] |
| **19** | [6, 7, 19] | 17 | [−2, 66] | | [12, 20, 33] | 36 | [−7, 137] |
| | [10, 15, 19] | 25 | [−6, 94] | **34** | [3, 6, 34] | 18 | [7, 79] |
| **20** | [5, 12, 20] | 20 | [−3, 77] | | [3, 7, 34] | 19 | [6, 82] |
| | [12, 17, 20] | 28 | [−7, 105] | | [6, 7, 34] | 22 | [3, 91] |
| **21** | [4, 16, 21] | 22 | [−3, 85] | | [6, 31, 34] | 38 | [−5, 147] |
| | [5, 20, 21] | 25 | [−4, 96] | | [18, 19, 34] | 40 | [−9, 151] |
| **22** | [3, 15, 22] | 21 | [−2, 82] | | [27, 31, 34] | 53 | [−14, 198] |
| | [7, 18, 22] | 26 | [−5, 99] | **35** | [2, 15, 35] | 25 | [2, 102] |
| | [18, 19, 22] | 34 | [−9, 127] | | [6, 14, 35] | 28 | [−1, 111] |
| **23** | [2, 3, 23] | 11 | [6, 50] | | [14, 15, 35] | 35 | [−6, 134] |
| | [2, 6, 23] | 14 | [3, 59] | | [14, 26, 35] | 42 | [−9, 159] |
| | [3, 6, 23] | 15 | [2, 62] | | [18, 23, 35] | 40 | [−10, 162] |
| | [11, 14, 23] | 27 | [−6, 102] | **36** | [1, 25, 36] | 31 | [0, 124] |
| **24** | [1, 12, 24] | 18 | [1, 73] | | [5, 29, 36] | 37 | [−4, 144] |
| | [5, 21, 24] | 27 | [−4, 104] | | [9, 17, 36] | 33 | [−4, 12] |
| | [12, 17, 24] | 30 | [−7, 113] | | [9, 32, 36] | 42 | [−7, 161] |
| | [13, 21, 24] | 33 | [−8, 124] | **37** | [4, 28, 37] | 36 | [−3, 141] |
| **25** | [1, 16, 25] | 21 | [0, 84] | | [13, 24, 37] | 41 | [−8, 156] |
| | [4, 9, 25] | 19 | [0, 76] | | [16, 36, 37] | 50 | [−11, 189] |
| **26** | [3, 14, 26] | 22 | [−1, 87] | | [21, 28, 37] | 49 | [−12, 184] |
| | [11, 15, 26] | 29 | [−6, 110] | **38** | [2, 3, 38] | 14 | [15, 71] |
| **27** | [2, 18, 27] | 24 | [−1, 95] | | [2, 15, 38] | 26 | [3, 107] |
| | [7, 10, 27] | 23 | [−2, 90] | | [2, 27, 38] | 34 | [−1, 135] |
| | [14, 26, 27] | 38 | [−9, 143] | | [3, 15, 38] | 27 | [2, 110] |
| | [18, 23, 27] | 39 | [−10, 146] | | [14, 27, 38] | 44 | [−9, 167] |
| **28** | [1, 4, 28] | 12 | [9, 57] | | [23, 30, 38] | 52 | [−13, 195] |
| | [1, 9, 28] | 17 | [4, 72] | ... | ... | ... | ... |

**Table 3: Case i) $q = c_3$. Proper solutions $[x, y] = [c_1, c_2]$ of eq. (35), and $[X, Y] = [y - x, y + x]$ of eq. (38), q, and rpapfs $t-$tuples, for $s \in [1, 313]$**

| s | [x, y] | [X, Y] | $q = c_3$ | $t-$tuples |
|---|---|---|---|---|
| **7** | [2, 3] | [1, 5] | 6 | $(-1,4,1),\ [(-5,1)]$ |
| **17** | [3, 10] | [7, 13] | 15 | $(-1,6,2,1),\ [(-6,2,2)]$ |
| **23** | [5, 12] | [7, 17] | 20 | $(-1,3,2,3,1),\ [(-3,3,2,2)]$ |
| **31** | [4, 21] | [17, 25] | 28 | $(-1,8,2,2,1),\ [(-8,3,2)]$ |
| **41** | [14, 15] | [1, 29] | 35 | $(-1,28,1),\ [(-29,1)]$ |
| **47** | [7, 30] | [23, 37] | 42 | $(-4,2,3,2,1),\ [(-1,4,3,3,2)]$ |
| <u>**49**</u> | [5, 36] | [31, 41] | 45 | $(-1,10,2_3,1),\ [(-10,4,2)]$ |
| **71** | [6, 55] | [49, 61] | 66 | $(-1,12,2_4,1),\ [(-12,5,2)]$ |
| **73** | [18, 35] | [17, 53] | 63 | $(-1,2_8,4,1),\ [(-2,9,2_3)]$ |
| **79** | [9, 56] | [47, 65] | 72 | $(-1,3_3,2,2,1),\ [(-3,2,3,4,2)]$ |
| **89** | [21, 44] | [23, 65] | 77 | $(-1,3,6,3,1),\ [(-3,2_4,3,2,2)]$ |
| **97** | [7, 78] | [71, 85] | 91 | $(-1,14,2_5,1),\ [(-14,6,2)]$ |
| **103** | [33, 40] | [7, 73] | 88 | $[(-1,3,2,11,1),\ [(-3,3,2_{10})]$ |
| **113** | [22, 63] | [41, 85] | 99 | $(-2,2_{13},3,1),\ [(-1,2,15,2,2)]$ |
| <u>**119**</u> | [24, 65] | [41, 89] | 104 | $(-6,2_5,3,1),\ [(-1,6,7,2,2)]$ |
| | [11, 90] | [79, 101] | 110 | $(-4,3,3,2_3,1),\ [(-1,4,2,3,5,2)]$ |
| **127** | [8, 105] | [97, 113] | 120 | $(-1,16,2_6,1),\ [(-16,7,2)]$ |
| **137** | [45, 52] | [7, 97] | 117 | $(1,6,14,1),\ [(-6,2_{14})]$ |
| **151** | [39, 70] | [31, 109] | 130 | $(-15,2,4,1),\ [(-1,15,3,2_3)]$ |
| <u>**161**</u> | [26, 99] | [73, 125] | 143 | $(-1,10,2,4,2,1),\ [(-10,3,2,3,2)]$ |
| | [9, 136] | [127, 145] | 153 | $(-1,18,2_7,1),\ [(-18,8,2)]$ |
| **167** | [13, 132] | [119, 145] | 156 | $(-1,3,4,3,2_4,1),\ [(-3,2,2,3,6,2)]$ |
| **191** | [30, 119] | [89, 149] | 170 | $(-1,2_{14},4,2,1),\ [(-2,15,2,3,2)]$ |
| **193** | [60, 77] | [17, 137] | 165 | $(-17,8,1),\ [(-1,17,2_8)]$ |
| **199** | [10, 171] | [161, 181] | 190 | $(-1,20,2_8,1),\ [(-20,9,2)]$ |
| <u>**217**</u> | [55, 102] | [47, 157] | 187 | $(-15,2,2,4,1),\ [(-1,15,4,2_3)]$ |
| | [30, 143] | [113, 173] | 195 | $(-1,3,3,2_6,3,2,1),\ [(-3,2,9,3,2)]$ |
| **223** | [15, 182] | [167, 197] | 210 | $(-4,4,3,2_5,1),\ [(-1,4,2,2,3,7,2)]$ |
| **233** | [33, 152] | [119, 185] | 209 | $(-1,13,2_3,3,2,1),\ [(-13,5,3,2)]$ |
| **239** | [84, 85] | [1, 169] | 204 | $(-1,168,1),\ [(-169,1)]$ |
| **241** | [11, 210] | [199, 221] | 231 | $(-1,22,2_9,1),\ [(-22,10,2)]$ |
| **257** | [68, 117] | [49, 185] | 221 | $(-1,5,2,5,4,1),\ [(-5,3,2,2,3,2_3)]$ |
| **263** | [60, 133] | [73, 193] | 228 | $(-5,5,3,3,1),\ [(-1,5,2_3,3,3,2,2)]$ |
| **271** | [51, 154] | [103, 205] | 238 | $(-1,102,2,1),\ [(-102,2,2)]$ |
| **281** | [34, 195] | [161, 229] | 255 | $(-3,2,4,4,2,2,1),\ [(-1,3,3,2,3,2,4,2)]$ |
| <u>**287**</u> | [17, 240] | [223, 257] | 272 | $(-1,3,5,3,2_6,1),\ [(-3,2,2,2,3,8,2)]$ |
| | [12, 253] | [241, 265] | 276 | $(-1,24,2_{10},1),\ [(-24,11,2)]$ |
| <u>**289**</u> | [91, 114] | [23, 205] | 247 | $(-2,11,9,1),\ [(-1,2_{10},3,2_8)]$ |
| **311** | [95, 126] | [31, 221] | 266 | $(-3,2_7,8,1),\ [(-1,3,9,2_7)]$ |
| **313** | [65, 168] | [103, 223] | 273 | $(-2,3,2_3,3,2,2,3,1),\ [(-1,2,2,6,5,2,2)]$ |
| ... | ... | ... | ... | ... |

## Table 4: Case ii) $q < c_3$.

| $c_3$ | $[X, \widehat{Y}]$ | $[c_1, c_2]$ | $q$ | $k$ | $t$ | $a$ | $[c_{4,-}, c_{4,+}]$ |
|---|---|---|---|---|---|---|---|
| **9** | [3, 7] | [1, 4] | 7 | 2 | 9 | 89 | [0, 28] |
| **11** | [4, 9] | [2, 6] | 10 | 1 | 12 | $146 = 2 \cdot 73$ | $[-1, 39]$ |
| **12** | [3, 9] | [1, 4] | 8 | 4 | 11 | $153 = 3^2 \cdot 17$ | [1, 33] |
| **14** | [3, 11] | [3, 6] | 12 | 2 | 15 | 233 | $[-1, 47]$ |
| **15** | [1, 11] | [2, 3] | 9 | 6 | 13 | 241 | [2, 38] |
| **16** | [8, 13] | [1, 9] | 13 | 3 | 16 | $274 = 2 \cdot 137$ | [0, 52] |
| **18** | [9, 15] | [2, 11] | 16 | 2 | 19 | $369 = 3^2 \cdot 41$ | $[-1, 63]$ |
| **19** | [1, 15] | [6, 7] | 17 | 2 | 21 | 449 | $[-2, 66]$ |
| **22** | [12, 19] | [3, 15] | 21 | 1 | 24 | $578 = 2 \cdot 17^2$ | $[-2, 82]$ |
| **23** | [1, 17] | [2, 3] | 11 | 12 | 17 | 577 | [6, 50] |
| | [4, 17] | [2, 6] | 14 | 9 | 20 | $562 = 2 \cdot 281$ | [3, 59] |
| | [3, 17] | [3, 6] | 15 | 8 | 21 | 569 | [2, 62] |
| **24** | [11, 19] | [1, 12] | 18 | 6 | 23 | 601 | [1, 73] |
| **25** | [15, 21] | [1, 16] | 21 | 4 | 25 | $657 = 3^2 \cdot 73$ | [0, 84] |
| | [5, 19] | [4, 9] | 19 | 6 | 25 | $697 = 17 \cdot 41$ | [0, 76] |
| **26** | [11, 21] | [3, 14] | 22 | 4 | 27 | 761 | $[-1, 87]$ |
| **27** | [16, 23] | [2, 18] | 24 | 3 | 28 | $802 = 2 \cdot 401$ | $[-1, 95]$ |
| | [3, 21] | [7, 10] | 23 | 4 | 29 | $873 = 3^2 \cdot 97$ | $[-2, 90]$ |
| **28** | [3, 21] | [1, 4] | 12 | 16 | 19 | $873 = 3^2 \cdot 97$ | [9, 57] |
| | [8, 21] | [1, 9] | 17 | 11 | 24 | $818 = 2 \cdot 409$ | [4, 72] |
| | [5, 21] | [4, 9] | 20 | 8 | 27 | 857 | [1, 81] |
| **29** | [4, 23] | [8, 12] | 26 | 3 | 32 | $1042 = 2 \cdot 521$ | $[-3, 101]$ |
| **30** | [5, 23] | [6, 11] | 24 | 6 | 31 | 1033 | $[-1, 95]$ |
| **31** | [19, 27] | [3, 22] | 29 | 2 | 33 | 1097 | $[-2, 114]$ |
| **32** | [15, 27] | [5, 20] | 30 | 2 | 35 | $1233 = 3 \cdot 137$ | $[-3, 117]$ |
| **33** | [3, 25] | [1, 4] | 13 | 20 | 21 | $1241 = 17 \cdot 73$ | [12, 64] |
| | [11, 25] | [1, 12] | 21 | 12 | 29 | 1129 | [4, 88] |
| | [8, 25] | [4, 12] | 24 | 9 | 32 | $1186 = 2 \cdot 593$ | [1, 97] |
| | [9, 27] | [8, 17] | 31 | 2 | 37 | $1377 = 3^4 \cdot 17$ | $[-4, 120]$ |
| **34** | [3, 25] | [3, 6] | 18 | 16 | 27 | $1241 = 17 \cdot 73$ | [7, 79] |
| | [4, 25] | [3, 7] | 19 | 15 | 28 | $1234 = 2 \cdot 617$ | [6, 82] |
| | [1, 25] | [6, 7] | 22 | 12 | 31 | 1249 | [3, 91] |
| **35** | [13, 27] | [2, 15] | 25 | 10 | 33 | 1289 | [2, 102] |
| | [8, 27] | [6, 14] | 28 | 7 | 36 | $1394 = 2 \cdot 17 \cdot 41$ | $[-1, 111]$ |
| **36** | [24, 31] | [1, 25] | 31 | 5 | 36 | $1346 = 2 \cdot 673$ | [0, 124] |
| | [8, 29] | [9, 17] | 33 | 3 | 40 | $1618 = 2 \cdot 809$ | $[-4, 128]$ |
| **37** | [24, 33] | [4, 28] | 36 | 1 | 40 | $1602 = 2 \cdot 3^2 \cdot 89$ | $[-3, 141]$ |
| ... | ... | ... | ... | ... | ... | ... | ... |

**Legend for Table 4:** $\mathbf{q = \sqrt{c_1\,c_2 + (c_1 + c_2)\,c_3}}$**, and case** $\mathbf{q = c_3 - k,\ k \geq 1}$**.**
**Solutions** $[X, \widehat{Y}]$ **of** $\mathbf{X^2 - 2\,\widehat{Y}^2 = -(t^2 + 2\,k^2) =: -a}$**, with** $\mathbf{0 < X < \widehat{Y} - k}$**.**

$$[c_1, c_2] = \left[\frac{1}{2}(\widehat{Y} - X - k),\ \frac{1}{2}(\widehat{Y} + X - k)\right],\ c_1^2 + c_2^2 + 6\,c_1\,c_2 + 4\,k\,(c_1 + c_2) = t^2.$$

$$c_3 = (\widehat{Y} + k + t)/2,\ q = (\widehat{Y} - k + t)/2.$$

**Table 5 Case ii) $q > c_3$.**

| $c_3$ | $[X, \widehat{Y}]$ | $[c_1, c_2]$ | q | k | t | a | $[c_{4,-}, c_{4,+}]$ |
|---|---|---|---|---|---|---|---|
| **7** | [3, 7] | [3, 6] | 9 | 2 | 9 | 89 | [−2, 34] |
| **10** | [4, 9] | [3, 7] | 11 | 1 | 12 | $146 = 2 \cdot 73$ | [−2, 42] |
| **12** | [3, 11] | [5, 8] | 14 | 2 | 15 | 233 | [−3, 53] |
| **13** | [8, 13] | [4, 12] | 16 | 3 | 16 | $274 = 2 \cdot 137$ | [−3, 61] |
| **15** | [3, 17] | [11, 14] | 23 | 8 | 21 | 569 | [−6, 86] |
| **16** | [9, 15] | [4, 13] | 18 | 2 | 19 | $369 = 3^2 \cdot 41$ | [−3, 69] |
| **17** | [1, 15] | [8, 9] | 19 | 2 | 21 | 449 | [−4, 72] |
| **19** | [5, 19] | [10, 15] | 25 | 6 | 25 | $697 = 17 \cdot 41$ | [−6, 94] |
| **20** | [5, 21] | [12, 17] | 28 | 8 | 27 | 857 | [−7, 105] |
| **21** | [12, 19] | [4, 16] | 22 | 1 | 24 | $578 = 2 \cdot 17^2$ | [−3, 85] |
| | [15, 21] | [5, 20] | 25 | 4 | 25 | $657 = 3^2 \cdot 73$ | [−4, 96] |
| **22** | [11, 21] | [7, 18] | 26 | 4 | 27 | 761 | [−5, 99] |
| | [1, 25] | [18, 19] | 34 | 12 | 31 | 1249 | [−9, 127] |
| **23** | [3, 21] | [11, 14] | 27 | 4 | 29 | $873 = 3^2 \cdot 97$ | [−6, 102] |
| **24** | [16, 23] | [5, 21] | 27 | 3 | 28 | $802 = 2 \cdot 401$ | [−4, 104] |
| | [5, 23] | [12, 17] | 30 | 6 | 31 | 1033 | [−7, 113] |
| | [8, 25] | [13, 21] | 33 | 9 | 32 | $1186 = 2 \cdot 593$ | [−8, 124] |
| **26** | [4, 23] | [11, 15] | 29 | 3 | 32 | $1042 = 2 \cdot 521$ | [−6, 110] |
| **27** | [12, 29] | [14, 26] | 38 | 11 | 36 | $1538 = 2 \cdot 769$ | [−9, 143] |
| | [5, 29] | [18, 23] | 39 | 12 | 37 | 1657 | [−10, 146] |
| **28** | [8, 27] | [13, 21] | 35 | 7 | 36 | $1394 = 2 \cdot 17 \cdot 41$ | [−8, 132] |
| | [3, 31] | [21, 24] | 42 | 14 | 39 | 1913 | [−11, 157] |
| **29** | [19, 27] | [5, 24] | 31 | 2 | 33 | 1097 | [−4, 120] |
| **30** | [15, 27] | [7, 22] | 32 | 2 | 35 | $1233 = 3^2 \cdot 137$ | [−5, 123] |
| **31** | [24, 31] | [6, 30] | 36 | 5 | 36 | $1346 = 2 \cdot 673$ | [−5, 139] |
| | [9, 27] | [10, 19] | 33 | 2 | 37 | $1377 = 3^4 \cdot 17$ | [−6, 126] |
| **32** | [8, 35] | [21, 29] | 47 | 15 | 44 | $2386 = 2 \cdot 1193$ | [−12, 176] |
| **33** | [8, 29] | [12, 20] | 36 | 3 | 40 | $1618 = 2 \cdot 809$ | [−2, 137] |
| | [13, 31] | [12, 25] | 39 | 6 | 41 | 1753 | [−8, 148] |
| **34** | [25, 33] | [6, 31] | 38 | 4 | 39 | 1533 | [−5, 147] |
| | [1, 31] | [18, 19] | 40 | 6 | 43 | $1921 = 17 \cdot 113$ | [−9, 151] |
| | [4, 39] | [27, 31] | 53 | 19 | 48 | $3026 = 2 \cdot 17 \cdot 89$ | [−14, 198] |
| **35** | [12, 33] | [14, 26] | 42 | 7 | 44 | $2034 = 2 \cdot 3^2 \cdot 113$ | [−9, 159] |
| | [5, 33] | [18, 23] | 43 | 8 | 45 | 2153 | [−10, 162] |
| ... | ... | ... | ... | ... | ... | ... | ... |

**Legend for Table 5:** $q = \sqrt{c_1\, c_2 + (c_1 + c_2)\, c_3}$, **and case** $q = c_3 + k,\ k \geq 1$.
**Solutions** $[X, \widehat{Y}]$ **of** $X^2 - 2\,\widehat{Y}^2 = -(t^2 + 2\,k^2) =: -a$, **with** $0 < X < \widehat{Y} + k$.

$[c_1, c_2] = \left[\frac{1}{2}(\widehat{Y} - X + k),\, \frac{1}{2}(\widehat{Y} + X + k)\right]$, $c_1^2 + c_2^2 + 6\,c_1\,c_2 - 4\,k\,(c_1 + c_2) = t^2$.

$c_3 = (\widehat{Y} - k + t)/2,\ q = (\widehat{Y} + k + t)/2$.

## Table 6: Primitive Cartesian triples with $c_{4,-} = 0$ and $c_3 = n^2, n = 2..24$

| n | [c1, c2, c3] | q | [c4,−, c4,+] | n | [c1, c2, c3] | q | [c4,−, c4,+] |
|---|---|---|---|---|---|---|---|
| **2** | [1, 1, 4] | 3 | [0, 12] | **17cont′d** | [9, 196, 289] | 247 | [0, 988] |
| **3** | [1, 4, 9] | 7 | [0, 28] | | [4, 225, 289] | 259 | [0, 1036] |
| **4** | [1, 9, 16] | 13 | [0, 52] | | [1, 256, 289] | 273 | [0, 1092] |
| **5** | [4, 9, 25] | 19 | [0, 76] | **18** | [49, 121, 324] | 247 | [0, 988] |
| | [1, 16, 25] | 21 | [0, 84] | | [25, 169, 324] | 259 | [0, 1036] |
| **6** | [1, 25, 36] | 31 | [0, 124] | | [1, 289, 324] | 307 | [0, 1228] |
| **7** | [9, 16, 49] | 37 | [0, 148] | **19** | [81, 100, 361] | 271 | [0, 1084 |
| | [4, 25, 49] | 39 | [0, 156] | | [64, 121, 361] | 273 | [0, 1092] |
| | [1, 36, 49] | 43 | [0, 172] | | [49, 144, 361] | 277 | [0, 1108] |
| **8** | [9, 25, 64] | 49 | [0, 196] | | [36, 169, 361] | 283 | [0, 1132] |
| | [1, 49, 64] | 57 | [0, 228] | | [25, 196, 361] | 291 | [0, 1164] |
| **9** | [16, 25, 81] | 61 | [0, 244] | | [16, 225, 361] | 301 | [0, 1204] |
| | [4, 49, 81] | 67 | [0, 268] | | [9, 256, 361] | 313 | [0, 1252] |
| | [1, 64, 81] | 73 | [0, 292] | | [4, 289, 361] | 327 | [0, 1308] |
| **10** | [9, 49, 100] | 79 | [0, 316] | | [1, 324, 361] | 343 | [0, 1372] |
| | [1, 81, 100] | 91 | [0, 364] | **20** | [81, 121, 400] | 301 | [0, 1204] |
| **11** | [25, 36, 121] | 91 | [0, 364] | | [49, 169, 400] | 309 | [0, 1236] |
| | [16, 49, 121] | 93 | [0, 372] | | [9, 289, 400] | 349 | [0, 1396] |
| | [9, 64, 121] | 97 | [0, 388] | | [1, 361, 400] | 381 | [0, 1524] |
| | [4, 81, 121] | 103 | [0, 412] | **21** | [100, 121, 441] | 331 | [0, 1324] |
| | [1, 100, 121] | 111 | [0, 444] | | [64, 169, 441] | 337 | [0, 1348] |
| **12** | [25, 49, 144] | 109 | [0, 436] | | [25, 256, 441] | 361 | [0, 1444] |
| | [1, 121, 144] | 133 | [0, 532] | | [16, 289, 441] | 373 | [0, 1492] |
| **13** | [36, 49, 169] | 127 | [0, 508] | | [4, 361, 441] | 403 | [0, 1612] |
| | [25, 64, 169] | 129 | [0, 516] | | [1, 400, 441] | 421 | [0, 1684] |
| | [16, 81, 169] | 133 | [0, 532] | **22** | [81, 169, 484] | 367 | [0, 1468] |
| | [9, 100, 169] | 139 | [0, 556] | | [49, 225, 484] | 379 | [0, 1516] |
| | [4, 121, 169] | 147 | [0, 588] | | [25, 289, 484] | 399 | [0, 1596] |
| | [1, 144, 169] | 157 | [0, 628] | | [9, 361, 484] | 427 | [0, 1708] |
| **14** | [25, 81, 196] | 151 | [0, 604] | | [1, 441, 484] | 463 | [0, 1852] |
| | [9, 121, 196] | 163 | [0, 652] | **23** | [121, 144, 529] | 397 | [0, 1588] |
| | [1, 169, 196] | 183 | [0, 732] | | [100, 169, 529] | 399 | [0, 1596] |
| **15** | [49, 64, 225] | 169 | [0, 676] | | [81, 196, 529] | 403 | [0, 1612] |
| | [16, 121, 225] | 181 | [0, 724] | | [64, 225, 529] | 409 | [0, 1636] |
| | [4, 169, 225] | 199 | [0, 796] | | [49, 256, 529] | 417 | [0, 1668] |
| | [1, 196, 225] | 211 | [0, 844] | | [36, 289, 529] | 427 | [0, 1708] |
| **16** | [49, 81, 256] | 193 | [0, 772] | | [25, 324, 529] | 439 | [0, 1756] |
| | [25, 121, 256] | 201 | [0, 804] | | [16, 361, 529] | 453 | [0, 1812] |
| | [9, 169, 256] | 217 | [0, 868] | | [9, 400, 529] | 469 | [0, 1876] |
| | [1, 225, 256] | 241 | [0, 964] | | [4, 441, 529] | 487 | [0, 1948] |
| **17** | [64, 81, 289] | 217 | [0, 868] | | [1, 484, 529] | 507 | [0, 2028] |
| | [49, 100, 289] | 219 | [0, 876] | **24** | [121, 169, 576] | 433 | [0, 1732] |
| | [36, 121, 289] | 223 | [0, 892] | | [49, 289, 576] | 457 | [0, 1828] |
| | [25, 144, 289] | 229 | [0, 916] | | [25, 361, 576] | 481 | [0, 1924] |
| | [16, 169, 289] | 237 | [0, 948] | | [1, 529, 576] | 553 | [0, 2212] |