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Four Sequences of Length 28 and the Gregorian Calendar

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Abstract

It is shown that each sequence giving the number of times a given day of the month falls on a certain day of the week for 400 successive years of the Gregorian cycle can be composed of various pieces of various length of one of 4 sequences of length 28, used periodically.

1 Introduction

The formula which determines the day of the week, $D = 0, 1, ..., 6$ for Sunday, Monday, ..., Saturday, for an admissible dth day of the month $d \in \{1, 2, ..., 31\}$ of a month $m \in\in \{1, 2, ..., 12\}$ in a year y of the *Gregorian* calendar [2], [4] (in use since October 15, 1582, a Friday) is well known. *e.g.*, [3], II, pp. 357 - 358.

$$D(d, m, y) \;=\; \Big(d \,+\, \lfloor 2.6\, M(m) \,-\, 0.2 \rfloor \,+\, T(y) \,+\, \left\lfloor \frac{T(y)}{4} \right\rfloor \,+\, \left\lfloor \frac{H(y)}{4} \right\rfloor \\ -\, 2\, H(y) \,-\, (L(y)\,+\,1) \left\lfloor \frac{M(m)}{11} \right\rfloor \Big) \bmod 7\,. \tag{1}$$

Here $\lfloor . \rfloor$ is the floor function, $M(m) = (m + 10) \bmod 12$ if $M(m) \neq 0$ and $M(2) = 12$. $T(y)$ are the last two digits of the year y (the 'tenth') and $H(y)$ are the first two digits of $y \in \{1583, ..., 9999\}$. $L(y) = 1$ if y is a leap year and $L(y) = 0$ otherwise. In the *Gregorian* calendar $L(y) = 0$ if and only if $y \bmod 4 \neq 0$ or $y \bmod 100 = 0$ but $y \bmod 400 \neq 0$. All other years are leap years. *E.g.* $L(1900) = 0$ but $L(2000) = 1$.
For the following application it is necessary to exclude nonsense days of a month which are: $d = 29$ if $m = 2$ and $L(y) = 0$, $d = 30$ if $m = 2$, and $d = 31$ if $m = 2, 4, 6, 9, 11$. Formula 1 satisfies $D(d, m, y) = D(d, m, y + 400)$. Therefore, it suffices to consider $y\, mod\, 400$, *i.e.* $y \in [0, 399]$ (keeping the start of the *Gregorian* calendar in mind). In the formula $y = 0$ (for 0000)uses $H(y) = 0$ and $T(y) = 0$.
Every *Gregorian* cycle of 400 consecutive years has $146\,097 = 7 \cdot 20871$ days. Such a cycle comprises 97 leap years and 303 ordinary years. Any day of the month $d \in \{1, 2, ..., 28\}$ appears $12 \cdot 400 = 4800$ times in such a *Gregorian* cycle. Because $4800 \bmod 7 \neq 0$ such a day of the month cannot be equally distributed over the days of the week. It is well known [1], [5] that *e.g.* $d = 13$ (hence $d = 6$ or 20 or 27) happens to fall more often on a Friday than on any other day of the week. For these days of the month the distribution is $[687, 685, 685, 687, 684, 688, 684]$ for the days of the week ordered from left to right for Sunday to Saturday. The mean value is $4800/7 \simeq 685.71$. Similarly, the first of a month, $d = 1$ (or $d = 8, 15, 22$), falls predominantly on Sunday. The distribution for these days is $[688, 684, 687, 685, 685, 687, 684]$, which is a cyclic permutation of the above given numbers because $d = 6$ on a Sunday corresponds to $d = 1$ on a Tuesday, *etc.*)
$d = 29$ appears $12 \cdot 97 + 11 \cdot 303 = 4497$ times during a 400 year cycle. For $d = 30$ this number is $11 \cdot 400 = 4400$, and for $d = 31$ it is $7 \cdot 400 = 2800$ Even though 2800 is divisible by 7 the distribution over the days of the week for day 31 is also not uniform.
See *Table 1* for the well known (see *e.g.*, [6]) distribution of the days of the month on the days of a week. Each day $d \in \{8, 9, ..., 28\}$ is congruent to one of the representative days of the month 1 to 7. The 7×7 sub-matrix is a symmetric *Toeplitz* matrix with maximum 688 only in the main diagonal. For $d = 29$ the maximum 644 appears for the days of the week $D = 0$ and 2. For $d = 30$ the maximum 631 appears for $D = 1$ and 3, and for $d = 31$ the maximum 402 appears for $D = 4$.

2 Multiplicities for days of the month on days of the week

The focus of this note is on the number of times a given day of the month falls on a day of the week for each of the 400 years of a *Gregorian* cycle. That is we wish to know, for example, how many Friday 13th happen in a given year, not only the total sum of such events over a 400 year cycle. To this end we consider sequences $M(d, D)$ of length 400 for the consecutive years $y = 0, 1, ..., 399$, with entry at position y giving the number of times an admissible day of the month $d \in \{1, 2, .., 31\}$ falls on a day of the week $D \in \{0, 1, ..., 6\}$. (This M should not be confused wit the M of eq. 1. $M(d, D)$ is then repeated periodically for values $y \geq 400$.

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The representatives of the conjugacy classes modulo 7 for the days $d \in \{1, 2, ..., 28\}$ are $d \in \{1, 2, ..., 7\}$. $d = 1$ represents the days from $\{1, 8, 15, 22\}$, *etc.* For every $d \in \{2, ..., 28\}$ the sequence $M(d, D)$ can be obtained from one of the seven $M(1, D)$ sequences. The precise relation is $M(d, D) = M(\bar{d}, D) = M(1, (D + 1 - \bar{d}) \bmod 7)$, with $\bar{d} := 7$ if $d \bmod 7 = 0$ and otherwise $d \bmod 7$. This is exhibited as a cyclic 7×7 matrix scheme in *Table* 2.
Therefore, only $4 \cdot 7 = 28$ such sequences of length 400 are needed, viz $M(1, D), M(29, D), M(30, D)$ and $M(31, D)$, for $D \in 0, 1, ..., 6$.
The main observation reported here is that each of these 28 basic sequences $M(d, D)$ of lenght 400, used periodically, can be put together from pieces of one of only 4 cyclic sequences of length 28. These four short sequences will be called S_1, S_{29}, S_{30} and S_{31}. They will be used periodically to build $M(1, D), M(29, D), M(30, D)$ and $M(31, D)$, respectively. They are given in the second row of *Table* 3. Note that the offset is 0 for these sequences.

$$S_1 = [1, 2, 2, 1, 2, 1, 2, 2, 1, 3, 1, 1, 3, 2, 1, 3, 1, 2, 2, 2, 2, 1, 1, 2, 2, 1, 3, 1], \tag{2}$$
$$S_{29} = [1, 2, 2, 1, 2, 1, 2, 2, 1, 2, 1, 1, 3, 2, 1, 2, 1, 2, 2, 2, 2, 1, 1, 2, 2, 1, 2, 1], \tag{3}$$
$$S_{30} = [3, 2, 1, 2, 1, 2, 2, 2, 2, 1, 1, 2, 2, 1, 2, 1, 1, 2, 2, 1, 1, 1, 2, 2, 1, 2, 1, 1], \tag{4}$$
$$S_{31} = [1, 0, 1, 1, 1, 1, 1, 0, 1, 1, 2, 1, 0, 1, 1, 1, 2, 1, 0, 1, 1, 2, 1, 1, 1, 1, 1, 2]. \tag{5}$$

These four sequences for multiplicities are obtained from the first 28 entries of $M(1, 0)$, $M(29, 0)$, $M(30, 0)$ and $M(31, 0)$, respectively (also with offset 0).
These sequences are used cyclically and shown as 'clocks' in the *Figure*s 1, 2, 3 and 4.
Later they will be considered without the outer brackets and denoted by $\bar{S}$.
The composition of the 28 sequences $M(i, D)$ for $i = 1, 29, 30, 31$ and $D = 0, 1, ... 6$, are encoded like shown in *Table* 3. This encoding is explained by giving examples for a certain day of the week D for each $M(i, D)$ in the following four subsections.

A) Example M(1, 2)

Here the cyclic sequence with period S_1 is used. For $D = 2$ (Tuesday) the start in the representative year $y = 0$ is seen from the number of entries 2 of the list $[D(1, m, 0)_{m=1..12}] = [6, 2, 3, 6, 1, 4, 6, 2, 5, 0, 3, 5]$ to be $M(1, 2)(0) = 2$. For the next years the multiplicity of 2 is found by starting with $S_1(4) = 2$ for the all together $100 = 3 \cdot 28 + 16$ entries, repeating S_1, shown in *Figure* 1 cyclically. This means on starts with $S_1(4)$ reaching first the end of the cycle $S_1(27)$, then repeating the whole S_1 twice, and in order to obtain $100 = (27 - 4 + 1) + 2 \cdot 28 + (4 + 16)$ entries, one keeps going for the missing $4 + 16 = 20$ entries to end up with $S_1(19) = 2$. Here one has to stop because the next entries in $M(1, 2)$ are 1, 3, 1, 1, 3 ..., but $S_1(20) = 2$, not 1.
This second piece of $M(1, 2)$ starts with $S_1(8)$, *i.e.*, one has to move from $S_1(19)$ to $S_1(8) = 1$ in 17 steps. Up to now the encoding is (4)100(17), where the numbers in brackets give the number of steps one has to move from the reached entry of S_1 to the start of the next valid sub-sequence read off the S_1 clock ((4) indicates the four steps from $S_1(0)$ to $S_1(4)$), and the following numbers give the length of the sequence cycling in S_1, before the next wrong number compared to $M(1, 2)$ appears. This second sub-sequence has $101 = (27-8+1) + 2 \cdot 28 + (8+17)$ entries, viz $S_1(8)$ to $S_1(27)$, then cycling S_1 twice and going fom $S_1(0)$ to $S_1(24) = 2$. The next entry $S(25) = 1$ does not fit $M(1, 2)(201) = 2$, which turns out to be $S_1(13)$, after again 17 steps. Up to now the encoding is (4)100(17)101(17).
The third piece of $M(1, 2)$ has $102 = (27 - 13 + 1) + 3 \cdot 28 + 3$ entries, viz $S_1(13)$ to $S_1(27)$, then cycling thrice, and continuing with $S_1(0)$ to $S_1(2)$. The next entry $S_1(3) = 1$ does not fit $M(1, 2)(303) = 2$. The correct fourth sub-sequence starts with $S_1(19) = 2$, after again (17) steps. The encoding continues therefore with (4)100(17)101(17)102(17).
The last piece of $M(1, 2)$ has the missing $97 = (27 - 19 + 1) + 3 \cdot 28 + 4$ entries, viz $S_1(19)$ to $S_1(27)$, cycling thrice and going from $S_1(0)$ to $S_1(3) = 1$. Then a new length 400 cycle will starts for $M(1, 2)$ with $S_1(4) = 2$ (after one step). Thus the complete encoding of this length 400 sequence is

$$\text{encoded } M(1, 2) = (4)100(17)101(17)102(17)97, \tag{6}$$

as given in *Table* 3.
Written in terms of the four S_1 pieces of lengths 100, 101, 102 and 97 this becomes

$$\begin{aligned} M(1, 2) = \quad & [S_1(4..27), \bar{S}_1, \bar{S}_1, S_1(0..19), \quad S_1(8..27), \bar{S}_1, \bar{S}_1, S_1(0..24), \quad S_1(13..27)\bar{S}_1, \bar{S}_1, \bar{S}_1, S_1(0..2), \\ & S_1(19..27)\bar{S}_1, \bar{S}_1, \bar{S}_1, S_1(0..3)], \end{aligned} \tag{7}$$

where $S_1(i..j)$ stands for the sub-sequence $S_1(k)_{k=i,i+1,...,j}$ (with entries separated by commas), and $\bar{S}_1$ is the sequence S_1 given in eq. 2 without the brackets [.].

B) Example M(29, 1)

After the detailed description of the previous example it suffices to give the encoding, from *Table* 3 and the corresponding decoding in terms of five pieces of the sequence S_{29}, shown in *Fig.* 2.

$$\text{encoded}\,\mathrm{M}(29,\,1)\;=\;(16)102(17)98(17)100(7)4(11)96\,. \tag{8}$$

$$\begin{aligned} M(29,\,1)\;=\;\; & [S_{29}(16..27), \bar{S}_{29}, \bar{S}_{29}, \bar{S}_{29}, S_{29}(0..5), \quad S_{29}(22..27), \bar{S}_{29}, \bar{S}_{29}\bar{S}_{29}, S_{29}(0..7), \\ & S_{29}(24..27), \bar{S}_{29}, \bar{S}_{29}, \bar{S}_{29}, S_{29}(0..11), \quad S_{29}(18..21), \quad S_{29}(4..27)\bar{S}_{29}, \bar{S}_{29}, S_{29}(0..15)]\,. \end{aligned} \tag{9}$$

C) Example M(30, 4)

This case consists of five pieces, see *Table* 3, and S_{30} is shown in *Fig.* 3.

$$\text{encoded}\,\mathrm{M}(30,\,4)\;=\;(8)100(17)100(7)5(11)96(17)99\,. \tag{10}$$

$$\begin{aligned} M(30,\,4)\;=\;\; & [S_{30}(8..27), \bar{S}_{30}, \bar{S}_{30}, S_{30}(0..23), \quad S_{30}(12..27), \bar{S}_{30}, \bar{S}_{30}, \bar{S}_{30}, \quad S_{30}(6..10), \\ & S_{30}(21..27), \bar{S}_{30}, \bar{S}_{30}, \bar{S}_{30}, S_{30}(0..4), \quad S_{30}(21..27), \bar{S}_{30}, \bar{S}_{30}, \bar{S}_{30}, S_{30}(0..7)]. \end{aligned} \tag{11}$$

D) Example M(31, 0)

This case consists of four pieces, see *Table* 3, and S_{31} is shown in *Fig.* 3.

$$\text{encoded}\,\mathrm{M}(31,\,0)\;=\;(0)102(17)99(17)99(17)(100)\,. \tag{12}$$

$$\begin{aligned} M(31,\,0)\;=\;\; & [\bar{S}_{31}, \bar{S}_{31}, \bar{S}_{31}, S_{31}(0..17), \quad S_{31}(6..27), \bar{S}_{31}, \bar{S}_{31}, S_{31}(0..20), \\ & S_{31}(9..27), \bar{S}_{31}, \bar{S}_{31}, S_{31}(0..23), \quad S_{31}(12..27), \bar{S}_{31}, \bar{S}_{31}, \bar{S}_{31}]. \end{aligned} \tag{13}$$

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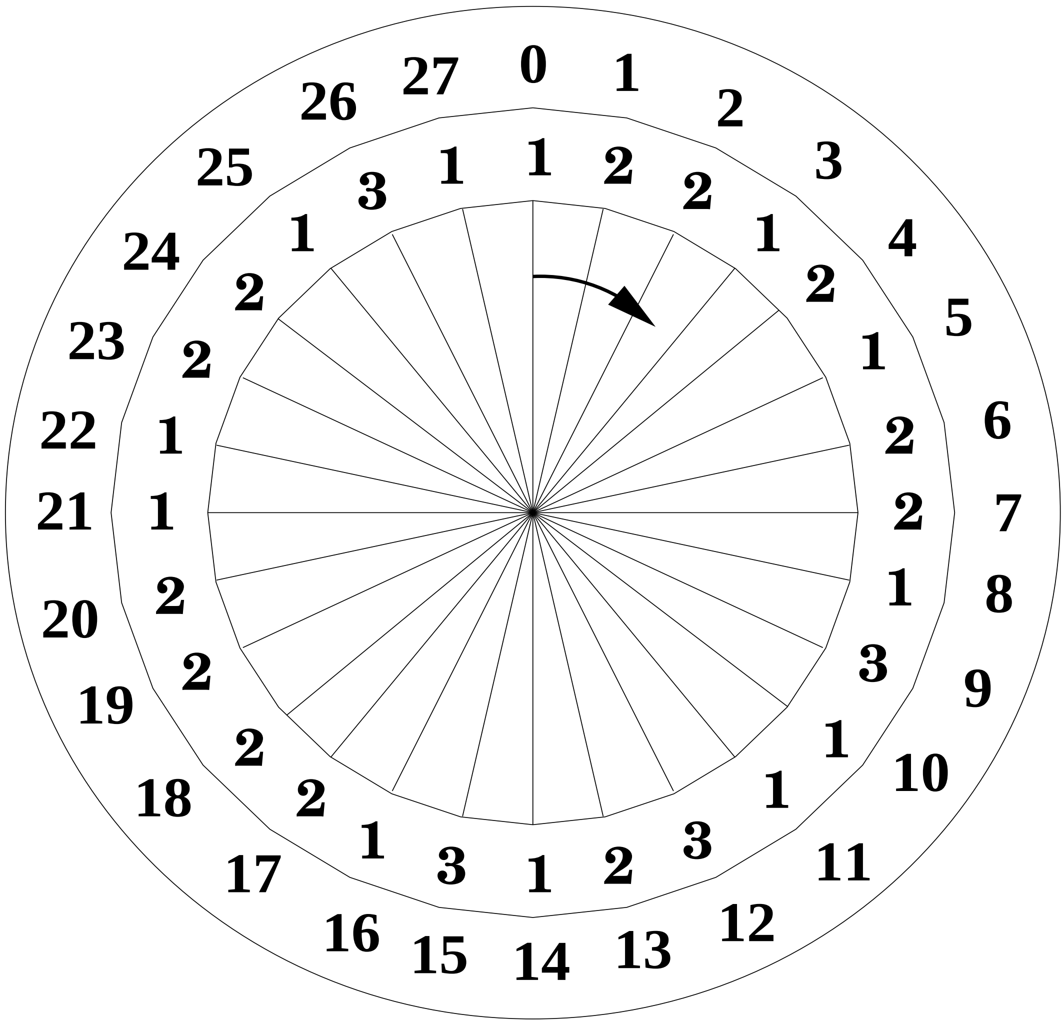

Fig. 1: S_1 sequence

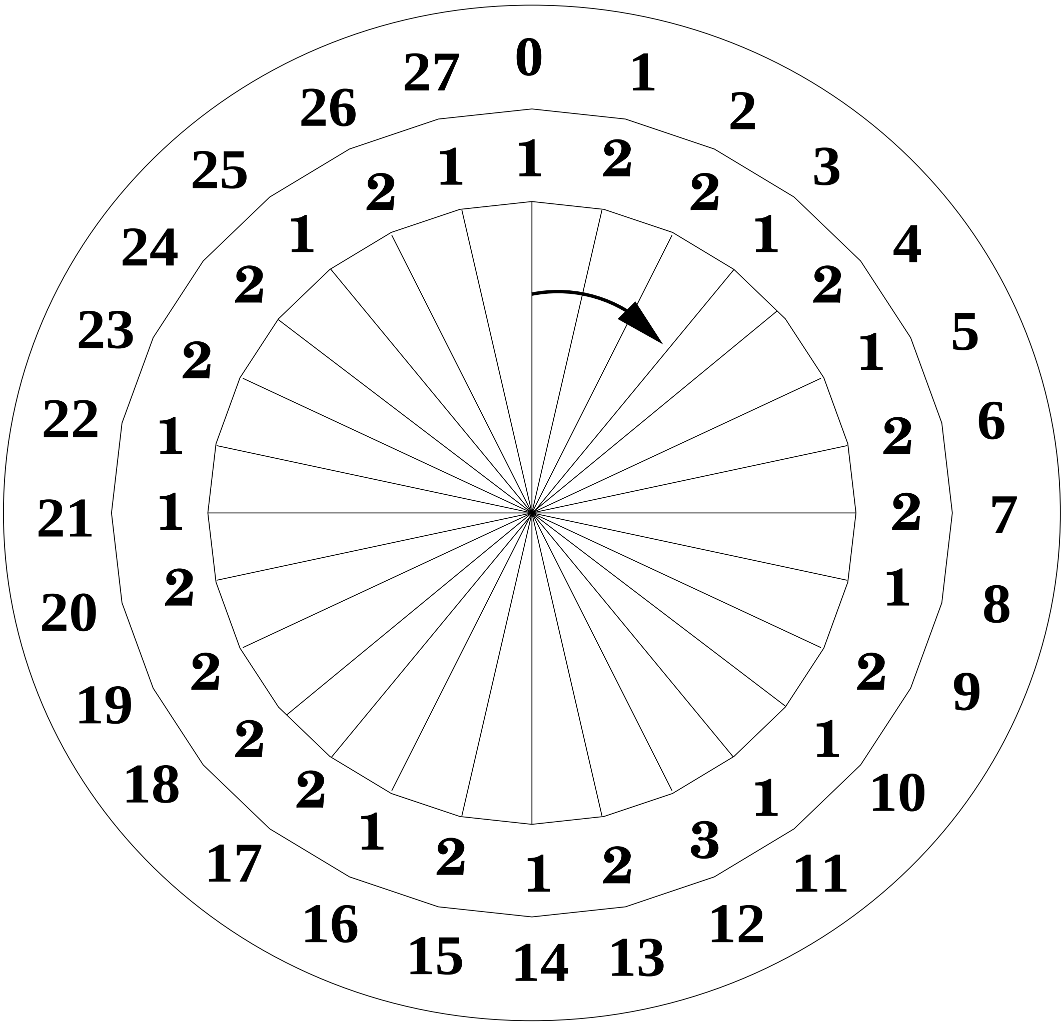

Fig. 2 : S_{29} sequence

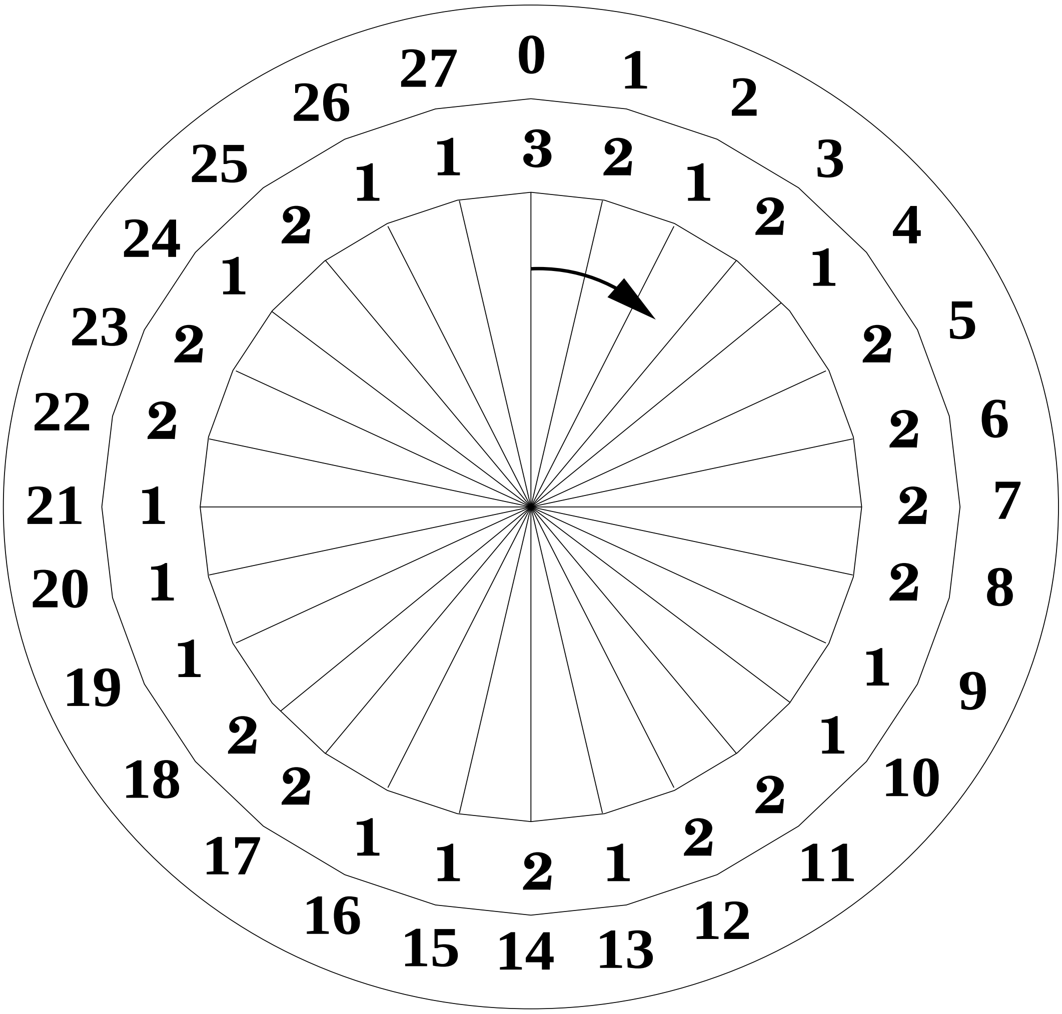

Fig. 3 : S $_{30}$ sequence

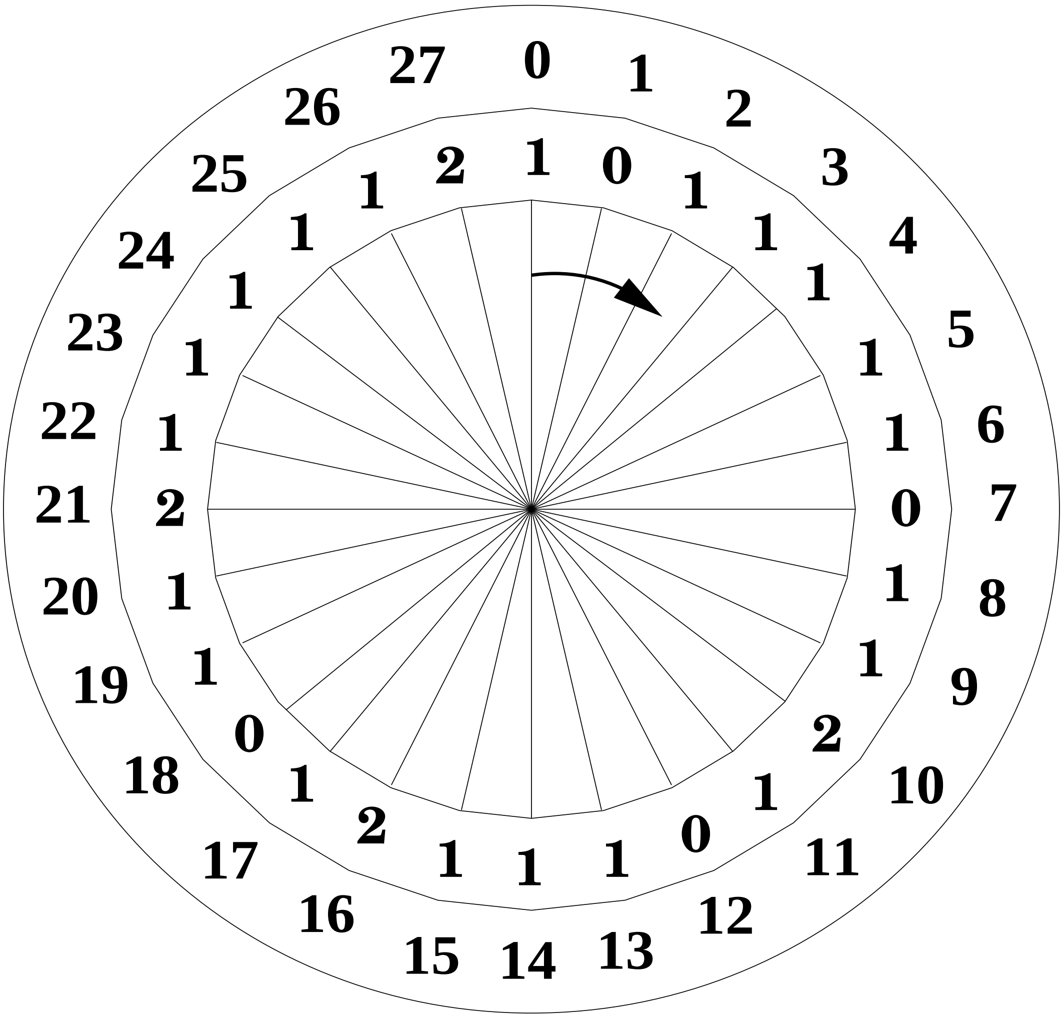

Fig. 4 : S_{31} sequence

Table 1: Occurrences for days of a month d at a weekday D for each Gregorian cycle of 400 years

d\D	0 Su	1 Mo	2 Tu	3 We	4 Th	5 Fr	6 Sa
1 (8, 15, 22)	688	684	687	685	685	687	684
2 (9, 16, 27)	684	688	684	687	685	685	687
3 (10, 17, 24)	687	684	688	684	687	685	685
4 (11, 18, 25)	685	687	684	688	684	687	685
5 (12, 19, 26)	685	685	687	684	688	684	687
6 (13, 20, 27)	687	685	685	687	684	688	684
7 (14, 21, 28)	684	687	685	685	687	684	688
29	644	641	644	642	642	643	641
30	627	631	626	631	627	629	629
31	400	399	401	398	402	399	401

Table 2: $T_7(\overline{d}, D)$ matrix for the identity $M(d, D) = M(1, T_7(\overline{d}, D))$ for $d \in \{1, 2, ..., 28\}$

$\overline{d}$\D	0	1	2	3	4	5	6
1	0	1	2	3	4	5	6
2	6	0	1	2	3	4	5
3	5	6	0	1	2	3	4
4	4	5	6	0	1	2	3
5	3	4	5	6	0	1	2
6	2	3	4	5	6	0	1
7	1	2	3	4	5	6	0

Example: $M(17, 4) = M(\overline{17}, 4) = M(3, 4) = M(1, 2)$
with $\overline{d} := 7$ if $d \equiv 0 \mod 7$, otherwise $d \mod 7$.

Table 3: Sequences for number of times a given days of the month d falls on days of a week within a 400 cycle

	$d \in \{1, 8, 15, 22\}$	$d = 29$	$d = 30$	$d = 31$
Period 28 sequence	S_1 = [1, 2, 2, 1, 2, 1, 2, 2, 1, 3, 1, 1, 3, 2, 1, 3, 1, 2, 2, 2, 2, 1, 1, 2, 2, 1, 3, 1]	S_{29} = [1, 2, 2, 1, 2, 1, 2, 2, 1, 2, 1, 1, 3, 2, 1, 2, 1, 2, 2, 2, 2, 1, 1, 2, 2, 1, 2, 1]	S_{30} = [3, 2, 1, 2, 1, 2, 2, 2, 2, 1, 1, 2, 2, 1, 2, 1, 1, 2, 2, 1, 1, 1, 2, 2, 1, 2, 1, 1	S_{31} = [1, 0, 1, 1, 1, 1, 1, 0, 1, 1, 2, 1, 0, 1, 1, 1, 2, 1, 0, 1, 1, 2, 1, 1, 1, 1, 1, 2]
Sunday (0)	(0)101(17)99(17)101(17)99	(0)101(17)99(17)101(17)99	(0)103(17)97(17)100(17)100	(0)102(17)99(17)99(17)100
Monday (1)	(16)100(23)6(23)94(17)100(7)4(11)96	(16)102(17)98(17)100(7)4(11)96	(16)101(17)99(17)101(17)99	(16)101(17)101(17)98(17)100
Tuesday (2)	(4)100(17)101(17)102(17)97	(4)100(17)101(17)102(17)97	(4)100(17)100(17)100(7)5(11)95	(4)101(17)99(17)101(17)99
Wednesday (3)	(20)100(17)100(7)4(11)97(17)99	(20)100(17)100(7)4(11)97(17)99	(20)100(17)101(17)102(17)97	(20)102(17)98(17)102(17)98
Thursday (4)	(8)101(17)102(17)97(23)6(23)94	(8)101(17)102(17)99(17)98	(8)100(17)100(7)5(11)96(17)99	(8)100(17)101(17)100(17)99
Friday (5)	(24)100(7)4(11)97(17)99(17)100	(24)100(7)4(11)97(17)100(16)99	(24)101(17)102(17)97(17)100	(24)100(17)102(17)99(17)99
Saturday (6)	(12)103(17)97(23)6(23)94(17)100	(12)103(17)99(17)103(18)95	(12)100(7)5(11)96(17)100(16)99	(12)101(17)100(17)101(17)98