Karlsruhe
March 28 2018

On a Conformal Mapping of Regular Hexagons and the Spiral of its Centers

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Abstract

A sequence of regular hexagons used in a geometrical proof of the incommensurability of the shorter diagonal and the side of a hexagon is obtained by iteration of a conformal mapping. The centers form a discrete spiral and are interpolated by two continuous spirals, one with discontinuous curvature the other one a logarithmic spiral.

1 Introduction

A geometrical proof by contradiction of the incommensurability of the shorter diagonal of a regular hexagon and its side can be given by considering an infinite process of ever smaller hexagons. This is explained in *Havil*'s book [2] on irrationals. It shows the irrationality of $\sqrt{3}$, the length ratio between the a shorter diagonal and the side of a regular hexagon. We use this geometrical construction of a sequence of translated, rotated and down-scaled hexagons (always regular ones) $\{H_k\}_{k=0}^{\infty}$ inscribed in circles $\{C_k\}_{k=0}^{\infty}$ of radius $\sigma^k\, r_0$, with $\sigma = -1 + \sqrt{3}$ and centers $\{O_k\}_{k=0}^{\infty}$. These centers build a discrete spiral. The interpolation of the centers by a continuous curve is immediately given by patching together circular arcs of radius σ^k with one of the H_k vertices as centers. The curvature of this spiral is therefore discontinuous. Due to a conformal mapping of the loxodromic type whose iteration produces the sequence of hexagons an interpolating logarithmic spiral ensues with the finite fixed point S as its center. These two spirals are analogous to the ones in a regular pentagon with a sequence of golden triangles (or rectangles) shown, *e.g.*, in the book of *Livio* [4], as figures 40 and 41 on p. 119. For these triangles the conformal mapping has been given in [3]. The completion of the hexagon sequence and the spirals using negative k values is also considered.

2 Hexagon Descent

For the following geometrical construction see *Figure 1* with $k = 0$. One starts with a circle C_0 with center O_0 and radius r_0 (this will be taken in the sequel as length unit. Hence, lengths will always be lengths ratios *w.r.t.* r_0), and inscribes a regular hexagon (the standard construction with a pair of compasses). The vertices of the hexagon (only regular hexagons will be considered) are denoted by $V_k(j)$, for $j = 0, 1, ..., 5$, taken in the positive (anti-clockwise) sense. The choice of $V_0(0)$ defines the non-negative x_0 axis as prolongation of $\overline{O_0,\, V_0(0)}$. These Cartesian coordinates are named $(x_0,\, y_0)$ (or in the complex plane $z = x_0 + y_0\, i$).

The next (smaller) hexagon H_1 is inscribed in a circle C_1 with center O_1 and radius $r_1 = \sigma := -1 + \sqrt{3}$. This center is obtained by drawing the smaller diagonal in H_0, *viz*, $D_0 = \overline{V_0(0),\, V_0(2)}$, which has length $\sqrt{3}$, intersecting it with a circle of radius 1 around $V_0(2)$. Then on the circle $C_1(O_1, r_1)$, with radius

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$r_1 = \overline{O_1, V_0(0)} = \sigma = -1 + \sqrt{3}$, the vertex $V_1(3)$ of H_1 is the intersection point with the x_0 axis, *i.e.*, the prolongation of $\overline{O_0\, V_0(0)}$ or $\overline{V_0(3)\, V_0(0)}$. From this vertex $V_1(3)$ one finds the vertex $V_1(0)$ as antipode on C_1. $V_1(5)$ coincides with $V_0(0)$.

In the second step the new center O_2 of H_2 is constructed in the same way by drawing the smaller diagonal $D_1 = \overline{V_1(0)\, V_1(2)}$ ($V_1(2)$ happens to lie on the diagonal D_0, and D_1 is parallel to the x_0 axis). Then the circle around $V_1(2)$ with radius r_1 intersects D_1 at O_2. The vertex $V_2(3)$ on $C_2(O_2, r_2)$, with $r_2 = \overline{O_2, V_1(0)} = \sigma\, r_1 = \sigma^2$), is the point of intersection of C_2 with the x_1 axis (prolongation of $\overline{O_1, V_1(0)}$). The antipode of $V_2(3)$ on C_2 is $V_2(0)$, etc.

This construction implies the following data (besides some obvious ones for a hexagon).

Lemma 1

1) $|V_0(2), V_0(0)| = \sqrt{3}$, $|O_1, V_0(0)| = \sigma := -1 + \sqrt{3}$. $|V_1(3), O_0| = \dfrac{\sigma^2}{2} = 2 - \sqrt{3}$.

2) The two circles C_0 and C_1 intersect at $(1, 0)$ and $S = (0, 1)$.

Proof: (In Cartesian coordinates (x_0, y_0))

1) $V_0(2) = \left(-\dfrac{1}{2}, \dfrac{\sqrt{3}}{2}\right)$, hence $\angle(V_0(2), V_0(1), O_0) = \frac{\pi}{6}$. Therefore, $O_1 = \left(\frac{\sigma}{2}, \frac{\sigma}{2}\right)$, and $\angle(V_0(0), O_0, O_1) = \dfrac{\pi}{4}$. $\angle(V_0(0), V_1(3), O_1) = \dfrac{\pi}{6}$. From $\triangle(V_1(3), O_1, V_0(0))$ one has $|V_1(3), V_0(0)| = 2 \cdot \left(\dfrac{\sigma}{2}\sqrt{3}\right)$. On the other hand, the y_0 component of $V_1(0)$ is $\sin\left(\dfrac{\pi}{6}\right) 2\sigma = \sigma$, hence $V_0(0) = V_1(5)$, and $\overline{V_1(0), V_0(0)}$ is parallel to the y_0-axis. Therefore $\overline{V_1(0), V_1(2)}$ is parallel to the x_0-axis, and $V_1(2)$ with y_0-component σ lies on the diagonal D_0. $|V_1(3), O_0| = \sigma\dfrac{\sqrt{3}}{2} - \dfrac{\sigma}{2} = \dfrac{\sigma^2}{2} = 2 - \sqrt{3}$.

2) With $C_0 : x_0^2 + y_0^2 = 1$ and $C_1 : \left(x_0 - \dfrac{\sigma}{2}\right)^2 + \left(y_0 - \dfrac{\sigma}{2}\right)^2 = \sigma^2$ one finds the intersections $(1, 0)$ and $S = (0, 1)$. □

Thus the new hexagon H_1 is obtained from the old one, H_0, by a translation with $\vec{v}_0 := \overrightarrow{O_0, O_1} = \sigma\,(1, 1)^\top$ (a column vector), followed by a rotation about the axis perpendicular to the plane (the $z-$axis) through O_1 by the angle $\angle(V_1(0), V_1(2), V_1(5)) = \dfrac{\pi}{6}$ and scaling down by a factor σ. This process is iterated to find H_{k+1} from H_k, for $k = 0, 1, \ldots$ (see *Figure 1*).

Next, the vectors $\vec{v}_k = \overrightarrow{O_{k-1}, O_k}$ are given in polar coordinates.

Lemma 2: Vectors $\vec{v}_k$, $k = 1, 2, \ldots$

$$\vec{v}_k \doteq v_k \begin{pmatrix} \cos\alpha_k \\ \sin\alpha_k \end{pmatrix}, \quad \text{with} \quad v_k = \sigma^k\,\frac{\sqrt{2}}{2}, \quad \text{and} \quad \alpha_{\rm k} = (2\,{\rm k} + 1)\,\frac{\pi}{12}, \text{ for k} \in \mathbb{N}, \tag{1}$$

$$v_k = (a_k + b_k\sqrt{3})\,\frac{\sqrt{2}}{2}, \quad \text{where} \quad a_k = (-1)^k\, \text{A026150}(k), \text{ and } b_k = (-1)^{k+1}\, \text{A002605}(k)\,.$$

For the first a_k and b_k entries see *Table 6*, column r_k. For the components of the first twelve vectors $\vec{v}_k$ see *Table 1*.

Proof:

i) The polar angle α is obtained recursively from $\alpha_k = \alpha_{k-1} + \frac{\pi}{6}$, for $k = 2, 3, \ldots$, with input $\alpha_1 = \frac{\pi}{4}$ which follows from the rotation by an angle of $\frac{\pi}{6}$ to obtain H_k from H_{k-1}.

ii) The length v_k is obtained recursively from $v_k = v_{k-1}\,\sigma$ for $k = 2, 3, \ldots$ with input $v_1 = \sigma\sqrt{2}$. One may take formally $v_0 = \frac{\sqrt{2}}{2}$ and then $v_k = \sigma^k\, v_0$, for $k = (0), 1, 2, \ldots$. For $\{a_k\}_{k=0}^{\infty}$ and $\{b_k\}_{k=0}^{\infty}$ one obtains the mixed recurrence $a_k = -a_{k-1} + 3\,b_{k-1}$ and $b_k = a_{k-1} - b_{k-1}$, for $k = 0, 1, \ldots$, and inputs $a_0 = 1$ and $b_0 = 0$. This decouples, inserting $b_k + b_{k-1} = a_{k-1}$ into $a_k + a_{k-1}$, to the three term recurrences $b_k = 2\,(-b_{k-1} + b_{k-2})$ with inputs $b_0 = 0$ and $b_1 = 1$, and $a_k = 2\,(-a_{k-1} + a_{k-2})$ with inputs

$a_0 = 1$ and $a_1 = -1$. The *Binet* formulae are, with $\tau := \frac{2}{\sigma} = 1 + \sqrt{3} =: -\overline{\sigma}$, $a_k = \frac{1}{2}\left(\sigma^k + (-\tau)^k\right)$ and $b_k = \frac{1}{2\sqrt{3}}\left(\sigma^k - (-\tau)^k\right)$. The *o.g.f.*s (ordinary generating functions) are $Ga(x) = \frac{1 + x}{1 + 2x - 2x^2}$ and $Gb(x) = \frac{x}{1 + 2x - 2x^2}$. This explains the given result involving A026150 and A002605. □

In Cartesian coordinates one can write the recurrence as

$$\vec{v}_k = \sigma\, \mathbf{R}\, \vec{v}_{k-1}, \quad k = 2, 3, \ldots \text{ with } \vec{v}_1 \doteq \frac{\sigma}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \mathbf{R} \doteq \frac{1}{2}\begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix}. \tag{2}$$

$\mathbf{R}$ is the rotation matrix for angle $\frac{\pi}{6}$. This leads to

$$\vec{v}_{k+1} = (\sigma\, \mathbf{R})^k\, \vec{v}_1, \quad \text{for } k = (0), 1, 2, \ldots \tag{3}$$

The powers of σ have been given above as $\sigma^k = a_k + b_k\sqrt{3}$.
The powers of R are found as an application of the *Cayley – Hamilton* theorem, *e.g.*, [8],[7]:

$$\mathbf{R}^k = S_{k-1}(\sqrt{3})\, \mathbf{R} - S_{k-2}(\sqrt{3})\, \mathbf{1}_2, \quad \text{for } k = 1, 2, \ldots, \tag{4}$$

Where $S_n(x)$ is the *Chebyshev* polynomial with coefficients given in A049310 with $S_{-1}(x) = 0$ and $S_{-2}(x) = -1$. Here $S_{2l}(\sqrt{3}) =$ A057079(l) and $S_{2l+1}(\sqrt{3}) =$ A019892$(l)\sqrt{3}$, for $k = 0, 1, \ldots$. A057079 and A019892 are period length 6 sequences, repeat$(1, 2, 1, -1, -2, -1)$ and repeat$(1, 1, 0, -1, -1, 0)$, respectively. *I.e.*, $S_n(\sqrt{3}) = s_n + t_n\sqrt{3}$, with $\{s_n\}_{n=0}^{\infty} =$ repeat$(1, 0, 2, 0, 1, 0, -1, 0, -2, 0, -1, 0)$ and $\{t_n\}_{n=0}^{\infty} =$ repeat$(0, 1, 0, 1, 0, 0, 0, -1, 0, -1, 0, 0)$.

Corollary 1: $\vec{v}_k$ Periodicity modulo 12 up to scaling

$$\vec{v}_{k+12\,l} = \sigma^{12\,l}\, \vec{v}_k, \quad \text{for } k \in \mathbb{N},\ l \in \mathbb{N}_0\,. \tag{5}$$

This follows from the periodicity of the angle α_k in eq. (1).

The calculation of the $\vec{v}_{2\,l}$ and $\vec{v}_{2\,l+1}$ components *w.r.t.* the (x_0, y_0) coordinate system leads to

Proposition 1: Components of $\vec{v}_k$, $k = 1, 2, \ldots$

$$\vec{v}_{2\,l} \doteq \frac{1}{4}\begin{pmatrix} ve1(l) + we1(l)\sqrt{3} \\ ve2(l) + we2(l)\sqrt{3} \end{pmatrix},\ l \geq 1, \quad \vec{v}_{2\,l+1} \doteq \frac{1}{4}\begin{pmatrix} vo1(l) + wo1(l)\sqrt{3} \\ vo2(l) + wo2(l)\sqrt{3} \end{pmatrix},\ l \geq 0, \tag{6}$$

$$\text{with } ve1(l) = -a_{2\,l}\, A(l-1) + 3\, b_{2\,l}\, (A(l-1) - 2\, B(l-2))\,, \tag{7}$$

$$we1(l) = +a_{2\,l}\, (A(l-1) - 2\, B(l-2)) - b_{2\,l}\, A(l-1)\,, \tag{8}$$

$$ve2(l) = +a_{2\,l}\, A(l-1) + 3\, b_{2\,l}\, (A(l-1) - 2\, B(l-2))\,, \tag{9}$$

$$we2(l) = +a_{2\,l}\, (A(l-1) - 2\, B(l-2)) + b_{2\,l}\, A(l-1)\,, \tag{10}$$

$$\text{and } vo1(l) = a_{2\,l+1}\, (3\, B(l-1) - 2\, A(l-1)) - 3\, b_{2\,l+1}\, B(l-1)\,, \tag{11}$$

$$wo1(l) = -a_{2\,l+1}\, B(l-1) + b_{2\,l+1}\, (3\, B(l-1) - 2\, A(l-1))\,, \tag{12}$$

$$vo2(l) = +a_{2\,l+1}\, (3\, B(l-1) - 2\, A(l-1)) + 3\, b_{2\,l+1}\, B(l-1)\,, \tag{13}$$

$$wo2(l) = +a_{2\,l+1}\, B(l-1) + b_{2\,l+1}\, (3\, B(l-1) - 2\, A(l-1))\,, \tag{14}$$

where $A(l) = S_{2\,l}(\sqrt{3})$, $B(l) = S_{2\,(l-1)}(\sqrt{3})/\sqrt{3}$,
and a_k and b_k are given in Lemma 2.

See *Table 1* for the coordinates of $\vec{v}_k$ for $k = 1, 2, \ldots, 12$.

The center O_k of hexagon H_k, the endpoint of the vector $\vec{O}_k := \overrightarrow{O_0,\, O_k}$, is obtained from (undefined sums are set to 0)

$$\vec{O}_k \;=\; \sum_{j=1}^{k} \vec{v}_j\,, \quad k \,=\, 1,\, 2,\, \dots \quad \text{and} \quad \vec{O}_0 \,=\, \vec{0}, \tag{15}$$

$$\vec{O}_k \;=\; \left(\mathbf{1}_2 \,+\, \sum_{j=1}^{k-1} (\sigma\,\mathbf{R})^j\right) \vec{v}_1\,. \tag{16}$$

In the coordinate system $(x_0,\, y_0)$ the components of center O_k follow from *Proposition 1.*

Corollary 2: Components of O_k, $k \,=\, 1,\, 2,\, \dots$

$$(O_k)_{x_0} \;=\; \frac{1}{4}\left(\sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} (ve1(j) \,+\, we1(j)\,\sqrt{3}\,) \,+\, \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} (vo1(j) \,+\, wo1(j)\,\sqrt{3}\,)\right),$$

$$(O_k)_{y_0} \;=\; \frac{1}{4}\left(\sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} (ve2(j) \,+\, we2(j)\,\sqrt{3}\,) \,+\, \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} (vo2(j) \,+\, wo2(j)\,\sqrt{3}\,)\right). \tag{17}$$

See *Table 1* for the components of O_k for $k \,=\, 1,\, 2,\, \dots,\, 12$. It seems that the centers $O_{6\,l}$, for $l \,=\, 0,\, 1,\, \dots$ lie on the y_0 axis. This will be proved in the next section in *Proposition 4*.
The relation between $\vec{O}_{k+12\,l}$ and Q_k will also be considered in the next section in *Proposition 6*, part 7), in the complex plane. It is a periodicity modulo 12 up to a scaling and a translation.

The vertices $V_k(j)$, for $j \,=\, 0,\, 1,\, \dots,\, 5$, of the hexagon H_k follow from $\vec{V}_k(j) := \overrightarrow{O_0,\, V_k(j)}$.

Proposition 2: Vertices of hexagons H_k

$$\vec{V}_k(j) \,=\, \vec{O}_k \,+\, \sigma^k\,\mathbf{R}^{k+2\,j}\begin{pmatrix}1\\0\end{pmatrix}, \quad \text{for} \;\; k \,=\, 0,\, 1,\, \dots\,, \quad \text{and} \;\; j \,=\, 0,\, 1,\, \dots,\, 5\,. \tag{18}$$

Proof:
For the hexagon H_k the vector $\overrightarrow{O_k,\, V_k(0)}$ is obtained from the unit vector in x_0 direction of the original coordinate system $(x_0,\, y_0)$ for the first hexagon H_0 by k-fold rotation with $\mathbf{R} \,=\, \mathbf{R}(\frac{\pi}{6})$ and down-scaling by σ as

$$\overrightarrow{O_k,\, V_k(0)} \,=\, (\sigma\,\mathbf{R})^k \begin{pmatrix}1\\0\end{pmatrix}. \tag{19}$$

Then the vectors for the other vertices are obtained by repeated rotation of 60°, *i.e.*, by application of $\mathbf{R}^2$ leading to the assertion. □

For the $(x_0,\, y_0)$ components of $\vec{V}_k(0)$, for $0,\, 1,\, \dots,\, 12$, see *Table 2*, and for the other vertices, for $j \,=\, 1,\, 2,\, \dots,\, 5$, see *Tables 3*, *4* and *5*.

Lemma 3: Triangles T_k

The triangle $T_k \,=\, \triangle(O_k,\, V_k(2),\, O_{k+1})$, for $k \,=\, 0,\, 1,\, \dots\,$, is isosceles with basis $v_{k+1} \,=\, \frac{1}{\sqrt{2}}\,\sigma^{k+1}$ and two sides of length $r_k \,=\, \sigma^k$. The angles are $\angle(O_{k+1},\, V_k(2),\, O_k) \,=\, \frac{\pi}{6} \,\hat{=}\, 30°$ and twice $\frac{5\,\pi}{12} \,\hat{=}\, 75°$.

Proof: This is clear from the construction and the values for v_k given above in *Lemma 2* and r_k. See *Figure 1.* □

The polar coordinates of O_k, the center of hexagon H_k are given as follows. Note that $\varphi \in [0, 2\pi)$. The number of revolutions, using also $\varphi \geq 2\pi$ (sheets in the complex plane), will be considered in the next section.

Corollary 3: Polar coordinates of O_k

In the complex plane $O_k \hat{=} z_k = \rho_k\, exp(i\,\varphi_k)$ with $\rho_k = \left|\overrightarrow{O_0,\, O_k}\right|$, one has

$$\rho_k = \sqrt{((O_k)_{x0})^2 + ((O_k)_{y0})^2}, \quad \text{with eq.(17)} \tag{20}$$

$$\varphi_k = \hat{\varphi}_k \text{ in quadrant I}, = \hat{\varphi}_k + \pi \text{ in quadrants II and III}, = \hat{\varphi}_k + 2\pi \text{ in quadrant IV}, \quad \text{with}$$

$$\hat{\varphi}_k = \arctan\left(\frac{(O_k)_{y0}}{(O_k)_{x0}}\right). \tag{21}$$

ρ_k^2 is integer in the real quadratic number field $\mathbb{Q}(\sqrt{3})$. For the values for $k = 0, 1, ..., 12$, see *Table 2*. The corresponding angles are $(\varphi_k\, 180/\pi)^\circ$. The values for tan $\hat{\varphi}_k$ are elements of $\mathbb{Q}(\sqrt{3})$. For their components see also Table 2, for $k = 1, 2, ..., 12$ (for $k = 0$, with $z_0 = 0$, the value of $\hat{\varphi}_0$ is arbitrary; in *Table 2* we have set it to 0).

3 Conformal mapping and the Hexagon Spiral

The discrete spiral formed by the hexagon centers O_0 and O_k given in eq. (17) for $k = 0, 1, \ldots$, are shown as dots in Figure 2 for $k = 0, 1, ..., 11$. In the complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \infty$ these centers will be called $z_k = (O_k)_{x0} + (O_k)_{y0}\, i$. The construction of these hexagon described in sect. 1 is obtained by repeated application of a conformal *Möbius* transformation. It is determined by mapping the triangle T_0 of H_0 with vertices $z(1) = V_0(2) = \frac{1}{2}(-1 + \sqrt{3}\,i)$, $z(2) = z_0 = 0 + 0\,i$ and $z(3) = z_1 = \frac{1}{2}(1 + 1\,i)$ to the translated, rotated and scaled triangle T_1 of H_1 with vertices $w(1) = V_1(2) = (-2 + \sqrt{3}) + (-1 + \sqrt{3}\,)\,i$, $w(2) = z_1 = \frac{1}{2}\left(-1 + \sqrt{3} + (-1 + \sqrt{3})\,i\right)$ and $w(3) = z_2 = (-3 + 2\sqrt{3}\,) + (-1 + \sqrt{3})\,i$. See *Figure 1* for these two triangles, setting $k = 0$. In general triangle T_k is mapped to T_{k+1} by this conformal transformation, especially $w(z_k) = z_{k+1}$, for $k = 0, 1, \ldots$. The unique *Möbius* transformation which maps the vertices of T_0 to those of T_1 is given by solving the double quotient equation for $w = w(z)$ (see. *e.g.*, [6], [9])

$$DQ(w(1),\, w(2),\, w(3),\, w) = DQ(z(1),\, z(2),\, z(3),\, z), \quad \text{with } DQ(z1,\, z2, z3, z4) := \frac{z4 - z3}{z4 - z1} \Big/ \frac{z2 - z3}{z2 - z1}. \tag{22}$$

The solution is a *Möbius* transformation of the loxodromic type, having besides one fixed point at ∞ another finite one S with $(w - S) = a\,(z - S)$, where a is not real non-negative, and $|a| \neq 1$.

$$\begin{aligned} w(z) &= \frac{A}{D}\,z + \frac{B}{D}, \quad \text{with} \\ A &= 2\left((-2 + \sqrt{3}\,) + (-7 + 4\sqrt{3}\,)\,i\right), \\ B &= (-9 + 5\sqrt{3}\,) + (5 - 3\sqrt{3}\,)\,i, \\ D &= (1 - \sqrt{3}\,) + (-5 + 3\sqrt{3}\,)\,i. \end{aligned} \tag{23}$$

The determinant of this transformation is $A\,D = 8\,(-19 + 11\sqrt{3})$. A, B and D are integers in $\mathbb{Q}(\sqrt{3}\,)$. This is rewritten in the following *Proposition*.

Proposition 3: Loxodromic map w

1) The unique conformal *Möbius* transformation w which maps the corners of triangle T_0 to those of T_1 (keeping the orientation), and hence $T_k = \triangle(V_k(2),\, O_k,\, O_{k+1})$ to T_{k+1}, is given by the loxodromic map

$$\begin{aligned} w(z) &= a\,z + b, \quad \text{with} \\ a &= \frac{1}{2}\left((3 - \sqrt{3}\,) + (-1 + \sqrt{3}\,)\,i\right), \\ b &= \frac{1}{2}(-1 + \sqrt{3}\,)\,(1 + i) = (1 - a)\,i\,. \end{aligned} \tag{24}$$

2) $a = \sigma\, e^{i\,\frac{\pi}{6}}$, and $|a| = \sigma = -1 + \sqrt{3} \neq 1$. The finite fixed point of this map is $S = i$. S is the common intersection point of all circles C_k.

Proof:

1) This is clear from the construction and the previous form of w from eq. (23) , and the computation has been checked with the help of Maple [5].

2) The values of a and $|a|$ show that this *Möbius* transformation is loxodromic with finite fixed point $S = i$. S has to lie on each circle C_k, for $k = 0, 1, \ldots$, because w maps C_k to C_{k+1}. □

Corollary 4: Inverse map $w^{[-1]}$

The inverse of map $w^{[-1]}$ of w is given by

$$\begin{aligned} w^{[-1]}(z) &= a^{-1}\,z + (1 - a^{-1})\,i \\ &= \frac{1}{4}\left[\left((3 + \sqrt{3}\,) - (1 + \sqrt{3}\,)\,i\right)\,z + \left(-(1 + \sqrt{3}\,) + (1 - \sqrt{3}\,)\,i\right)\right], \text{ for } z \in \overline{\mathbb{C}}\,. \end{aligned} \tag{25}$$

Check: $w^{[-1]}(w(z)) \equiv z$.

With the help of the conformal map w it is now easy to prove that points $z_{6\,j}$ (corresponding to the centers $O_{6\,j}$) lie on the imaginary axis (the y_0-axis).

Proposition 4: Centers $z_{6\,j}$ lie on the imaginary axis

$$\Re(z_{6\,j}) = 0, \text{ for } j \in \mathbb{N}_0\,.$$

Proof:

Compute $w^6(z)$ for z on the imaginary axis, $z = y\,i$, with real y: $w^6(y\,i) = \left(y + (209 - 120\sqrt{3}\,)\,(1 - y)\right)i$ $= (y + (O_6)_{y_0}\,(1 - y))\,i$. See the last column of *Table 1* for $(O_6)_{y_0}$. Therefore, points on the non-negative imaginary axis are mapped by w^6 again on this axis. Because $z_0 = 0$ lies on the imaginary axis also $z_{6\,j}$, for $j = 1, 2, \ldots$, have to lie on the imaginary axis. □

Corollary 5: Number of centers for each revolution of the spiral

The number of centers 0_k for each revolution is 12.

See *Figure 4* for the first revolution, except for 0_{12} on the imaginary axis where the second revolution starts.

The discrete hexagon spiral can be interpolated between O_k and O_{k+1} by circular arcs A_k of the circles $\hat{C}_k(V_k(2), r_k)$. See *Figure 4*. These arcs A_k belong to a sector of $\hat{C}_k$ of angle $\frac{5\,\pi}{12}$ (see *Lemma 3*). The precise form is given by

Proposition 5: Interpolating circular arcs A_k

The circular arc with center $V_k(2)$ and radius $r_k = \sigma^k$ which interpolates between the centers O_k and O_{k+1} of the hexagon H_k is given by

$$A_k = \text{arc}\left(V_k(2),\, r_k,\, \frac{(k-2)\,\pi}{6},\, \frac{(k-1)\,\pi}{6}\right) . \tag{26}$$

Proof:

From *Lemma 3* the range of the angle φ is $\frac{\pi}{6}$. The angles are counted in the positive sense with respect to the horizontal line, defined by the x_0-axis. It is therefore sufficient to know the angle for one of the lines $\overline{V_k(2),\, O_{k+1}}$ which corresponds to the larger of the angles for arc A_k , For $k = 1$ this angle vanishes because the y_0 components of $V_1(2)$ and O_2 coincide, they are $\sigma\, r_0$. Hence the angle for arc A_2 starts with 0 ($V_2(2)$ is on the line segment $\overline{V_1(2),\, O_2}$) and ends with $\frac{\pi}{6}$. This proves the given range for each A_k. □

This interpolation by circular arcs is continuous but has discontinuous curvature with increases at each center O_k by a factor of $1/\sigma = \frac{\tau}{2} = \frac{1}{2}(1 + \sqrt{3}) \approx 1.366025403$.

An interpolation with continuous curvature is given by the equal angle spiral (the logarithmic) spiral (*Jacob I Bernoulli: spira mirabilis*), defined in the complex plane by $LS(\phi) = r(\phi)\, exp(i\,\phi)$, with $r(\phi) = r(0)\, exp(-\kappa\,\phi)$ where the constant κ defines the constant angle α between the radial ray and the tangent (taken in the direction of increasing angle ϕ) at any point of the spiral by $\alpha = \text{arccot}(-k)$. Here the center of the logarithmic spiral is at the finite fixed point S and we choose a coordinate system $(X,\, Y)$ with the positive X direction along the vertical line (the y_0-axis in the negative sense) and the positive Y axis in the horizontal direction to the right, parallel to the positive x_0 axis. *I.e.*, $X = -y_0 + 1$ and $Y = x_0$. In this system $0_0 = (1,\, 0)$ and $r(0) = r_0 = 1$. The angle ϕ_1 for $0_1 = \left(\frac{2-\sigma}{2},\, \frac{\sigma}{2}\right)$ (in the $(x_0,\, y_0)$ system) becomes in the $(X,\, Y)$ system $\frac{\pi}{6}$ because $\tan(\phi_1) = \frac{\sigma}{2-\sigma} = \frac{\sqrt{3}}{3}$. $r\left(\frac{\pi}{6}\right) = r_1 = \sigma$. Therefore the constant of the logarithmic spiral is $\kappa = -\frac{6}{\pi}\log(\sigma) \approx -0.5956953531$. This corresponds to $\text{arccot}(-\kappa) \approx 1.033548020$, corresponding to about $59.216°$. To summarize:

Proposition 6: Logarithmic Spiral for non-negative k

1) In the coordinate system $(X,\, Y)$ of the logarithmic spiral with origin S and $X = -y_0 + 1$, $Y = x_0$ the spokes $Sp_k = \overline{S,\, O_k}$ have lengths $r_k = \sigma^k$. The angles ϕ_k are obtained by $\sin(\phi_k) = (O_k)_{x_0}\,\sigma^{-k}$ where $\sigma^{-k} = \left(\frac{\tau}{2}\right)^k = a_{-k} + b_{-k}\sqrt{3}$, where $\tau = 1 + \sqrt{3} = -\overline{\sigma}$ and $a_{-k} =$ A002531$(k)/2^{\lfloor\frac{k+1}{2}\rfloor}$, $b_{-k} =$ A002530$(k)/2^{\lfloor\frac{k+1}{2}\rfloor}$ for $k = 0, 1, \ldots$. *I.e.*, $\{\sin(\phi_k)\}_{k=0}^{\infty} =$ repeat $\left(0,\, \frac{1}{2},\, \frac{1}{2}\sqrt{3},\, 1,\, \frac{1}{2}\sqrt{3},\, \frac{1}{2},\, 0,\, -\frac{1}{2},\, -\frac{1}{2}\sqrt{3},\, -1,\, -\frac{1}{2}\sqrt{3},\, -\frac{1}{2}\right)$. The first period applies to the first revolution of the spiral (sheet S_1 in the complex plane). The corresponding angles are for the N-th revolution (sheet S_N in the complex plane) $\phi_k = 2\,\pi\,(N-1) + \frac{\pi}{6}\,k \pmod{12}$, *I.e.*, an addition of $\frac{\pi}{6}$ or $30°$ from spoke Sp_k to Sp_{k+1} for each $k = 0, 1, \ldots$. The periodicity modulo 12 is proved in part **6**).

2) In the coordinate system $(X,\, Y)$ with origin S the hexagon centers are $LS(\phi_k) = Z_k = \sigma^k\, exp(i\,\phi_k) = (a_k + b_k\sqrt{3}\,)\, exp(i\frac{\pi}{6}\,k)$, for $k = 0, 1, \ldots$. This becomes with the help of the *de Moivre* formula, expressed in terms of *Chebyshev*'s S polynomials evaluated at $\sqrt{3}$:

$$\begin{aligned} Z_k \;=\; & \frac{1}{2}\Big((3\,b_k\,S_{k-1}(\sqrt{3}\,) - 2\,a_k\,S_{k-2}(\sqrt{3}\,)) + (a_k\,S_{k-1}(\sqrt{3}\,) - 2\,b_k\,S_{k-2}(\sqrt{3}\,))\sqrt{3} + \\ & (a_k + b_k\sqrt{3}\,)\,S_{k-1}(\sqrt{3}\,)\,i\Big) = (O_k)_X + (O_k)_Y\,i\,, \end{aligned} \tag{27}$$

where a_k and b_k have been given in *Lemma 2*, and *Chebyshev*'s $S_n(\sqrt{3}\,)$ polynomials entered in connection with eq. (4). See *Table 6* for the *Cartesian* coordinates $((O_k)_X,\,(O_k)_Y)$ for $k\,=\,0,\,1,\,...,\,12$.

3) The curvature $K(\phi)$ of the logarithmic spiral $r(\phi)\,=\,exp(-\kappa\,\phi)$ is itself a logarithmic spiral

$$K(\phi)\;=\;\frac{1}{\sqrt{1\,+\,\kappa^2}}\,exp(+\kappa\,\phi)\quad\text{with}\quad\kappa\;=\;-\frac{6}{\pi}\,log(\sigma)\;. \tag{28}$$

$\kappa\approx-0.5956953531$ and $K(0)\;=\;\frac{1}{\sqrt{1+\kappa^2}}\approx 0.8591201770.$

4) The conformal map $W(Z)$ and its inverse $W^{[-1]}$ in the S-system are for $Z\,\in\,\overline{\mathbb{C}}$ given by

$$W(Z)\;=\;\frac{1}{2}\,\Big((3\,-\,\sqrt{3}\,)\,+\,(-1\,+\,\sqrt{3}\,)\,i\Big)\;Z\;=\;a\,Z, \tag{29}$$

$$W^{[-1]}(Z)\;=\;\frac{1}{4}\,\Big((3\,+\,\sqrt{3}\,)\,-\,(1\,+\,\sqrt{3}\,)\,i\Big)\;Z\;=\;a^{-1}\,Z\,. \tag{30}$$

5) The relation between the conformal maps w and W is

$$W(Z)\;=\;i\,w(z(Z))\,+\,1,\quad\text{or}\quad w(z)\;=\;i\,(1\,-\,W(Z(z)), \tag{31}$$

$$\text{with}\quad z(Z,\overline{Z})=z(Z)=i\,(1-Z),\quad\text{or}\quad Z(z)=1\,+\,i\,z\,. \tag{32}$$

6) Periodicity modulo 12 up to scaling for Z_k:

$$Z_{k\,+\,12\,l}\;=\;\sigma^{12\,l}\,Z_k\,,\quad\text{for}\quad k\,\in\,\mathbb{N}_0,\;l\,\in\,\mathbb{N}_0\;. \tag{33}$$

7) Periodicity modulo 12 up to scaling and translation for z_k:

$$z_{k\,+\,12\,l}\;=\;\sigma^{12\,l}\,z_k\,+\,i\,(1\,-\,\sigma^{12\,l}),\quad\text{for}\quad k\,\in\,\mathbb{N}_0,\;l\,\in\,\mathbb{N}_0\;. \tag{34}$$

Proof:

1) The length ratio of the spokes is clear: S is the intersection of all circles C_k, for $k\,=\,0,\,1,\,...\,,$ and O_k is the center of C_k. The periodicity modulo 12 of the angles ϕ_k follows conjecturally from the $\sin(\phi_k)$ formula if the x_0 component of O_k from eq. (17) is inserted. Later, under part **6**), this is proved. The values for the first revolution then show that in general $\phi_{k+1}\,=\,\phi_k\,+\,\frac{\pi}{6}$. One has to take into account the quadrants when interpreting the angles from the $\sin(\phi_k)$ result.

2) This uses a standard reformulation of the trigonometric quantities obtained from the *de Moivre* formula in terms of *Chebyshev*'s polynomials (they are the circular harmonics). The powers of σ have already been treated in *Lemma 2*.

3) The formula for the curvature K of a curve in two-dimensional polar coordinates $r\,=\,r(\phi)$ is $K(\phi)\,=\,\frac{r^2\,+\,2\,r'^{\,2}\,-\,r\,r''}{(r^2\,+\,r'^{\,2})^{3/2}}$, *e.g.*, [1]. As explained in the preamble to this *Proposition* the logarithmic spiral is $r(\phi)\,=\,exp(-\kappa\,\phi)$, and with $r_1\,=\,r\left(\frac{\pi}{6}\right)\,=\,\sigma$ one determines the constant $-\kappa$. The curvature K becomes itself a logarithmic spiral with $K(0)\,=\,\frac{1}{\sqrt{1\,+\,\kappa^2}}$ and the constant $+\kappa$.

4) Like for the conformal map w, the unique *Möbius* transformation W which maps the points $(S\,=\,0,\,Z_0,\,Z_1)$ to $(S,\,Z_1,\,Z_2)$ is obtained by solving the double quotient equation $DQ(0,\,Z_0,\,Z_1,\,Z)\,=\,DQ(0,\,Z_1,\,Z_2,\,W)$ for $W\,=\,W(Z)$. The real and imaginary parts of Z_k, for $k\,=\,0,\,1,\,...,\,12$ are shown in *Table 6* as $(O_k)_X$ and $(O_k)_Y$. In general $W(Z_k)\,=\,Z_{k+1}$, for $k\,=\,0,\,1,\,...$. The same a as in eq. (24) appears. The inverse map $W^{[-1]}$ satisfies $W^{[-1]}(W(Z))\,=\,Z$, identically. Note that, in contrast to w, the map W, hence $W^{[-1]}$, is linear.

5) The coordinate transformation $X = 1 - y_0$ and $Y = x_0$ leads for $z = x_0 + y_0\, i$ and $Z = X + Y\, i$ to $z(Z, \overline{Z}) = \dfrac{Z - \overline{Z}}{2\,i} + \left(1 - \dfrac{Z + \overline{Z}}{2}\right) i = i\,(1 - Z) + 0\,\overline{Z} = i\,(1 - Z) = z(Z)$. With $w(z) = a\,z + (1-a)\,i$ from eq. (24) , one obtains $w(z(Z)) = a\,(1-Z)\,i + (1-a)\,i = i\,(1 - a\,Z) = i\,(1 - W(Z))$. *I.e.*, $W(Z) = i\,w(z(Z)) + 1$. Or, with $Z(z) = 1 + z\,i$, $W(Z(z)) = i\,(a\,z + b) + (-i\,b + a) = i\,w(z) + 1$, because $a - i\,b = 1$. Therefore, $w(z) = i\,(1 - W(Z(z))$.

6) The linearity of W means that $W^{[p]}(Z) = a^p\,Z$ for the p-fold iterated map W for $Z \in \overline{\mathbb{C}}$. Now , with $Z_0 = 1$, one has $Z_{k+12\,l} = W^{[k+12\,l]}(1) = W^{[12\,l]}(W^{[k]}(1)) = W^{[12\,l]}(Z_k)$ By linearity this is $a^{12\,l}\,Z_k = (\sigma^{12})^l\,Z_k$. Here $a^{12} = \sigma^{12}$ even though $a \neq \sigma$. This follows from $Z_{12} = W^{[12]}(1) = a^{12}\,1 = a^{12}$, and by computation (see the last two columns of *Table 6*) $Z_{12} = 86464 - 49920\,\sqrt{3} + 0\,i = \sigma^{12}$ by the first column of this *Table*.

7) This periodicity modulo 12 up to scaling translates into a periodicity modulo 12 up to translation and scaling for the centers z_k of the circles C_k in the coordinate system (x_0, y_0) due to the transformation given in part **5)** applied to these centers, *viz*, $z_k(Z_k) = i\,(1 - Z_k)$ for $k \in \mathbb{N}_0$. Therefore, $z_{k+12\,l} = i\,(1 - Z_{k+12\,l}) = i\,(1 - \sigma^{12\,l}\,Z_k)$ from part **4)**. With $Z = Z_k(z_k) = 1 + z_k\,i$ this becomes $z_{k+12\,l} = \sigma^{12\,l}\,z_k + i\,(1 - \sigma^{12\,l})$. □

4 Hexagon Ascent

It is straightforward to continue the discrete spiral and its interpolations to negative k values. In the coordinate system $(x_0,\, y_0)$ with origin $O_0 = 0$ the vectors $\vec{v}_{-k} = \overrightarrow{O_{-(k+1)},\, O_{-k}}$ have polar coordinates following from extending eq. (1) .

$$\vec{v}_{-k} \doteq v_{-k}\begin{pmatrix}\cos\,\alpha_{-k}\\ \sin\,\alpha_{-k}\end{pmatrix},\ \ \text{with}\ \ v_{-k} = \sigma^{-k}\,\frac{\sqrt{2}}{2},\ \ \text{with}\ \ \alpha_{-k} = (1 - 2\,\mathrm{k})\,\frac{\pi}{12}\ \text{for k} \in \mathbb{N}_0, \tag{35}$$

$$v_{-k} = (a_{-k} + b_{-k}\,\sqrt{3}\,)\,\frac{\sqrt{2}}{2},\ \ \text{where}\ \ a_{-k} = \text{A002531}(k)/2^{\lfloor\frac{k+1}{2}\rfloor},\ \text{and}\ b_{-k} = \text{A002530}(k)/2^{\lfloor\frac{k+1}{2}\rfloor}\,.$$

σ^{-k} appeared already in *Proposition 5*, part 1). See also the second column of *Table 3* for $\{a_{-k},\, b_{-k}\}$ for $k = 0,\, 1,\, ...,\, 12$.

This can be written as

$$\vec{v}_{-k} = (\sigma\,\mathbf{R}^{-1})^{k+1}\,\vec{v}_1,\ \ \text{with}\ \ \mathbf{R}^{-1} \doteq \frac{1}{2}\begin{pmatrix}\sqrt{3} & 1\\ -1 & \sqrt{3}\end{pmatrix},\ \ \text{for}\ \ k \in \mathbb{N}_0\,. \tag{36}$$

For $\vec{v}_1$ and $\mathbf{R}$ see eq. (2) . *E.g.*, $\vec{v}_0 \doteq \frac{1}{4}\begin{pmatrix}\tau\\ \sigma\end{pmatrix}$.

The formula eq. (4) can be used to obtain $\mathbf{R^{-k}}$ with the *Chebyshev* polynomials $S_{-n}(x) = -S_{n-2}(x)$, for $n \in \mathbb{N}_0$, with $S_{-1}(x) = 0$.

$$\mathbf{R}^{-k} = -S_{k-1}(\sqrt{3}\,)\,\mathbf{R} + S_k(\sqrt{3}\,)\,\mathbf{1}_2,\ \ \text{for}\ \ k = 0,\, 1,\, 2,\, ...\,. \tag{37}$$

The components of $\vec{v}_{-k}$ can be computed from this. Similarly to *Corollary 1* these vectors are periodic modulo 12 up to scaling:

Corollary 6 = 1′: $\vec{v}_{-k}$ periodicity up to scaling

$$\vec{v}_{-(k+12\,l)} = (\sigma)^{-12\,l}\,\vec{v}_{-k} = \left(\frac{\tau}{2}\right)^{12\,l}\,\vec{v}_{-k},\ \ \text{for}\ \ k \in \mathbb{N}_0,\ l \in \mathbb{N}_0\,. \tag{38}$$

In order to obtain components of $\vec{v}_{-k}$ which are integers in the real quadratic number field $\mathbb{Q}(\sqrt{3}\,)$ the largest denominator $2^{s(k)}$ with $s(k)$ = A300068(k) has been multiplied. This sequence $\{s(k)\}_{k\,\geq\,0}$ is obtained from the periodic sequence A300067, repeat(0, 0, 0, 1, 2, 2,).

Lemma 4: Sequence s

The formula for the members of sequence s and its *o.g.f.* is

$$s(k) \,=\, 2 \,+\, \left\lfloor\frac{k \pmod 6}{3}\right\rfloor + \left\lfloor\frac{k \pmod 6}{4}\right\rfloor + 3\left\lfloor\frac{k}{6}\right\rfloor, \quad \text{for} \quad k \,\in\, \mathbb{N}_0\,,$$

$$\text{O.g.f.}: \quad G(x) \,=\, \frac{2 \,+\, x^3 \,+\, x^4 \,-\, x^6}{(1 \,-\, x^6)\,(1 \,-\, x)}\,. \tag{39}$$

Proof:

Due to the periodicity up to scaling (*Corollary 6*) it is sufficient to consider $s(k)$, for $k \,=\, 0,\, 1,\, ...,\, 11$. These values are given from the first twelve vectors $\vec{v}_{-k}$ by the second column of *Table 7* with the first six members 2, 2, 2, 3, 4, 4, and the other six ones are obtained by adding 3 to each member. The scaling factor $\sigma^{-12\,l}$ (see *Proposition 6*, part 1) and $\mathbf{r}_{-k}$ in *Table 6*) has the denominator $2^{\left\lfloor\frac{12\,l+1}{2}\right\rfloor} \,=\, 2^{6\,l}$ because gcd(A002531(k), A002530(k)) = 1 due to the fact that they are denominators and numerators in lowest terms of fractions (they give the continued fraction convergents of $\sqrt{3}$). Therefore, for each period of length 12 a new factor 2^6 has to be multiplied, which means for the exponents that $s(k + 12\,l) \,=\, 6\,l\,s(k)$. Because in the first period 3 is added to the first six entries of s this results in a period of length 6 and the periodicity up to scaling formula for s becomes $s(k \,+\, 6\,l) \,=\, 3\,l\,s(k)$. This explains the last term in the explicit formula for s. The second and third terms result from A300067, repeat(0, 0, 0, 1, 2, 2,), and the 2 has then to be added to produce the first six entries of the sequence s. The *o.g.f.* of $\{s(k) \,-\, 2\}_{k\,\geq\,0}$ is found from the obvious ones of A300067 and $3\,\left\lfloor\frac{k}{6}\right\rfloor$. ☐

For the scaled vectors components $\vec{v}_{-k}$, for $k \,=\, 0,\, 1,\, ...,\, 12$, see *Table 7*.

The centers O_{-k} are then given by

$$\vec{O}_{-k} \,=\, \overrightarrow{O_0,\, O_{-k}} \,=\, -\sum_{j\,=\,0}^{k-1}\,\vec{v}_{-j}\,, \quad \text{for} \quad k \,\in\, \mathbb{N}\,, \text{ and } \vec{O}_{-0} \,=\, \vec{0}\,. \tag{40}$$

Again, some scaling $2^{t(k)}$ is applied to obtain integers in $\mathbb{Q}(\sqrt{3}\,)$ for the components of $\vec{O}_{-k}$. For $k \,=\, 0$, the zero-vector $\vec{0}$, no scaling is needed and $t(0) \,=\, 0$. The above reasoning for sequence s does not apply immediately because O_{-k}, like O_k, is not periodic up to scaling, but in the y_0 component also a translation appears (for O_k, in the complex plane called z_k, see the *Proposition 6*, part 7)). Later, in *Proposition 9*, part 5), it will be seen that for Z_{-k}, in the coordinate system $(X,\, Y)$ with origin S, the same sequence t is used to obtain integers in $\mathbb{Q}(\sqrt{3}\,)$ for the real and imaginary parts of $2^{t(k)}\,Z_{-k}$. Then by the coordinate transformation $x_0 \,=\, Y \,=\, \Im(Z)$ and $y_0 \,=\, 1 \,-\, X \,=\, 1 \,-\, \Re(Z)$ this will imply integer coordinates in $\mathbb{Q}(\sqrt{3}\,)$ also for O_{-k}. It is therefore again sufficient to consider $t(k)$ for the first period $k \,=\, 1,\, 2,\, ...,\, 12$. These values are given in the fifth column of *Table 7* as 2, 2, 2, 3, 4, 3, and the next six numbers are obtained by adding 3 to these members. This results in the following formula based on the period length 6 sequence A300069, repeat(0, 0, 0, 1, 2, 1,) (but there the offset is 0, not 1).

Lemma 5: Sequence t

The formula for the members of sequence t and its *o.g.f.* is

$$t(0) \,=\, 0, \quad \text{and}$$

$$t(k) \,=\, 2 \,+\, \left\lfloor\frac{k-1 \pmod 6}{3}\right\rfloor + \left\lfloor\frac{k \pmod 6}{5}\right\rfloor + 3\left\lfloor\frac{k-1}{6}\right\rfloor \,=\, 2 \,+\, \text{A174257}(k), \quad \text{for} \quad k \,\in\, \mathbb{N}\,.$$

$$\text{O.g.f.}: \quad G(x) \,=\, \frac{x\,(2 \,+\, 2\,x \,-\, x^3)}{(1 \,+\, x \,-\, x^3 \,-\, x^4)\,(1 \,-\, x)}\,. \tag{41}$$

The proof is analogous to the one of the preceding *Lemma 4* but the different offset has to be taken into account.

For the scaled vectors components $2^{t(k)}\,\vec{O}_{-k}$, for $k\,=\,0,\,1,\,...,\,12$, see *Table 7*.
The square of the lengths $2^k\,\rho_{-k}^2$ are given in *Table 5*.

The vertices of the hexagons H_{-k}, for $k\,\in\,\mathbb{N}_0$, are given in the obvious extension of *Proposition 2* with $\sigma^{-1}\,=\,\frac{\tau}{2}$ as follows.

Proposition 7: Vertices of hexagons $H_{-k},\ k\,\in\,\mathbb{N}_0$,

$$\vec{V}_{-k}(j)\,=\,\vec{O}_{-k}\,+\,\left(\frac{\tau}{2}\right)^k\,\mathbf{R}^{-k+2j}\begin{pmatrix}1\\0\end{pmatrix},\quad\text{for}\ \ k\,=\,0,\,1,\,...\,,\ \ \text{and}\ \ j\,=\,0,\,1,\,...,\,5\,. \tag{42}$$

In order to obtain integers in $\mathbb{Q}(\sqrt{3}\,)$ after some scaling of the components of $\vec{V}_{-k}(j)$ it turns out that one needs only the three scaling sequences $2^{v0(k)}$, $2^{v1(k)}$, $2^{v2(k)}$ for $\vec{V}_{-k}(0)$, $\vec{V}_{-k}(1)$, $\vec{V}_{-k}(2)$, which also work for $\vec{V}_{-k}(3)$, $\vec{V}_{-k}(4)$, $\vec{V}_{-k}(5)$, respectively. Again it is sufficient for the sequences $v0$, $v1$ and $v2$ to concentrate on the first six entries besides the values for $k=0$ (the original hexagon H_0) which are 0, 1 and 1, respectively (for $\vec{V}_0(0)$ see *Table 2* for $k\,=\,0$ which does not need a scaling). The other six values are obtained by adding 3, and for each new period of length 12 (starting with $k=1$) another 3 is added. We skip the proof (see the one for the sequence t which is similar), and give the results for these three sequences.

Lemma 6: Sequences $v0$, $v1$, $v2$

$$\begin{aligned}
v0(0)\,&=\,0,\quad\text{and}\\
v0(k)\,&=\,1\,+\,\left\lfloor\frac{k\pmod 6}{2}\right\rfloor\,+\,2\left\lfloor\frac{(k-1)\pmod 6}{5}\right\rfloor\,+\,3\left\lfloor\frac{k-1}{6}\right\rfloor\\
&=\,1\,+\,A300076(k-1),\quad\text{for}\ \ k\,\in\,\mathbb{N}\,. &&(43)\\
v0(k)\,&=\,A300068(k+2),\quad\text{for}\ \ k\,\in\,\mathbb{N}_0.\\
\text{O.g.f.}:\ \ G0(x)\,&=\,\frac{x\,(1\,+\,x\,+\,x^3)}{(1\,-\,x^6)\,(1\,-\,x)}\,. &&(44)\\
v1(0)\,&=\,1,\quad\text{and}\\
v1(k)\,&=\,1\,+\,(k-1)\pmod 6\,-\,\left\lfloor\frac{(k-1)\pmod 6}{3}\right\rfloor\,-\,\left\lfloor\frac{(k-1)\pmod 6}{5}\right\rfloor\,+\,3\left\lfloor\frac{k-1}{6}\right\rfloor\\
&=\,1\,+\,A300068(k+1),\quad\text{for}\ \ k\,\in\,\mathbb{N}\,. &&(45)\\
\text{O.g.f.}:\ \ G1(x)\,&=\,\frac{1\,+\,x^2\,+\,x^3\,+\,x^5\,-\,x^6}{(1\,-\,x^6)\,(1\,-\,x)}\,. &&(46)\\
v2(0)\,&=\,1,\quad\text{and}\\
v2(k)\,&=\,2\,+\,2\left\lfloor\frac{(k-1)\pmod 6}{5}\right\rfloor\,+\,\left\lfloor\frac{k\pmod 6}{3}\right\rfloor\,+\,3\left\lfloor\frac{k-1}{6}\right\rfloor\\
&=\,2\,+\,A300293(k-1),\quad\text{for}\ \ k\,\in\,\mathbb{N}\,. &&(47)\\
\text{O.g.f.}:\ \ G2(x)\,&=\,\frac{1\,+\,x\,+\,x^3}{(1\,-\,x^6)\,(1\,-\,x)}\,. &&(48)
\end{aligned}$$

The *o.g.f.*s show that $v2(k)\,=\,v0(k+1)$, for $k\,\in\,\mathbb{N}_0$.

The discrete hexagon spiral with points O_{-k} can again be interpolated by circular arcs A_{-k} between O_{-k} and O_{-k+1}. The centers of the circles are $\hat{C}_{-k}\,=\,V_{-k}(2)$ and the radius is $r_{-k}\,=\,\sigma^{-k}\,=\,\left(\frac{\tau}{2}\right)^k$ (see *Table 6* for $2^{\frac{k+1}{2}}\,r_{-k}$. The precise statement is given in

Proposition 8: Interpolating circular arcs A_{-k}, $k \in \mathbb{N}_0$

The circular arcs A_{-k} interpolation between the centers O_{-k} and O_{-k+1} of the discrete hexagon spiral are, for $k \in \mathbb{N}$ given by

$$A_{-k} = \text{arc}\left(V_{-k}(2),\, r_{-k},\, \frac{-(k+2)\,\pi}{6},\, \frac{-(k+1)\,\pi}{6}\right). \tag{49}$$

In *Figure 6* this interpolation by arcs is shown in dashed blue (almost coinciding with the later discussed logarithmic spiral shown there in solid red).

Proof:

This is simply the generalization of eq. (26) for negative k. The angle $-\frac{2\pi}{6}$, the first angle for A_0 becomes the second angle for A_{-1} and then $-\frac{\pi}{6}$ has to be added in order to obtain the first angle. This continues for each step $A_{-k} \to A_{-(k+1)}$. □

Proposition 9: Logarithmic Spiral for non-positive k

1) The centers of the circles C_{-k} are

$$Z_{-k} = (W^{[-1]})^{[k]}(1) = (a^{-1})^k, \text{ for } k \in \mathbb{N}_0, \text{ and } a_{-1} = \frac{\tau}{2}\, e^{-i\,k\,\frac{\pi}{6}}. \tag{50}$$

2) The spokes $Sp_k = \overline{S\,Z_{-k}}$ have lengths $\left(\frac{\tau}{2}\right)^k$ and the angles $\phi_{-k} = -k\,\frac{\pi}{6}$, for $k \in \mathbb{N}_0$. For $\sigma^{-k} = \left(\frac{\tau}{2}\right)^k$ see *Proposition 6*, part 1).

3) The explicit form, using *de Moivre*'s formula expressed in terms of the *Chebyshev*'s S polynomials with negative index $S_{-n}(x) = -S_{n-2}(x)$ is like eq. (27) with $k \to -k$:

$$\begin{aligned} Z_{-k} \;&=\; \frac{1}{2}\Big((-3\,b_{-k}\,S_{k-1}(\sqrt{3}\,) + 2\,a_{-k}\,S_k(\sqrt{3}\,)) + (-a_{-k}\,S_{k-1}(\sqrt{3}\,) + 2\,b_{-k}\,S_k(\sqrt{3}\,))\,\sqrt{3}\, - \\ &\quad (a_{-k} + b_{-k}\,\sqrt{3}\,)\,S_{k-1}(\sqrt{3}\,)\,i\Big). \end{aligned} \tag{51}$$

4) The logarithmic spiral in the complex plane

$$LS(\phi) = e^{(-\kappa+i)\,\phi}, \text{ with } \kappa = -\frac{\pi}{6}\,\log(\sigma). \tag{52}$$

interpolates between all points Z_k for $k \in \mathbb{Z}$.

5) Periodicity modulo 12 up to scaling for Z_{-k}:

$$Z_{-(k+12\,l)} = \left(\frac{\tau}{2}\right)^{12\,l} Z_{-k}, \text{ for } k \in N_0,\, l \in \mathbb{N}_0. \tag{53}$$

Therefore one has eq. (33) with $k \in \mathbb{Z}$ and $l \in \mathbb{Z}$.

Proof:

1) With the map $W^{[-1]}$ from Proposition 6, eq. (30), with a^{-1} from eq. (25), the hexagon centers O_{-k}, in the complex plane denoted by Z_{-k}, satisfy

$$Z_{-k} = W^{[-1]}(Z_{k-1}) = W^{[-k]}(Z_0) = W^{[-k]}(1) = (a^{-1})^k = a^{-k}, \quad \text{for } k \in \mathbb{N}_0. \tag{54}$$

2) This is clear from part **1**).

3) This is also clear, repeating the steps which led to *Proposition 6*, part **2)**, and the rewriting of S polynomials with negative index, as given.

4) The logarithmic spiral, by construction of the maps W and $W^{[-1]}$, interpolates between all hexagon centers Z_k, for $k \in \mathbb{Z}$.

5) The periodicity up to scaling is obvious from part **1)**. □

References

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Keywords: Conformal Mapping, Hexagon, Spiral

Concerned with OEIS sequences A002530, A002531, A002605, A019892, A026150, A049310, A057079, A174257, A300067, A300068, A300069, A300076, A300293.

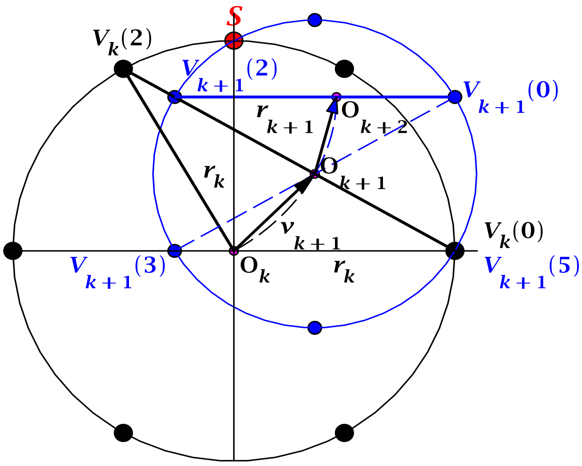

Figure 1

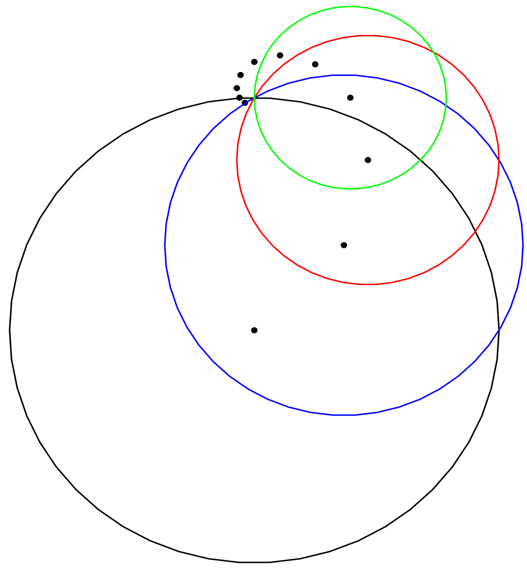

Figure 2

Figure 1: Construction $H_k \to H_{k+1}$: $C_k(O_k,\, r_k)$, $V_k(0)$, $(x_k,\, y_k), D_k = \overline{V_k(0),\, V_k(2)}$, $\overline{V_k(2),\, O_{k+1}} = r_k$, $C_{k+1}(O_{k+1},\, r_{k+1} = \sigma^k$, $V_{k+1}(3)$, $V_{k+1}(0)$, $(x_{k+1},\, y_{k+1}), \ldots$

Figure 2: The first four circles and the first eleven centers.

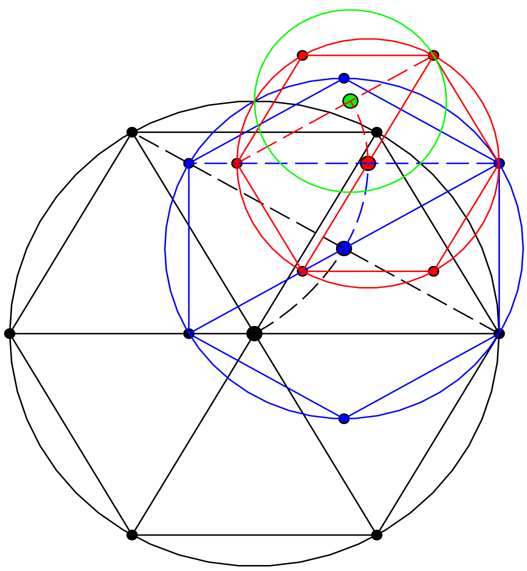

Figure 3

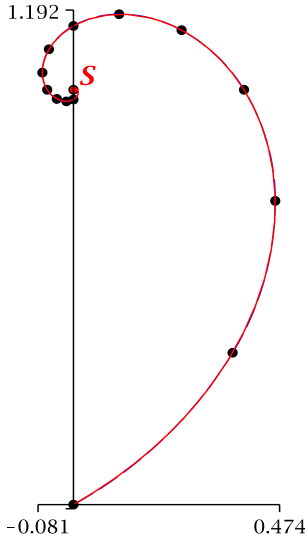

Figure 4

Figure 3: The first three hexagons and the first four circles.

Figure 4: The discreet hexagon spiral of the first 13 centers. The interpolating circular arcs (dashed blue) and the logarithmic spiral (solid red) are almost indistinguishable).

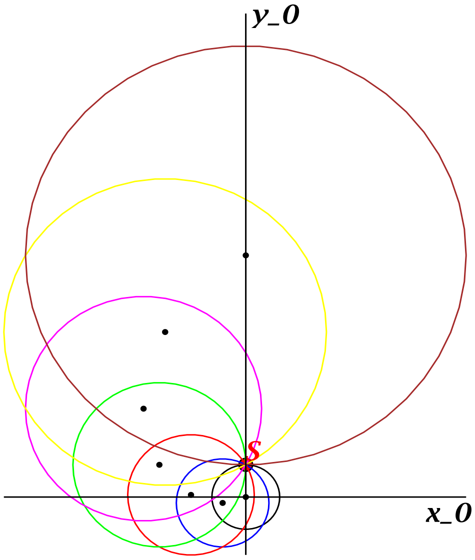

Figure 5

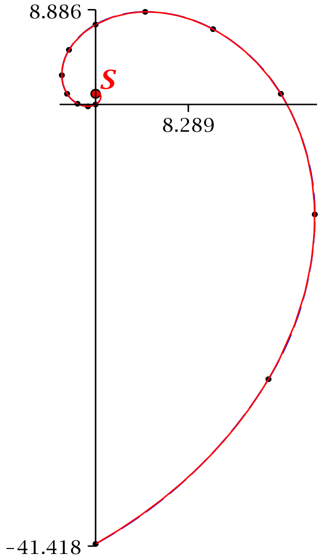

Figure 6

Figure 5: The circle $C_0(0,\, 1)$ and the first six circles $C_{-k}(O_{-k},\, \sigma^{-k})$, for $k\, =\, 1,\, 2,\, ...,\, 6$.

Figure 6: The fixed point S, the center $O_0\, =\, 0$, the first 12 centers O_{-k} with $k\, =\, 1,\, 2,\, ...,\, 12$. The interpolating circular arcs (dashed blue) and the logarithmic spiral (solid red) are almost indistinguishable.

In the following tables all length have been divided by r_0.

Table 1

k	$(\overrightarrow{v}_k)_{x_0}$ $\cdot\mathbf{1}\ ,\ \cdot\sqrt{\mathbf{3}}$	$(\overrightarrow{v}_k)_{y_0}$ $\cdot\mathbf{1}\ ,\ \cdot\sqrt{\mathbf{3}}$	$(O_k)_{x_0}$ $\cdot\mathbf{1}\ ,\ \cdot\sqrt{\mathbf{3}}$	$(O_k)_{y_0}$ $\cdot\mathbf{1}\ ,\ \cdot\sqrt{\mathbf{3}}$
1	$-1/2\ ,\ 1/2$	$-1/2\ ,\ 1/2$	$-1/2\ ,\ 1/2$	$-1/2\ ,\ 1/2$
2	$-5/2\ ,\ 3/2$	$-1/2\ ,\ 1/2$	$-3\ ,\ 2$	$-1\ ,\ 1$
3	$-7\ ,\ 4$	$2\ ,\ -1$	$-10\ ,\ 6$	$1\ ,\ 0$
4	$-14\ ,\ 8$	$14\ ,\ -8$	$-24\ ,\ 14$	$15\ ,\ -8$
5	$-14\ ,\ 8$	$52\ ,\ -30$	$-38\ ,\ 22$	$67\ ,\ -38$
6	$38\ ,\ -22$	$142\ ,\ -82$	$0\ ,\ 0$	$209\ ,\ -120$
7	$284\ ,\ -164$	$284\ ,\ -164$	$284\ ,\ -164$	$493\ ,\ -284$
8	$1060\ ,\ -612$	$284\ ,\ -164$	$1344\ ,\ -776$	$777\ ,\ -448$
9	$2896\ ,\ -1672$	$-776\ ,\ 448$	$4240\ ,\ -2448$	$1\ ,\ 0$
10	$5792\ ,\ -3344$	$-5792\ ,\ 3344$	$10032\ ,\ -5792$	$-5791\ ,3344$
11	$5792\ ,\ -3344$	$-21616\ ,\ 12480$	$15824\ ,\ -9136$	$-27407\ ,15824$
12	$-15824\ ,\ 9136$	$-59056\ ,\ 34096$	$0\ ,\ 0$	$-86463\ ,49920$
...				

Table 2

k	$(\overrightarrow{V}_k(0))_{x_0}$ $\cdot\mathbf{1}\ ,\ \cdot\sqrt{\mathbf{3}}$	$(\overrightarrow{V}_k(0))_{y_0}$ $\cdot\mathbf{1}\ ,\ \cdot\sqrt{\mathbf{3}}$	$\rho_k^2 = \left\lvert\overrightarrow{O_0, O_k}\right\rvert^2$ $\cdot\mathbf{1}\ ,\ \cdot\sqrt{\mathbf{3}}$	$tan\,\hat{\varphi}_k$ $\cdot\mathbf{1}\ ,\ \cdot\sqrt{\mathbf{3}}$
0	$1\ ,\ 0$	$0\ ,\ 0$	$0\ ,\ 0$	$0\ ,\ 0$
1	$1\ ,\ 0$	$-1\ ,\ 1$	$2\ ,\ -1$	$1\ ,\ 0$
2	$-1\ ,\ 1$	$-4\ ,\ 3$	$25\ ,\ -14$	$1\ ,\ 1/3$
3	$-10\ ,\ 6$	$-9\ ,\ 6$	$209\ ,\ -120$	$5/4\ ,\ 3/4$
4	$-38\ ,\ 22$	$-9\ ,\ 6$	$1581\ ,\ -912$	$2\ ,\ 3/2$
5	$-104\ ,\ 60$	$29\ ,\ -16$	$11717\ ,\ -6764$	$19/4\ ,\ 15/4$
6	$-208\ ,\ 120$	$209\ ,\ -120$	$87881\ ,\ -50160$	∞
7	$-208\ ,\ 1204$	$777\ ,\ -448$	$646361\ ,\ -373176$	$-71/8\ ,\ -49/8$
8	$568\ ,\ -328$	$2121\ ,\ -1224$	$4818705\ ,\ -2782080$	$-7\ ,\ -35/8$
9	$4240\ ,\ -2448$	$4241\ ,\ -24488$	$35955713\ ,\ -20759040$	$-265/32\ ,\ -153/32$
10	$15824\ ,\ -9136$	$4241\ ,\ -2448$	$268365505\ ,\ -154940896$	$-209/16\ ,-173/24$
11	$43232\ ,\ -24960$	$-11583\ ,\ 6688$	$2003139041\ ,\ -1156512864$	$-989/32\ ,-539/32$
12	$86464\ ,\ -49920$	$-86463\ ,\ 49920$	$14951869569\ ,-8632465920$	∞
...				

Table 3

k	$(\overrightarrow{V}_k(1))_{x_0}$ $\cdot\mathbf{1}\ ,\ \cdot\sqrt{\mathbf{3}}$	$(\overrightarrow{V}_k(1))_{y_0}$ $\cdot\mathbf{1}\ ,\ \cdot\sqrt{\mathbf{3}}$	$(\overrightarrow{V}_k(2))_{x_0}$ $\cdot\mathbf{1}\ ,\ \cdot\sqrt{\mathbf{3}}$	$(\overrightarrow{V}_k(2))_{y_0}$ $\cdot\mathbf{1}\ ,\ \cdot\sqrt{\mathbf{3}}$
0	1/2 , 0	0 , 1/2	−1/2 , 0	0 , 1/2
1	−1/2 , 1/2	−3/2 , 3/2	−2 , 1	−1 , 1
2	−5 , 3	−4 , 3	−7 , 4	−1 , 1
3	−19 , 11	−4 , 3	−19 , 11	6 , −3
4	−52 , 30	15 , −8	−38 , 22	39 , −22
5	−104 , 60	105 , −60	−38 , 22	143 , −82
6	−104 , 60	389 , −224	104 , −60	389 , −224
7	284 , −164	1061 , −612	776 , −448	777 , −448
8	2120 , −1224	2121 , −1224	2896 , −1672	777 , −448
9	7912 , −4568	2121 , −1224	7912 , −4568	−2119 , 1224
10	21616 , −12480	−5791 , 3344	15824 , −9136	−15823 , 9136
11	43232 , −24960	−43231 , 24960	15824 , −9136	−59055 , 34096
12	43232 , −24960	−161343 , 93152	−43232 , 24960	−161343 , 93152
...				

Table 4

k	$(\overrightarrow{V}_k(3))_{x_0}$ $\cdot\mathbf{1}\ ,\ \cdot\sqrt{\mathbf{3}}$	$(\overrightarrow{V}_k(3))_{y_0}$ $\cdot\mathbf{1}\ ,\ \cdot\sqrt{\mathbf{3}}$	$(\overrightarrow{V}_k(4))_{x_0}$ $\cdot\mathbf{1}\ ,\ \cdot\sqrt{\mathbf{3}}$	$(\overrightarrow{V}_k(4))_{y_0}$ $\cdot\mathbf{1}\ ,\ \cdot\sqrt{\mathbf{3}}$
0	−1 , 0	0 , 0	−1/2 , 0	0 , −1/2
1	−2 , 1	0 , 0	−1/2 , 1/2	1/2 , −1/2
2	−5 , 3	2 , −1	−1 , 1	2 , −1
3	−10 , 6	11 , −6	−1 , 1	6 , −3
4	−10 , 6	39 , −22	4 , −2	15 , −8
5	28 , −16	105 , −60	28 , −16	29 , −16
6	208 , −120	209 , −120	104 , −60	29 , −16
7	776 , −448	209 , −120	284 , −164	−75 , 44
8	2120 , −1224	−567 , 328	568 , −328	−567 , 328
9	4240 , −2448	−4239 , 2448	568 , −328	−2119 , 1224
10	4240 , −2448	−15823 , 9136	−1552 , 896	−5791 , 3344
11	−11584 , 6688	−43231 , 24960	−11584 , 6688	−11583 , 6688
12	−86464 , 49920	−86463 , 49920	−43232 , 24960	−11583 , 6688
...				

Table 5

k	$(\overrightarrow{V}_k(5))_{x_0}$ $\cdot\mathbf{1}\ ,\ \cdot\sqrt{\mathbf{3}}$	$(\overrightarrow{V}_k(5))_{y_0}$ $\cdot\mathbf{1}\ ,\ \cdot\sqrt{\mathbf{3}}$	$2^k\,\rho_{-k}^2 \;=\; 2^k\left\|\overrightarrow{O_0,\,O_{-k}}\right\|^2$ $\cdot\mathbf{1}\ ,\ \cdot\sqrt{\mathbf{3}}$	$\cdot\mathbf{1}\ ,\ \cdot\sqrt{\mathbf{3}}$
0	1/2 , 0	0 , −1/2	0 , 0	0 , −1/2
1	1 , 0	0 , 0	1 , 0	1/2 , −1/2
2	1 , 0	−1 , 1	7 , 2	2 , −1
3	−1 , 1	−4 , 3	34 , 15	6 , −3
4	−10 , 6	−9 , 6	141 , 72	15 , −8
5	−38 , 22	−9 , 6	526 , 285	29 , −16
6	−104 , 60	29 , −16	1831 , 1020	29 , −16
7	−208 , 120	209 , −120	6154 , 3479	−75 , 44
8	−208 , 120	777 , −448	20625 , 11760	−567 , 328
9	568 , −328	2121 , 1224	70738 , 40545	−2119 , 1224
10	4240 , −2448	4241 , −2448	251527 , 144628	−5791 , 3344
11	15824 , −9136	4241 , −2448	925354 , 533071	−11583 , 6688
12	43232 , −24960	−11583 , 6688	3481569 , 2007720	−11583 , 6688
...				

Table 6

k	$r_k \;=\; \left\|\overrightarrow{S,\,O_k}\right\| \;=\; \sigma^k$ $\cdot\mathbf{1}\ ,\ \cdot\sqrt{\mathbf{3}}$	$2^{\left\lfloor\frac{k+1}{2}\right\rfloor}\,\mathbf{r}_{-k}$ $\cdot\mathbf{1}\ ,\ \cdot\sqrt{\mathbf{3}}$	$(O_k)_X$ $\cdot\mathbf{1}\ ,\ \cdot\sqrt{\mathbf{3}}$	$(O_k)_Y$ $\cdot\mathbf{1}\ ,\ \cdot\sqrt{\mathbf{3}}$
0	1 , 0	1 , 0	1 , 0	0 , 0
1	−1 , 1	1 , 1	3/2 , −1/2	−1/2 , 1/2
2	4 , −2	2 , 1	2 , −1	−3 , 2
3	−10 , 6	5 , 3	0 , 0	−10 , 6
4	28 , −16	7 , 4	−14 , 8	−24 , 14
5	−76 , 44	19 , 11	−66 , 38	−38 , 22
6	208 , −120	26 , 15	−208 , 120	0 , 0
7	−568 , 328	71 , 41	−492 , 284	284 , −164
8	1552 , −896	97 , 56	−776 , 448	1344 , −776
9	−4240 , 2448	265 , 153	0 , 0	4240 , −2448
10	11584 , −6688	362 , 209	5792 , −3344	10032 , −5792
11	−31648 , 18271	989 , 571	27408 , −15824	15824 , −9136
12	86464 , −49920	1351 , 780	86464 , −49920	0 , 0
...				

Table 7

k	$s(k)$	$2^{s(k)}\,(\overrightarrow{v}_{-k})_{x_0}$ $\cdot\mathbf{1}\ ,\ \cdot\sqrt{\mathbf{3}}$	$2^{s(k)}\,(\overrightarrow{v}_{-k})_{y_0}$ $\cdot\mathbf{1}\ ,\ \cdot\sqrt{\mathbf{3}}$	$t(k)$	$2^{t(k)}\,(O_{-k})_{x_0}$ $\cdot\mathbf{1}\ ,\ \cdot\sqrt{\mathbf{3}}$	$2^{t(k)}\,(O_{-k})_{y_0}$ $\cdot\mathbf{1}\ ,\ \cdot\sqrt{\mathbf{3}}$
0	2	$1\ ,\ 1$	$-1\ ,\ 1$	0	$0\ ,\ 0$	$0\ ,\ 0$
1	2	$2\ ,\ 1$	$-1\ ,\ 0$	2	$-1\ ,\ -1$	$1\ ,\ -1$
2	2	$2\ ,\ 1$	$-2\ ,\ -1$	2	$-3\ ,\ -2$	$2\ ,\ -1$
3	3	$2\ ,\ 1$	$-7\ ,\ -4$	2	$-5\ ,\ -3$	$4\ ,\ 0$
4	4	$-5\ ,\ -3$	$-19\ ,\ -11$	3	$-12\ ,\ -7$	$15\ ,\ 4$
5	4	$-19\ ,\ -11$	$-19\ ,\ -11$	4	$-19\ ,\ -11$	$49\ ,\ 19$
6	5	$-71\ ,\ -41$	$-19\ ,\ -11$	3	$0\ ,\ 0$	$34\ ,\ 15$
7	5	$-97\ ,\ -56$	$26\ ,\ 15$	5	$71,\ 41$	$155\ ,\ 714$
8	5	$-97\ ,\ -56$	$97\ ,\ 56$	5	$168\ ,\ 97$	$126\ ,\ 56$
9	6	$-97\ ,\ -56$	$382\ ,\ 2098$	5	$265\ ,\ 153$	$32\ ,\ 0$
10	7	$265\ ,\ 153$	$989\ ,\ 571$	6	$627\ ,\ 362$	$-298\ ,-209$
11	7	$989\ ,\ 571$	$989\ ,\ 571$	7	$989\ ,\ 571$	$-1585\ ,-989$
12	8	$3691\ ,\ 2131$	$989\ ,\ 571$	6	$0\ ,\ 0$	$-1287\ ,-780$
...						

Table 8

k	$v0(k)$	$2^{v0(k)}\,(\overrightarrow{V}_{-k}(0))_{x_0}$ $\cdot\mathbf{1}\ ,\ \cdot\sqrt{\mathbf{3}}$	$2^{v0(k)}\,(\overrightarrow{V}_{-k}(0))_{y_0}$ $\cdot\mathbf{1}\ ,\ \cdot\sqrt{\mathbf{3}}$	$2^{v0(k)}\,(\overrightarrow{V}_{-k}(3))_{x_0}$ $\cdot\mathbf{1}\ ,\ \cdot\sqrt{\mathbf{3}}$	$2^{v0(k)}\,(\overrightarrow{V}_{-k}(3))_{y_0}$ $\cdot\mathbf{1}\ ,\ \cdot\sqrt{\mathbf{3}}$
0	0	$1\ ,\ 0$	$0\ ,\ 0$	$-1\ ,\ 0$	$0\ ,\ 0$
1	1	$1\ ,\ 0$	$0\ ,\ -1$	$-2\ ,\ -1$	$1\ ,\ 0$
2	2	$-1\ ,\ -1$	$-1\ ,\ -3$	$-5\ ,\ -3$	$5\ ,\ 1$
3	2	$-5\ ,\ -3$	$-1\ ,\ -3$	$-5\ ,\ -3$	$9\ ,\ 3$
4	3	$-19\ ,\ -11$	$3\ ,\ -3$	$-5\ ,\ -3$	$27\ ,\ 11$
5	3	$-26\ ,\ -15$	$15\ ,\ 4$	$7\ ,\ 4$	$34\ ,\ 15$
6	3	$-26\ ,\ -15$	$34\ ,\ 15$	$26\ ,\ 15$	$34\ ,\ 15$
7	4	$-26\ ,\ -15$	$113\ ,\ 56$	$97\ ,\ 56$	$42\ ,\ 15$
8	5	$71\ ,\ 41$	$297\ ,\ 153$	$256\ ,\ 153$	$-39\ ,\ -41$
9	5	$265\ ,\ 153$	$297\ ,\ 153$	$265\ ,\ 153$	$-233\ ,\ -153$
10	6	$989\ ,\ 571$	$329\ ,\ 153$	$265\ ,\ 153$	$-925\ ,-571$
11	6	$1351\ ,\ 780$	$-298\ ,\ -209$	$-362\ ,\ -209$	$-1287\ ,-780$
12	6	$1351\ ,\ 780$	$-1287\ ,\ -780$	$-1351\ ,\ -780$	$-1287\ ,-780$
...					

Table 9

k	$v1(k)$	$2^{v1(k)}\ (\overrightarrow{V}_{-k}(1))_{x_0}$ $\cdot\mathbf{1}\ ,\ \cdot\sqrt{\mathbf{3}}$	$2^{v1(k)}\ (\overrightarrow{V}_{-k}(1))_{y_0}$ $\cdot\mathbf{1}\ ,\ \cdot\sqrt{\mathbf{3}}$	$2^{v1(k)}\ (\overrightarrow{V}_{-k}(4))_{x_0}$ $\cdot\mathbf{1}\ ,\ \cdot\sqrt{\mathbf{3}}$	$2^{v1(k)}\ (\overrightarrow{V}_{-k}(4))_{y_0}$ $\cdot\mathbf{1}\ ,\ \cdot\sqrt{\mathbf{3}}$
0	1	1 , 0	0 , 1	−1 , 0	0 , −1
1	1	1 , 0	1 , 0	−2 , −1	0 , −1
2	2	1 , 0	2 , −1	−7 , −4	2 , −1
3	3	−1 , −1	3 , −3	−19 , −11	13 , 3
4	3	−5 , −3	3 , −3	−19 , −11	27 , 11
5	4	−19 , −11	11 , −3	−19 , −11	87 , 41
6	4	−26 , −15	23 , 4	26 , 15	113 , 56
7	4	−26 , −15	42 , 15	97 , 56	113 , 56
8	5	−26 , −15	129 , 56	362 , 209	129 , 56
9	6	71 , 41	329 , 153	989 , 571	−201 , −153
10	6	265 , 153	329 , 153	989 , 571	−925 , −571
11	7	989 , 571	393 , 153	989 , 571	−3563 , −2131
12	7	1351 , 780	−234 , −209	−1351 , −780	−4914 , −2911
...					

Table 10

k	$v2(k)$	$2^{v2(k)}\ (\overrightarrow{V}_{-k}(2))_{x_0}$ $\cdot\mathbf{1}\ ,\ \cdot\sqrt{\mathbf{3}}$	$2^{v2(k)}\ (\overrightarrow{V}_{-k}(2))_{y_0}$ $\cdot\mathbf{1}\ ,\ \cdot\sqrt{\mathbf{3}}$	$2^{v2(k)}\ (\overrightarrow{V}_{-k}(5))_{x_0}$ $\cdot\mathbf{1}\ ,\ \cdot\sqrt{\mathbf{3}}$	$2^{v2(k)}\ (\overrightarrow{V}_{-k}(5))_{y_0}$ $\cdot\mathbf{1}\ ,\ \cdot\sqrt{\mathbf{3}}$
0	1	−1 , 0	0 , 1	1 , 0	0 , −1
1	2	−1 , −1	3 , 1	−1 , −1	−1 , −3
2	2	−1 , −1	5 , 1	−5 , −3	−1 , −3
3	3	−1 , −1	13 , 3	−19 , −11	3 , −3
4	3	2 , 1	15 , 4	−26 , −15	15 , 4
5	3	7 , 41	15 , 4	−26 , −15	34 , 15
6	4	26 , 15	23 , 4	−26 , −15	113 , 56
7	5	71 , 41	13 , −11	71 , 41	297 , 153
8	5	71 , 41	−39 , −41	265 , 153	297 , 153
9	6	71 , 41	−201 , −153	989 , 571	329 , 153
10	6	−97 , −56	−298 , −209	1351 , 780	−298 , −209
11	6	−362 , −209	−298 , −209	1351 , 780	−1287 , −780
12	7	−1351 , −780	−234 , −209	1351 , 780	−4914 , −2911
...					