Karlsruhe
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# Memoir on Napoléon's Theorem

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For the history on the so called *Napoléon*'s Theorem see [1]. This theorem and the related *Napoléon* points are treated in Wikipedia [5], [6] and Weisstein's *Word of Mathematics* [3], [4].

*Figures 1* and *2* show the outer and inner version of this theorem, respectively.

**a) Outer Napoléon's triangle and -theorem; Napoléon point and Fermat-Torricelli point**

The original general triangle is $T := \triangle(A,\, B,\, C)$ with sides $a$, $b$, $c$. The input parameters are the base $c$, the side $a$ and the angle $\beta = \angle(A,\, B,\, C)$. The *Cartesian* coordinate representation is

$$A \doteq [0,\, 0],\quad B \doteq [c,\, 0],\quad C \doteq [c - a\cos\beta,\, a\sin\beta] \doteq [b\cos\alpha,\, b\sin\alpha], \tag{1}$$

giving angle

$$\alpha = \angle(C,\, A,\, B) = \arctan(a\sin\beta/(c - a\cos\beta)), \tag{2}$$

and side

$$b = \frac{\sin\beta}{\sin\alpha}\, a = \sqrt{a^2 + c^2 - 2\,a\,c\,cos\,\beta}, \tag{3}$$

where the second formula used the cos theorem.

The centroid (or barycenter, in German *Schwerpunk*, $X(2)$ in the ETC [2],[7]) of $T$, the intersection of the three medians defined *via* the midpoint of the sides $M_a$, $M_b$ and $M_c$, is $S \doteq [S_x,\, S_y]$ with $S_x = \frac{c}{2}\tan\tau/(\tan\tau - \tan\sigma)$ and $S_y = \tan\tau\, S_x$, with $\tau := \angle(M_a,\, A,\, B)$ and $\sigma := \angle(C,\, M_c,\, B)$. *I.e.*, $\tan\tau = \frac{a}{2}\sin\beta/(c - \frac{a}{2}\cos\beta)$ and $\tan\sigma = a\sin\beta/(\frac{c}{2} - a\cos\beta)$. In the *Figures* this is the black point $S$.

The outside equilateral triangles (oriented positively like $T$) over the sides $a$, $b$, $c$ have apices $A'$, $B'$ and $C'$, respectively, with Cartesian coordinates:

$$A' \doteq \left[c - a\cos\left(\frac{\pi}{3} + \beta\right),\, a\sin\left(\frac{\pi}{3} + \beta\right)\right], \tag{4}$$

$$B' \doteq \left[b\cos\left(\frac{\pi}{3} + \alpha\right),\, b\sin\left(\frac{\pi}{3} + \alpha\right)\right], \tag{5}$$

$$\doteq \left[-b\sin\left(\frac{\pi}{3} - \beta'\right),\, b\cos\left(\frac{\pi}{3} - \beta'\right)\right],$$

$$C' \doteq \left[c/2,\, -\sqrt{3}\,c/2\right]. \tag{6}$$

Here the angle $\beta' := \angle(A,\, B,\, S_b)$ has also been used, where $S_b$ is the centroid of the equilateral triangle with base $b$. The coordinates of these centroids are

$$S_a \doteq \left[c - \frac{\sqrt{3}}{3}\, a\cos\left(\beta + \frac{\pi}{6}\right),\, \frac{\sqrt{3}}{3}\, a\sin\left(\beta + \frac{\pi}{6}\right)\right], \tag{7}$$

$$S_b \doteq \left[\frac{\sqrt{3}}{3}\, b\cos\left(\alpha + \frac{\pi}{6}\right),\, \frac{\sqrt{3}}{3}\, b\sin\left(\alpha + \frac{\pi}{6}\right)\right], \tag{8}$$

$$S_c \doteq \left[\frac{c}{2},\, -\frac{\sqrt{3}}{6}\, c\right]. \tag{9}$$

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Thus

$$\tan\beta' = \frac{(S_b)_y}{c - (S_b)_x}\,. \tag{10}$$

**Outer Napoléon's theorem**:

**a)** *The outer Napoléon triangle* $T_S := \triangle(S_a, S_b, S_c)$ *(oriented positively like* $T$*) is equilateral with side length* $s$*.*

**b)** $s^2 = \frac{1}{6}(a^2 + b^2 + c^2) + \frac{2}{\sqrt{3}}F$, *where the area of* $\triangle(A, B, C)$ *is* $F = \frac{1}{2}\,c\,a\,\sin\beta = \frac{1}{2}\,c\,b\,\sin\alpha$*.*

**c)** Hence the area of equilateral triangle $T_S$ is

$$F_S = \sqrt{3}\left(\frac{s}{2}\right)^2 = \frac{\sqrt{3}}{24}(a^2 + b^2 + c^2) + \frac{1}{2}F = \frac{1}{6}\sum_{i=1}^{3}F_i + \frac{1}{2}F\,, \tag{11}$$

where the areas of the equilateral triangles with centroids $S_a$, $S_b$ and $S_c$ are called $F_1$, $F_2$ and $F_3$, respectively.

**Proof**: By brute force and applying symmetry properties. First the square of length $\overline{S_c,\,S_b}$ is computed.

$$s(c,b)^2 := \overline{S_c,\,S_b}^{\,2} = \frac{1}{3}(b^2 + c^2) + \frac{1}{3}\,b\,c\left(\sqrt{3}\,\sin\alpha - \cos\alpha\right)\,, \tag{12}$$

after application of the addition theorems for sin and cos. Using $b\,c\,\sin\alpha = A$ and the cos theorem $2\,c\,b\,\cos\alpha = b^2 + c^2 - a^2$ one obtains the claimed formula $s(c,b)^2 = \frac{1}{6}(a^2 + b^2 + c^2) + \frac{2}{\sqrt{3}}F$.

Then, by exchange of $b \to a$ and $\beta \to \alpha$ (due to a sign change under the square signs):

$$s(c,a)^2 := \overline{S_c,\,S_a}^{\,2} = \frac{1}{3}(a^2 + c^2) + \frac{1}{3}\,a\,c\left(\sqrt{3}\,\sin\beta - \cos\beta\right)\,. \tag{13}$$

which also leads to the claimed formula for $s^2 \to s(c,a)^2$.

The same for $s(a,b)^2 := \overline{S_a,\,S_b}^{\,2}$ would becomes more tedious but this computation is not needed, because a rotation of the original triangle around $B$ with angle $\pi - \beta$ in the negative sense (such that the new coordinate of $C$ becomes $[c + a, 0]$) leads to a new triangle with rôle interchange compared to the calculation of $s(c,b)^2$ like $c \to a$, $b \to c$ and $\alpha \to \beta$ (the shift from the origin to $B$ is not of interest for length computations). This exchanges in $s(c,b)^2$ from eq. (12) shows then, with the formulae for $a\,c\,\sin\beta$ and $2\,c\,a\,\cos\beta$, that $s(c,b)^2$ leads to the same final formula as for $s(a,b)^2$. □

**Outer Napoléon point N1**

**Claim**: *The three straight lines connecting the points* $A, B, C$ *to the points* $S_a, S_b, S_c$*, respectively, intersect at the outer (or first) Napoléon point* $N_1$ *(*$X(17)$ *ETC [2], [7]) with Cartesian coordinates*

$$N_1 \doteq [N1_x,\, N1_y = \tan\alpha'\,N1_x], \quad \textit{with}\quad N1_x = c\left(\frac{\tan\beta'}{\tan\alpha' + \tan\beta'}\right)\,, \tag{14}$$

*where*

$$\tan\alpha' = \frac{(S_a)_y}{(S_a)_x}, \tag{15}$$

*and* $\tan\beta'$ *from eq. (10).*

**Proof:**

The straight line through $A$ and $S_a$ and the one through $B$ and $S_b$ intersect at the given point $N1$, which is elementary.

To prove that this $N1$ point also lies on the straight line through $C$ and $S_c$ one does not need a (lengthy) computation. One resorts again to the above used symmetry argument: consider also the rotated triangle around point $B$ with angle $\pi - \beta$ in the negative sense. This is the rotation $\mathbf{R}^{-1}(\pi - \beta) = -\mathbf{R}^{-1}(\beta)$ around $B$. The point $N1$ found above rotates to a point $N1'$. But this point is now found also in the same way as above in the old triangle by intersecting the straight line through the new origin $O' = B$ and $S_{a'}$ in the new equilateral triangle over base $a'$ of length $b$, and the one through $B'$ (the rotated $C$) and $S_{b'}$ in the equilateral triangle over base $b'$ of length $c$. Therefore, the line through $B'$ and $S_{b'}$ with point $N1'$ rotates back to the line through $C$ and $S_c$ through $N1$. □

**Outer Fermat-Torricelli point F1**

**Claim:** *The three straight lines connecting the triangle vertices $A$, $B$, $C$ to the respective apices $A'$, $B'$, $C'$ of the equilateral triangles intersect at the so-called outer (or first) Fermat-Torricelli point $F1$ ($X(13)$ in the ETC). Its Cartesian coordinates are*

$$F1 \doteq [F1_x, F1_y = \tan\,\alpha''\, F1_x], \quad with \;\; F1_x = c\left(\frac{\tan\,\beta''}{\tan\,\alpha'' + \tan\,\beta''}\right), \tag{16}$$

*where*

$$\tan\,\alpha'' = \frac{A'_y}{A'_x}, \quad and \quad \tan\,\beta'' = \frac{B'_y}{(c - B'_x)}. \tag{17}$$

**Proof:**

This runs along the same lines as the one given for $N1$ above if the corresponding angles are replaced. □

**b) Inner Napoléon's triangle, and -theorem; Napoléon point and Fermat-Torricelli point**

For this case the above three equilateral triangles are mirrored at their corresponding base sides. In *Figure 2* we use the same notations as in the outer case, but here a tilde indicates the inner case.

The Cartesian coordinates are obtained by changing $\frac{\pi}{3} \to -\frac{\pi}{3}$ in eqs. (4) to (6) and $\frac{\pi}{6} \to -\frac{\pi}{6}$ in eqs. (7) to (9). Also the $y-$coordinate of $\tilde{C}'$ is now positive.

$$\begin{aligned}
\tilde{A}' &\doteq \left[c - a\,\cos\left(\beta - \frac{\pi}{3}\right),\, a\,\sin\left(\beta - \frac{\pi}{3}\right)\right], & (18)\\
\tilde{B}' &\doteq \left[b\,\cos\left(\alpha - \frac{\pi}{3}\right),\, b\,\sin\left(\alpha - \frac{\pi}{3}\right)\right], & (19)\\
&\doteq \left[b\,\sin\left(\tilde{\beta}' + \frac{\pi}{3}\right),\, b\,\cos\left(\tilde{\beta}' + \frac{\pi}{3}\right)\right], &\\
\tilde{C}' &\doteq \left[c/2,\, -\sqrt{3}\,c/2\right]. & (20)
\end{aligned}$$

Here $\tan\tilde{\beta}'$ is defined by eq. (10) , but with the following tilde centroids.

$$\begin{aligned}
\tilde{S}_a &\doteq \left[c - \frac{\sqrt{3}}{3}\,a\,\cos\left(\beta - \frac{\pi}{6}\right),\, \frac{\sqrt{3}}{3}\,a\,\sin\left(\beta - \frac{\pi}{6}\right)\right], & (21)\\
\tilde{S}_b &\doteq \left[\frac{\sqrt{3}}{3}\,b\,\cos\left(\alpha - \frac{\pi}{6}\right),\, \frac{\sqrt{3}}{3}\,b\,\sin\left(\alpha - \frac{\pi}{6}\right)\right], & (22)\\
\tilde{S}_c &\doteq \left[\frac{c}{2}, \frac{\sqrt{3}}{6}\,c\right]. & (23)
\end{aligned}$$

The equilateral inner *Napoléon* triangle $T_{\tilde{S}} = \triangle(\tilde{S}_a, \tilde{S}_b, \tilde{S}_c)$ is now oriented in the negative sense. The square of its side length is $\tilde{s}$ given by

$$\tilde{s}^2 = \frac{1}{6}\,(a^2 + b^2 + c^2) - \frac{2}{\sqrt{3}}\,F\,, \tag{24}$$

*i.e.*, the (non-oriented) area of the inner *Napoléon* triangle is

$$|F_{\tilde{S}}| = \sqrt{3}\left(\frac{\tilde{s}^2}{2}\right)^2 = \frac{1}{6}\sum_{i=1}^{3} F_i - \frac{1}{2}F\ . \tag{25}$$

This can be computed from $\tilde{s}(c,b) = \overline{\tilde{S}_c\,\tilde{S}_b}^2$ as above, and the sign change follows.
Together with eq. (11) this yields the following area theorem.

**Area Theorem:**

**a)** *For the oriented areas:* $F_S + F_{\tilde{S}} = F$ .

**b)** *For the non-oriented areas* $F_S - |F_{\tilde{S}}| = \frac{1}{3}\sum_{i=1}^{3} F_i$ .

The formulae eqs. (14) and (15) also work for the **inner (or second) Naplo'eon point N2** ($X(18)$ in the *ETC*) but with the angles $\tilde{\alpha}'$ and $\tilde{\beta}'$ given now by the $\tilde{S}_a$ and $\tilde{S}_b$ coordinates from eqs. (21) and (22). Similarly, the formulae eqs. (16) and (17) also work for the **inner (or second) Fermat-Toriccelli point F2** ($X(14)$ in the *ETC*) but now with the angles taken from the coordinates of $\tilde{A}'$ and $\tilde{B}'$ from eqs. (18) and (19), respectively.

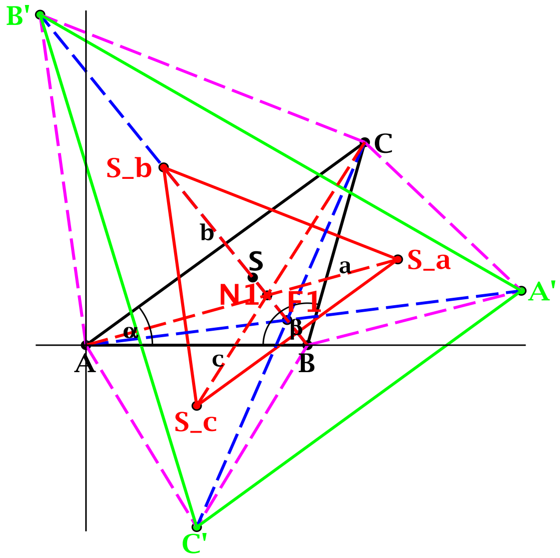


**Figure 1**

Outer *Napoléon*'s Theorem

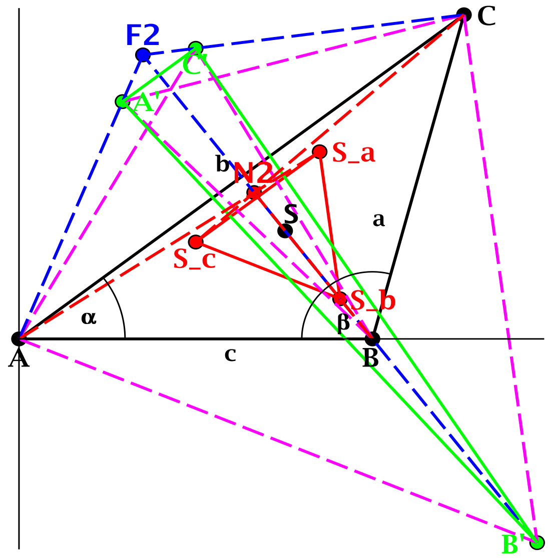


**Figure 2**

Inner *Napoléon*'s Theorem

In some (different) length units: $c = 1$, $a = 1$, $\beta = \frac{7\pi}{12}$

## References


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