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On a Certain Family of Sidi Polynomials

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Abstract

A family of a *Sidi* polynomials system $\{PS_N(n,\,x)\}$, for integer N, and their coefficient number triangles $\{TS_N(n,\,m)\}$, are studied. For all N the row sums of the triangles are $n!$. The exponential generating functions of the triangles are shown to involve derivatives of the *Lambert* W-function.

1 Introduction

A special family of *Sidi*'s one variable polynomial systems [3] which originally depended on three integers, is here reduced to only one integer N and studied in detail.
This family of polynomial systems is denoted by $\{PS_N(n,x)\}_{n\,\geq\,0}$.The corresponding number triangles TS_N and their exponential generating functions (*e.g.f.*) ETS_N are computed. For $N\,=\,0$ these *e.g.f.* s involve the derivative of *Lambert*'s W-function. For non-vanishing N the derivative of the N-fold convolution of $W(-x)/(-x)\,=\,exp(-W(-x))$ enters.
The *Jabotinsky* type *Sheffer* polynomials $(1,\,-W(-x))$ are essential for evaluating the case of non-vanishing N. They are identified with special $Abel$ polynomials.
A salient feature of this $N-$family of *Sidi* polynomials is the N independent row sum $n!$ for row n of each triangle TS_N.
The interest in this work started with the $N\,=\,0$ triangle OEIS [1] A075513 after a question by Harlan J. Brothers for a proof of the row sums.

2 Sidi N-polynomials and number triangles

The general *Sidi* polynomials [3], Theorem 4.2., p. 862, are for integers k, n, m, with $k\,\geq\,0$ and $m\,\geq\,0$

$$D_{k,n,m}(z)\,=\,\sum_{j=0}^{k}(-1)^j\binom{k}{j}(n\,+\,j)^m\,z^{n+j-1}. \tag{1}$$

They can also be computed as given in [3], eq. (4.11) , p. 862.

$$D_{k,n,m}(z)\,=\,\left(\frac{d}{dz}\,z\right)^m\,(z^{n-1}\,(1-z)^k)\,. \tag{2}$$

This can be rewritten, using the *Euler* derivative $E_z\,:=\,z\,\frac{d}{dz}$ as

$$D_{k,n,m}(z)\,=\,\frac{1}{z}\,E_z^m\,(z^n\,(1-z)^k)\,. \tag{3}$$

This shows immediately eq. (1) using the binomial sum for $(1-z)^k$ and the eigen-equation $E_z^m\,z^j\,=\,j^m\,z^j$.

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Here we consider the special $N-$family of polynomials $\{PS_N(n,\, x)\}$ with integer N, namely

$$PS_N(n, x) \,:=\, \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (k \,+\, N \,+\, 1)^n \, x^k \,=\, \frac{(-1)^n}{x^N} \, D_{n,N+1,n}(x)\,. \tag{4}$$

From eq. (3) this can be written as

$$PS_N(n,\, x) \,=\, \frac{1}{x^{N+1}} \, E_x^n (x^{N+1} \, (x \,-\, 1)^n)\,. \tag{5}$$

A simple computation shows that the instance $N \,=\, 0$ can also be obtained by

$$PS_0(n,\, x) \,=\, \frac{1}{(n+1)\, x} \, E_x^{n+1} (x \,-\, 1)^{n+1}\,. \tag{6}$$

The number triangles TS_N of the coefficients of PS_N are

$$TS_N(n,\, k) \,=\, (-1)^{n-k} \binom{n}{k} (k \,+\, N \,+\, 1)^n\,, \text{ for } \; n \,\geq\, 0, \text{ and } k \,=\, 0,\, 1,\, ...,\, n\,. \tag{7}$$

For $n \,<\, k$ one sets $TS_N(n,\, k) \,=\, 0$,
The *e.g.f.*s of the columns of these triangles, *i.e.*, $E_N(k,\, x) \,:=\, \sum_{n=k}^{\infty} TS_N(n,\, k)\, x^n$ (one can start with $n \,=\, 0$), are

Proposition 1:

$$E_N(k,\, x) \,=\, e^{-(k+N+1)\, x} \, \frac{((k + N + 1)\, x)^k}{k!} \quad \text{for} \;\; k \,\geq\, 0. \tag{8}$$

Proof:
$E_N(k,\, x) \,=\, (-1)^k \, g(k,\, -z)\Big|_{z\,=\,(k+N+1)\,x}$, with $g(k,\, x) \,:=\, \sum_{n=k}^{\infty} \binom{n}{k} \, x^n/n! \,=\, e^x \, x^k/k!$.
This follows from the fact that the *e.g.f.* of the kth column (with leading zeros) of the *Pascal* triangle, OEIS [1] A007318, can be obtained from the ordinary generating function $G_k(x) \,=\, x^k/(1 \,-\, x)^k$ by an inverse *Laplace* transformation, namely $\mathcal{L}^{[-1]}(G_k(1/p)/p) \,=\, \mathcal{L}^{[-1]}(1/(p \,-\, 1)^{k+1}) \,=\, e^t \, t^k/k!$. This also shows that the *Pascal* triangle, the *Riordan* triangle of the *Bell type* $(1/(1 \,-\, x),\, x/(1 \,-\, x)))$ is also the *Sheffer* triangle (sometimes called exponential *Riordan* triangle) of the *Appell* type $(exp(x),\, x)$.
□

Proposition 2: The *e.g.f.* $ETS_N(x,\, z)$ of the row polynomials $\{PS_N(n,\, x)\}$, *i.e.*, the *e.g.f.* of triangle TS_N, is

$$\begin{aligned} ETS_N(x,\, z) \;\; &:= \;\; \sum_{n=0}^{\infty} P_N(n,\, x) \, \frac{z^n}{n!} \,=\, \sum_{k=0}^{\infty} x^k \, E_N(k,\, x) \\ &= \;\; e^{-(N+1)\, z} \sum_{k=0}^{\infty} ((k \,+\, N \,+\, 1)\, x\, z\, e^{-z})^k/k!\,. \end{aligned} \tag{9}$$

Proof: This follows from $P_N(n,\, x) \,=\, \sum_{k=0}^{n} TS_N(n,\, k)\, x^k$, an interchange of the summation variables n and k, and the definition of $E_N(k,\, x)$ with the result eq. (8) . □

Because the instance $N \,=\, 0$ will turn out to be special we treat this case first. See A075513, but there the triangle has offset 1. (A-numbers will be given henceforth without the OEIS reference.)

Proposition 3:

$$ETS_0(x,\,z) \,=\, e^{-z}\,\sum_{k=0}^{\infty}(k\,+\,1)^k\,\frac{y^k}{k!}\Big|_{y=x\,z\,e^{-z}} \,=\, e^{-z}\,\frac{d}{dy}\,(-W(-y))\Big|_{y=x\,z\,e^{-z}} \,=\, \frac{e^{-(z\,+\,W(-x\,z\,e^{-z}))}}{1\,+\,W(-x\,z\,e^{-z})}\,, \tag{10}$$

where $W(y)$ is the principal branch of the *Lambert* W-function, (see,*e.g.*, [8], [5]) defined by the identity $W(y)\,exp(W(y)) \,=\, y$, with derivative $\frac{d}{dy}W(y) \,=\, exp(-W(y))/(1\,+\,W(y))$.

The proof uses the following *e.g.f.* of $\{k^{k-1}\}_{k>=1}$ = A000169.

Lemma 1:

$$-\,W(-y) \,=\, \sum_{k=1}^{\infty} k^{k-1}\frac{y^k}{k!}\;. \tag{11}$$

Proof:

The *Lagrange* inverse of $g(x) \,=\, x\,e^{-x}$ is $g^{[-1]}(y) \,=\, \sum_{n=1}^{\infty} g_n\,y^n/n!$ with $g_n \,=\, (d^{n-1}/dt^{n-1})(1/exp(-t))^n\Big|_{t=0} \,=\, n^{n-1}$. See A000169, and *Stanley* [4]. But this compositional inverse of $g(x)$ is $-W(-y)$ because, from the definition of W, $W(-y)\,exp(W(-y)) \,=\, -y$, or $(-W(-y))\,exp(-(-W(-y)) \,=\, y$.

For a proof that $-W(-y)$ is the compositional inverse of $x\,exp(-x)$ one can alternatively use the rule for the derivative of the compositional inverse $-W(-y)$ of $x\,e^{-x}$ and compare this with the known derivative of $-W(-y)$ (see above). □

Proof of eq. (10) :

The first step is eq. (9) for $N \,=\, 0$. From *Lemma* 1 follows the second step, after a change of the summation variable $k \,\to\, k+1$, $\frac{d}{dy}\,(-W(-y)) \,=\, \sum_{n=0}^{\infty}(k\,+\,1)^k\,\frac{y^k}{k!}$. The third step uses the above given result for $\frac{d}{dy}\,W(y)$ for $y \,\to\, -y$. □

The result of eq. (9) for non-vanishing integer N is, after evaluation of the sum:

Theorem:

For $N \,\in\, \mathbb{Z}\setminus\{0\}$:

$$ETS_N(x,\,z) \,=\, e^{-(N+1)\,z}\,\frac{1}{N}\left[\frac{d}{dy}\left(\frac{W(-y)}{(-y)}\right)^N\right]\Bigg|_{y=x\,z\,e^{-z}} \,=\, e^{-(N+1)\,z}\left[\frac{e^{(N+1)\,(-W(-y))}}{1\,-\,(-W(-y))}\right]\Bigg|_{y=x\,z\,e^{-z}}. \tag{12}$$

For the proof we need the following *Proposition* for the exponential (sometimes called binomial) convolution of $W(-y)/(-y) \,=\, e^{-W(-y)}$ (this identity follows from the definition of $W(x)$ with $x \,\to\, -y$).

Proposition 4:

a) The *e.g.f.* of $(k\,+\,1)^{k-1}$ = A000272 $(k+1)$, for $k \,\geq\, 0$, is $W(-y)/(-y)$, *i.e.*,

$$e^{-W(-y)} \,=\, \frac{W(-y)}{(-y)} \,=\, \sum_{k=0}^{\infty}(k\,+\,1)^{k-1}\,\frac{y^k}{k!}\,. \tag{13}$$

b) The special *Sheffer* triangle (or infinite matrix with upper diagonal part vanishing) of the *Jabotinsky* type $(1,\,-W(-x))$ has row polynomials

$$JW(n,\,x) \,:=\, \sum_{m=0}^{n} J(n,\,m)\,x^m \text{ with } e.g.f.\;EJW(x,\,z) \,=\, e^{-x\,W(-z)}\,. \tag{14}$$

c) The a-family of *Abel* polynomial systems $A(a;n,x) \,:=\, x\,(x\,-\,a\,n)^{n-1}$, for $n \,\geq\, 0$ and $a \,\in\, \mathbb{Z}$, [[2], [6], [9]] are *Sheffer* polynomials of the *Jabotinsky* type $(1,\,f^{[-1]}(a;\,y))$, with the compositional inverse $f^{[-1]}(a;\,y)$ of $f(a;\,x) = x\,e^{a\,x}$. Hence the $JW(n,\,x)$ polynomial is identified as the member $A(-1;n,x)$ of this *Abel* family.

d) The *e.g.f.* of $(W(-y)/(-y))^N = exp(-N\,W(-y))$ is defined by $\sum_{n=0}^{\infty} c_N(n)\, y^n/n!$, and $c_N(n)$ is a polynomial in N of degree n (with $0^0 := 1$), but later used only for integer $N \neq 0$), *i.e.*,

$$c_N(n) \,=\, \sum_{m=0}^{n} a(n,\, m)\, N^m, \text{ for } n \,\geq\, 0, \tag{15}$$

where the number triangle $\{a(n,m)\}$ is the *Jabotinsky* triangle $\{J(n,\, m)\}$, given by the unsigned triangle |A137452| . Hence

$$c_N(n) \,=\, JW(n,\, N). \tag{16}$$

e) The triangle entries $a(n,\, m) \,=\, J(n,\, m)$ are

$$a(0,\, 0) \,=\, 1;\; a(n,\, 0) \,=\, 0, \text{ and } a(n,\, m) \,=\, \binom{n-1}{m-1} n^{n-m}, \text{ for } n \,\geq\, 1 \text{ and } m \,=\, 1,\, 2,\, ...,\, n\,. \tag{17}$$

f) The explicit form of $c(N,\, n)$ is

$$c(N,\, 0) \,=\, 1, \text{ and } c(N,\, n) \,=\, N\,(n+N)^{n-1}, \text{ for } n \,\geq\, 1\,. \tag{18}$$

This shows that $c(N,\, n) \,=\,$ A232006$(n+N,\, N)$ for $N \,\geq\, 1$, and $n \,\geq\, 0$.

g) *Faà di Bruno*'s formula [7] for $c_N(n)$:
$c_N(0) \,=\, 1$, and for $n \,\geq\, 1$, with partitions of n of m parts, written as $n \,=\, \sum_{j=1}^{n} j\, e_j$ and $m \,=\, \sum_{j=1}^{n} e_j$. ($e_j$ is the non-negative exponent of part j, however, j^0 means that part j is absent) one obtains:

$$c_N(n) \,=\, \frac{d^n}{dy^n} e^{N\,(-W(y))}\Big|_{y=0} \,=\, n!\, \sum_{m=1}^{n} N^m \sum_{e_1\, e_2,\, ...,\, e_n} \prod_{j=1}^{n} \left(\frac{j^{j-1}}{j!}\right)^{e_j} \frac{1}{e_j!}\,. \tag{19}$$

Proof

a) The first equation follows from the definition of $W(x = -y)$. The second one follows from *Lemma 1* after a shift in the summation index.

b) This is a known result for the *e.g.f.* of general *Sheffer* $(g(x),\, f(x))$ polynomials with $g(0) \,=\, 1$ and $f(0) \,=\, 0$ (see, *e.g.*, the *Sheffer* part in the W. L. link '*Sheffer* a- and z-sequence' in A006232, with details and references). Here $g(x) \,=\, 1$ and $f(x) = -W(-x)$.

c) That the *Abel* polynomials are *Sheffer* polynomials of the *Jabotinsky* type is proved in *Roman* [2] (in a notation where f is the present $f^{[-1]}$). Here we give a proof using the known recurrence relation for *Jabotinsky* polynomials J, (also given in [2], Corollary 3.7.2., p. 50) namely

$$J(n,x) \,=\, x\left[\frac{1}{\frac{d}{dt}(f^{[-1]}(t))}\right]\Bigg|_{t\,=\,d/dx} J(n-1,\, x), \text{ for } n \,\geq\, 1, \text{ and } J(0,\, x) \,=\, 1. \tag{20}$$

Hence

$$A(-1;n,x) \,=\, x\left[\frac{e^t}{1\,-\,t}\right]\Bigg|_{t\,=\,d/dx} A(-1;\, n-1,\, x), \text{ for } n \,\geq\, 1, \text{ and } A(-1;0,\, x) \,=\, 1 \tag{21}$$

will be proved.
This uses the expansion ($n^{\underline{k}}$ is a falling factorial)

$$\frac{e^t}{1\,-\,t} \,=\, \sum_{n=0}^{\infty} a(n)\, \frac{t^n}{n!}, \text{ with } a(n) \,=\, \sum_{k=0}^{n} n^{\underline{k}} \,=\, n!\, \sum_{k=0}^{n} \frac{1}{k!}\,, \tag{22}$$

The proof of the $a(n)$ is done by expanding the *l.h.s.* and picking coefficients of $t^n/n!$, for $n \,\geq\, 0$ (using induction over n).

The recurrence relation is $a(n+1) = (n+1)\,a(n) + 1$, for $n \geq 0$, and $a(0) = 1$. For $\{a(n)\}_{n=0}$ see A000522.
In addition one needs higher derivatives of $A(-1; n-1, x)$.

Lemma 2

$$\left(\frac{d}{dx}\right)^k A(-1; n-1, x) = (n-1)^{\underline{k}}\,(x+k)\,(x+n-1)^{n-(k+2)},\ \text{for } k \geq 0 \text{ and } n \geq 1. \tag{23}$$

Proof of the Lemma

By induction over k for fixed n. The case $k = 0$ is satisfied because $(n-1)^{\underline{0}} := 1$. The induction step for $(\frac{d}{dx})^{k+1} A(-1; n-1, x)$ uses $(n-1)^{\underline{k}}\,(n-(k+1)) = (n-1)^{\underline{k+1}}$. □

Continuing with the proof of part **c**) we start with the binomial expansion $A(-1; n, x) = x\,((x+n-1)+1)^{n-1} = x\sum_{j=0}^{n-1}\binom{n-1}{j}(x+n-1)^j$, and eqs. (21) and (22). After division by x ($x \neq 0$) one wants to prove, for fixed $n \geq 1$,

$$\sum_{j=0}^{n-1}\binom{n-1}{j}(x+n-1)^j \stackrel{!}{=} \sum_{k=0}^{n-1}\frac{a(k)}{k!}\,\frac{d^k}{dx^k}A(-1; n-1, x), \tag{24}$$

where the k-sum is cut off at the degree of $A(-1; n-1, x)$. Applying *Lemma 2* leads to the *r.h.s.* (*RHS*)

$$RHS = \sum_{k=0}^{n-1}\frac{a(k)}{k!}\,(n-1)^{\underline{k}}\,(x+k)\,(x+n-1)^{n-(k+2)}\,. \tag{25}$$

In order to compare powers of $x+n-1$ on both sides of eq. (24), one rewrites $x + k = (x+n-1) + (k-(n-1))$ for each term, except for $k = n-1$. This last term needs no rewriting, it is $a(n-1)$. The first term, $k = 0$, leads to a rewritten first part $a(0)\,(x+n-1)^{n-1}$, and an addition to the rewritten first part of term $k+1$, *i.e.*, $a(0)(-(n-1))\,(x+n-1)^{n-2}$. This $k = 0$ term is the only one consisting of only one rewritten part.
For $k = 0, 1, \ldots, n-2$ the (x independent) second part of the replacement leads to $(a(k)/k!)\,(n-1)^{\underline{k}}\,(k-(n-1))\,(x+n-1)^{n-k-2}$, which adds to the first part of the rewritten term for $k+1$ that will produce this power.
This means that each power $(x+n-1)^{n-k-2}$, for $k \in \{1, 2, \ldots, n-1\}$ consist of two terms: the first one from the first part of the rewritten k term and the second one from the second part of the rewritten $k-1$ term. The single rewritten $k = 0$ term is $a(0)\,(x+n-1)^{n-1}$, and it coincides with the $j = n-1$ term of the *l.h.s.* (*LHS*) of eq. (24) because $a(0) = 1$. The last term $k = n-1$ receives the additional second part of the $k = n-2$ term, *i.e.*, $a(n-2)\,(n-1)\,(-1)$. This results in $a(n-1) - (n-1)\,a(n-2) = 1$ (by the recurrence), coinciding with the $j = 0$ term of the *LHS*.
Thus the coefficient of $(x+n-1)^{n-k-2}$, for $k = \{1, 2, \ldots, n-1\}$, can be compared on both sides of eq. (24),

$$\binom{n-1}{n-k-2} = \frac{(n-1)^{\underline{k+1}}}{(k+1)!} \stackrel{!}{=} \frac{a(k+1)}{(k+1)!}\,(n-1)^{\underline{k+1}} - \frac{a(k)}{k!}\,(n-1)^{\underline{k}}\,(n-1-k)\,. \tag{26}$$

This can be rewritten with the relation between $(n-1)^{\underline{k}}$ and $(n-1)^{\underline{k+1}}$ used above in the proof of *Lemma 2* as

$$RHS = \frac{(n-1)^{\underline{k}}}{(k+1)!}\,(n-k-1)\,(a(k+1) - (k+1)\,a(k)), \tag{27}$$

which equals the *LHS* because of the recurrence $a(k+1) - (k+1)\,a(k) = 1$, and again using the falling factorial relation. This ends the proof of part **c**).

d) The proof that $c_N(n) = JW(n, x)|_{x=N}$ is shown for the corresponding *e.g.f.*s. By definition the *e.g.f.* of $\{c_N(n)\}_{n>=0}$ is $exp(-N\,W(-y))$. From **b)** the *e.g.f.* of the row polynomials $\{JW(n, x)\}_{n\geq 0}$ is $EJW(x, y) = exp(-x\,W(-y)$ (expansion in y). For $x = N$ the claim follows.

e) This follows from $JW(n,\,x) \,=\, A(-1;x,n)$ from **c)**, and the trivial computation of $x\,(x\,+\,n)^{n-1}$ by the binomial expansion, and a shift of the summation index, The case of the x^0 coefficient is separated, giving $a(0,\,0) = 1$.

f) for $N \,\neq\, 0$ and $n\,\geq\,1$, $N\,(n\,+\,N)^{n-1} \,=\, n^n\,(N/n)\,(1\,+\,(N/n))^{n-1} \,=\, n^n\,\sum_{m=0}^{n-1}\,\binom{n-1}{m}\,(N/n)^{m+1} \,=\, n^n\,\sum_{m=1}^{n}\,\binom{n-1}{m-1}\,(N/n)^m \,=\, \sum_{m=1}^{n}\,\binom{n-1}{m-1} n^{n\,-\,m}\,N^m \,=\, \sum_{m=1}^{n}\,a(n,\,m)\,N^m$, with $a(n,\,m)$ from part **e)**, hence this equals $c(N,n)$, because the $m\,=\,0$ term $a(n,\,0)\,=\,0$ for $n\,\geq\,1$.

g) The *Faà di Bruno* formula is for $\frac{d^n}{dy^n}\,f(g(y))$, and here f(x) = exp(N x) and $g(y) \,=\, -W(-y)$. Because for $c_N(n)$ the formula is evaluated at $y\,=\,0$, one needs $\frac{d^m}{dx^m}\,f(x)\big|_{x\,=\,g(0)=0} \,=\, N^m$ and $\frac{d^j}{dy^j}\,g(y)\big|_{y=0} \,=\, j^{j-1}$ from eq. (11). The multinomials $n!/\prod_{j=1}^{n}\,j!^{e_j}\,e_j!$ appearing in this formula are called $M_3 \,=\, M_3(\vec{e}(n,\,m))$, with $\vec{e}(n,\,m) \,:=\, \{e_1,\,e_2,\,...,e_n\}$, and the given restrictions on the non-negative exponents of $\vec{e}(n,\,m)$ These multinomials are shown in A036040 (see the *Abramowitz-Stegun* link there).
This shows that $a(n,\,m)$ in eq. (15) for $c_N(n)$ equals the sum of M_3-partition polynomials (ParPolM3) over the $p(n,m)$ =A008284$(n,\,m)$ partitions of n with m parts: $\sum_{k=1}^{p(n,\,m)}\,ParPolM3(n,m,k,\{x_j \,=\, j^{j-1}\}_{j=1..n})$.
Example: $n\,=\,3$, the partitions for $m\,=\,1,\,2,\,3$ are $3^1,1^1\,2^1,\,1^3$, respectively.
$c_N(3) = 3!\,(N^1\,3^2/3! + N^2\,(1^0/1!)\,(2^1/2!) + N^3\,(1^0/1!)^3/3!) \,=\, 9\,N\,+\,6\,N^2\,+\,N^3$. Compare this with row $n\,=\,3$ of |A137452|: [0, 9, 6, 1]. □

Proof of the Theorem

The step from the last equation of eq. (9) to the first equation of eq. (12), with $y \,=\, x\,z\,exp(-z)$, is proved with the help of *Proposition 4*, part **d)** and the explicit form of $a(n,\,m)$ from part **e)**.
The *e.g.f.* $\sum_{k=0}^{\infty}(k\,+\,N\,+\,1)^k\,y^k/k!$ is proved to be $(1/N)\,d/dy\,(1 + \sum_{k=1}^{\infty}\,c_N(k)\,y^k/k!)$, where $c_N(0)\,=\,1$ was used. This means, after comparing powers of y,

$$(k\,+\,N\,+\,1)^k \stackrel{!}{=} \frac{1}{N}\,c_N(k+1),\ \text{for}\ k\,\geq\,0\,. \tag{28}$$

Because $a(k+1,\,0)\,=\,0$ the *r.h.s.* becomes, with eq. (15) and an index shift in m, $\sum_{m=0}^{k}\,a(k{+}1,\,m{+}1)\,N^m$. From eq. (17) this becomes $\sum_{m=0}^{k}\,\binom{k}{m}\,(k{+}1)^{k-m}\,N^m$, but this is the binomial expansion of $((k\,+\,1)\,+\,N)^k$,
For the proof of the second equation of the *Theorem*, eq. (12), one uses the replacement $(W(-y)/(-y))^N$ by $exp(N\,(-W(-x)))$, and with $d/dy\,(-W(-y)) \,=\, exp(-W(-y))/(1\,-\,(-W(-y))$, one obtains

$$\frac{1}{N}\,\frac{d}{dy}\,e^{N\,(-W(-y))} \,=\, e^{N\,(-W(-y))}\,\frac{e^{-W(-y)}}{1\,-\,(-W(-y))} \,=\, \frac{e^{(N+1)\,(-W(-y))}}{1\,-\,(-W(-y))}\,. \tag{29}$$

□

We close with the result that for each integer N the row sum of the triangle TS_N is $n!$.

Proposition 5

$$\sum_{k\,=\,0}^{n} T_N(n,\,k) \,=\, PS_N(n,\,1) \,=\, n!,\ \text{for}\ N\,\in\,\mathbb{Z}. \tag{30}$$

Proof

We show that the *e.g.f.* of $\{PS(n,1)\}_{n>=0}$, *i.e.*, $ETS_N(1,\,z)$ from eq. (10) and eq. (12) becomes $1/(1-z)$, the *e.g.f.* of $\{n!\}_{n\,\geq\,0}$.
For $N\,=\,0$ one obtains for eq. (10) from $-W(-y)|_{y\,=\,z\,exp(-z)} \,=\, z$ (compositional inverse relation, see the proof of *Lemma 1*)

$$ETS_0(1,\,z) \,=\, e^{-z}\,\frac{e^z}{1\,-\,z} \,=\, \frac{1}{1\,-\,z}\,. \tag{31}$$

For integer $N \neq 0$ one uses in eq. (12) the previously mentioned compositionl inverse rule for $-W(-y)$ with $y = z\,exp(-z)$

$$ETS_N(1,\, z) \,=\, e^{-(N+1)\,z}\, e^{N\,z}\, \frac{e^z}{1\,-\,z} \,=\, \frac{1}{1\,-\,z}\,. \tag{32}$$

The dependence on $N \neq 0$ dropped out. □

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The latest version of the *Harlan J. Brothers* paper is called 'Pascal's Triangle: Infinite Paths to e', tbp.

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