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Notes on the Discrete Theodorus Spiral

Wolfdieter L a n g [1]

Abstract

The explicit form for the coordinates of the points of the discrete Theodorus spiral including its mirror points (the inner spiral) are given in Cartesian as well as polar coordinates. For computational purpose the Cartesian coordinates are expressed in terms of reduced angles. A conjecture relating points of the inner and outer discrete spiral is proved except for the fourth quadrant in the complex plane. For this region a stronger conjecture is proposed.

1 Coordinates

The discrete spiral with the radial length $r_n = \sqrt{n}$, $n \in \mathbb{N}$, built from rectangular triangles $\triangle(O,\, z_n,\, z_{n+1})$ with length $|\overline{z_n,\, z_{n+1}}| = \sqrt{1} = 1$ named after the Pythagorean *Theodorus* (ΘΕ$\mathrm{O}\Delta\Omega$ΡΟΣ) [5] due to the interpretation by *J. H. Anderhub* [1] (see also [2], [3], [4], [6], [8]). In the complex plane it is written in polar coordinates $z_n = r_n\, e^{i\,\phi_n}, n \in \mathbb{N}$. It can be continued inwards to points $\hat{z}_n$ by taking the mirror image of z_n on the hypotenuse $\overline{O,\, z_{n+1}}$ [7], Fig. 1 (where z_n and $\hat{z}_n$ are called F_n and G_n, respectively). See also the present Figure 3. One obtains $\hat{z}_n$ after rotating z_n around an axis through the origin O, perpendicular to the complex plane, with angle $\alpha_n = \phi_{n+1} - \phi_n = \arctan\left(\frac{1}{r_n}\right)$ in the positive sense ($\phi_1 = 0$).

The recurrence for $\{z_n\}_{n>=1}$ is

$$z_{n+1} = \frac{\sqrt{n+1}}{\sqrt{n}}\, e^{i\,\alpha_n}\, z_n, \quad n = 1,\, 2\ldots, \text{ with } z_1 = 1. \tag{1}$$

This becomes

$$z_{n+1} = \left(1 + \frac{1}{\sqrt{n}}\, i\right) z_n, \quad n = 1,\, 2\ldots, \text{ with } z_1 = 1. \tag{2}$$

For the Cartesian coordinates $z_n = R_n + I_n\, i$ one obtains the mixed recurrence:

$$R_{n+1} = R_n - \frac{1}{\sqrt{n}}\, I_n, \quad I_{n+1} = \frac{1}{\sqrt{n}}\, R_n + I_n, \tag{3}$$

with inputs $R_1 = 1$ and $I_1 = 0$.

Iteration leads immediately to a recurrence for R_n alone (undefined sums are put to 0),

$$R_{n+1} = R_n - \frac{1}{\sqrt{n}} \sum_{j=1}^{n-1} \frac{1}{\sqrt{n-j}}\, R_{n-j}\,, \quad n \in \mathbb{N}, \quad \text{with} \quad R_1 = 1. \tag{4}$$

[1] wolfdieter.lang@partner.kit.edu, http://www/kit.edu/~wl/

Then $I_n = -\sqrt{n}\,(R_{n+1} - R_n)$ becomes

$$I_n = \sum_{j=1}^{n-1} \frac{1}{\sqrt{n-j}}\, R_{n-j}\,, \quad n \in \mathbb{N}. \tag{5}$$

The proof is straightforward, by showing that the recurrences from eq. (3) are satisfied together with the inputs.
The explicit form can now be given.

Proposition 1: Cartesian Coordinates for $\mathbf{z_n}$

If the outer spiral points z_n are taken in the complex plane $\mathbb{C}$ as $z_n = R_n + I_n\, i$ one has

$$R_n = 1 + \sum_{j=2}^{n-1} (-1)^{\lfloor \frac{j}{2} \rfloor} \sum_{2 \leq i_2 < i_3 < \ldots < i_j \leq n-1} \frac{1}{\sqrt{i_2 i_3 \cdots i_j}},\ n \geq 3,\ R_1 = 1 = R_2, \tag{6}$$

$$I_n = 1 - \sum_{j=2}^{n-1} (-1)^{\lceil \frac{j}{2} \rceil} \sum_{2 \leq i_2 < i_3 < \ldots < i_j \leq n-1} \frac{1}{\sqrt{i_2 i_3 \cdots i_j}},\ n \geq 3,\ I_1 = 0,\ I_2 = 1. \tag{7}$$

Proof:

By induction on n one shows that the recurrences and the inputs are satisfied.The basis of the induction for $n = 1$ is clear. Assuming that formulae eq. (6) and eq. (7) hold for $k = 1, ..., n$, one shows with recurrence eq. (3) that the formulae hold also for $n + 1$.

a) For the R-recurrence one collects from R_n the term j, for $j = 2, ..., n-1$ and from the I_n sum the term $j - 1$, with the first ($j = 1$) term 1. The remaining term $j = n - 1$ in the I_n sum is (with the prefactor) $-(-1)^{\lfloor \frac{n-1}{2} \rfloor} \dfrac{1}{\sqrt{2\cdots(n-1)}}$. These I_n terms are to be multiplied with $\frac{1}{\sqrt{n}}$. For $j = 2$ one obtains $(-\dfrac{1}{\sqrt{2}} - \ ... - \dfrac{1}{\sqrt{n-1}}) - \dfrac{1}{\sqrt{n}}$. For $j \geq 3$ the R part has overall sign $(-1)^{\lfloor \frac{j}{2} \rfloor}$ and all $\dfrac{1}{\sqrt{\cdots}}$ terms with products of $j-1$ numbers from $\{2, ..., n-1\}$. From the $j-1$ term of the I_n with overall sign $(-1)^{\lceil \frac{j-1}{2} \rceil} = -(-1)^{\lfloor \frac{j}{2} \rfloor}$ (shown by taking j even or odd), one obtains all $\dfrac{1}{\sqrt{\cdots}}$ terms with $j-2$ numbers from $\{2, ..., n-1\}$ multiplied by $\frac{1}{\sqrt{n}}$, *i.e.*, all $\dfrac{1}{\sqrt{\cdots}}$ terms with $j-1$ numbers where one of the numbers is always n. Together this becomes, with the extra term $j = n-1$ in the I_n part given above,

$$1 + \sum_{j=2}^{n} (-1)^{\lfloor \frac{j}{2} \rfloor} \sum_{2 \leq i_2 < i_3 < \ldots < i_j \leq n} \frac{1}{\sqrt{i_2 i_3 \cdots i_j}},$$

This is the claimed formula for R_{n+1} of eq. (6).

b) The proof for the I-recurrence eq. (3) is done along the same line. Here the identity $(-1)^{\lfloor \frac{j-1}{2} \rfloor} = -(-1)^{\lceil \frac{j}{2} \rceil}$ is employed in the R part with index $j - 1$ which is taken together with the j term of the I part. The left over term in the R part produces in the second part of eq. (3) $(-1)^{\lfloor \frac{n-1}{2} \rfloor} \dfrac{1}{\sqrt{2\cdots(n-1)\,n}} = -(-1)^{\lceil \frac{n}{2} \rceil} \dfrac{1}{\sqrt{2\cdots(n-1)\,n}}$ from $j = n-1$. This leads to the claimed I_{n+1} of eq. (7). $\square$

Example 1:

$R_4 = -1 - \dfrac{1}{\sqrt{2}} - \dfrac{1}{\sqrt{3}} - \dfrac{1}{\sqrt{2\cdot 3}} \simeq -.6927053409$ (Maple 10 digits),

$I_4 = 1 + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} - \dfrac{1}{\sqrt{2\cdot 3}} \simeq 1.876208759$ (Maple 10 digits).

The number of terms of R_n and I_n in eq. (6) and eq. (7) grows exponentially: both numbers are 2^{n-2}, for $n >= 2$ and 1 for $n = 1$. Therefore the recurrence or the explicit formulae are not useful to compute R_n and I_n for large n (say, $n \geq 25$). Later a more efficient way to compute these quantities will be given based on the angles ϕ_n which satisfy a simple recurrence.

Corollary 1: Cartesian Coordinates for $\hat{\mathbf{z}}_\mathbf{n}$

If the inner spiral points $\hat{z}_n$ are taken in the complex plane $\mathbb{C}$ as $\hat{z}_n = \hat{R}_n + \hat{I}_n\, i$ one has in terms of points of the coordinates of the outer spiral

$$\hat{R}_n = \frac{(n-1)\, R_n - 2\sqrt{n}\, I_n}{n+1},\ n \in \mathbb{N}, \tag{8}$$

$$\hat{I}_n = \frac{(n-1)\, I_n + 2\sqrt{n}\, R_n}{n+1},\ n \in \mathbb{N}. \tag{9}$$

This follows directly from the mirror definition for $\hat{z}_n$, namely (see also [7], and the present Figure 3) by a rotation of $\overline{0\, z_n}$ with angle $2\,\alpha_n$. This implies $|\hat{z}_n| = |z_n|$.

$$\hat{z}_n = e^{i\, 2\, \arctan\left(\frac{1}{\sqrt{n}}\right)}\, z_n. \tag{10}$$

The square of the formula $exp\left(i\, \arctan\left(\frac{1}{\sqrt{n}}\right)\right) = \frac{\sqrt{n} + i}{\sqrt{n+1}}$ leads to

$$\hat{z}_n = \frac{n - 1 + 2\sqrt{n}\, i}{n+1}\, z_n\,. \tag{11}$$

This produces *Corollary* 1.

Example 2:

$\hat{R}_4 = \frac{1}{5}\left(-1 - \frac{7}{2}\sqrt{2} - \frac{7}{3}\sqrt{3} + \frac{1}{6}\sqrt{6}\right) \simeq -1.916590212$ (Maple 10 digits),

$\hat{I}_4 = \frac{1}{5}\left(7 - \frac{1}{2}\sqrt{2} - \frac{1}{3}\sqrt{3} - \frac{7}{6}\sqrt{6}\right) \simeq 0.5715609834$ (Maple 10 digits).

One can also give a mixed recurrence for $\hat{R}_n$ and $\hat{I}_n$ by looking at a drawing of the points z_{n-1}, z_n, z_{n+1} and $\hat{z}_{n-1}$, $\hat{z}_n$. The important angle is $\angle(\hat{z}_{n-1},\, O,\, z_{n+1}) = \alpha_{n-1} - \alpha_n$. Remember that $\alpha = \phi_{n+1} - \phi_n$, and note that $\alpha_{n-1} > \alpha_n$. Then the angle $\angle(\hat{z}_n,\, O,\, \hat{z}_{n-1}) = \alpha_n - (\alpha_{n-1} - \alpha_n) = 2\,\alpha_n - \alpha_{n-1}$. Thus, by a rotation and a scaling, to obtain the correct length $|\hat{z}_n| = |z_n| = r_n = \sqrt{n}$,

$$\hat{z}_n = \frac{\sqrt{n}}{\sqrt{n-1}}\, e^{i\,(2\,\alpha_n - \alpha_{n-1})}\, \hat{z}_{n-1}\,,\ n = 2,\, 3,\, ...,\quad \text{with}\ \ \hat{z}_1 = i\,. \tag{12}$$

A calculation of $\beta_{n-1} := 2\,\alpha_n - \alpha_{n-1} = 2\, \arctan\left(\frac{1}{\sqrt{n}}\right) - \arctan\left(\frac{1}{\sqrt{n-1}}\right)$, for $n \geq 2$, leads, with $2\, \arctan\left(\frac{1}{\sqrt{n}}\right) = \arctan\left(\frac{2\,n}{\sqrt{n}\,(n-1)}\right)$, and the difference formula for arctan to

$$\beta_{n-1} = \arctan\left(\frac{2\,n\,\sqrt{n-1} - \sqrt{n\,(n-1)}}{2\,n + (n-1)\,\sqrt{n\,(n-1)}}\right)\,. \tag{13}$$

This implies, after some simplifications,

$$\cos\beta_{n-1} = \frac{1}{\sqrt{1 + (\tan\,\beta_{n-1})^2}} = \frac{2\,n + (n-1)\,\sqrt{n\,(n-1)}}{n\,(n+1)}\,, \tag{14}$$

$$\sin\beta_{n-1} = \frac{\tan\,\beta_{n-1}}{\sqrt{1 + (\tan\,\beta_{n-1})^2}} = \frac{2\,n\,\sqrt{n-1} - \sqrt{n}\,(n-1)}{n\,(n+1)}\,. \tag{15}$$

This gives the following mixed recurrence for $\hat{R}_n$ and $\hat{I}_n$.

Proposition 2: Mixed recurrence for $\hat{\mathbf{R}}_\mathbf{n}$ and $\hat{\mathbf{I}}_\mathbf{n}$

$$\hat{R}_n = \frac{1}{(n+1)\sqrt{n\,(n-1)}}\,\Big((2\,n+(n-1)\,\sqrt{n\,(n-1)}\,)\,\hat{R}_{n-1}\,-\,(2\,n\,\sqrt{n-1}-(n-1)\,\sqrt{n}\,)\,\hat{I}_{n-1}\Big)\ , \quad (16)$$

$$\hat{I}_n = \frac{1}{(n+1)\sqrt{n\,(n-1)}}\,\Big((2\,n+(n-1)\,\sqrt{n\,(n-1)}\,)\,\hat{I}_{n-1}\,+\,(2\,n\,\sqrt{n-1}-(n-1)\,\sqrt{n}\,)\,\hat{R}_{n-1}\Big)\ , \quad (17)$$

for $n\,=\,2,\,3,\,...$, with inputs $\hat{R}_1\,=\,0$ and $\hat{I}_1\,=\,1$. The iteration, in order to obtain an unmixed recurrence and from this the explicit form, is not considered here, because the explicit form is obtained from eq. (8) and eq. (9) and the explicit forms of R_n and I_n given in eq. (6) and eq. (7).

Proposition 3: Explicit formulae for $\hat{\mathbf{R}}_\mathbf{n}$ and $\hat{\mathbf{I}}_\mathbf{n}$

$$\begin{aligned}\hat{R}_n \;=\; \frac{1}{n+1}\,\Bigg(&(n-1\,-\,2\sqrt{n}\,)\,+\,(n\,-\,1\,+\,2\sqrt{n}\,)\sum_{J=1}^{\lfloor\frac{n-1}{2}\rfloor}(-1)^J\sum_{2\le i_2<i_3<\,...\,<i_{2J}\,\le n-1}\frac{1}{\sqrt{i_2 i_3\cdots i_{2J}}}\\ &+(n\,-\,1\,-\,2\sqrt{n}\,)\sum_{J=1}^{\lfloor\frac{n-2}{2}\rfloor}(-1)^J\sum_{2\le i_2<i_3<\,...\,<i_{2J+1}\,\le n-1}\frac{1}{\sqrt{i_2 i_3\cdots i_{2J+1}}}\Bigg)\ ,\end{aligned} \quad (18)$$

$$\begin{aligned}\hat{I}_n \;=\; \frac{1}{n+1}\,\Bigg(&(n-1\,+\,2\sqrt{n}\,)\,+\,(-(n\,-\,1)\,+\,2\sqrt{n}\,)\sum_{J=1}^{\lfloor\frac{n-1}{2}\rfloor}(-1)^J\sum_{2\le i_2<i_3<\,...\,<i_{2J}\,\le n-1}\frac{1}{\sqrt{i_2 i_3\cdots i_{2J}}}\\ &+(n\,+\,1\,-\,2\sqrt{n}\,)\sum_{J=1}^{\lfloor\frac{n-2}{2}\rfloor}(-1)^J\sum_{2\le i_2<i_3<\,...\,<i_{2J+1}\,\le n-1}\frac{1}{\sqrt{i_2 i_3\cdots i_{2J+1}}}\Bigg)\ .\end{aligned} \quad (19)$$

Proof:

From eq. (8) with eqs. (6) and (7) inserted, with replacement $(-1)^{\lceil\frac{j}{2}\rceil}\,=\,-(-1)^{\lfloor\frac{j-1}{2}\rfloor}$, one splits the two sums into even and odd j: $j\,=\,2\,J$, with $j\,=\,1,\,...,\,\left\lfloor\frac{n-1}{2}\right\rfloor$ and $j\,=\,2\,J\,+\,1$, with $j\,=\,1,\,...,\,\left\lfloor\frac{n-2}{2}\right\rfloor$. The following splitting is used.

$$(n-1)\,(-1)^{\lfloor\frac{j}{2}\rfloor}\,-\,2\sqrt{n}\,(-1)^{\lfloor\frac{j-1}{2}\rfloor}\,=\,\begin{cases}(-1)^J\,(n\,-\,1\,+\,2\sqrt{n}\,), & \text{if}\ \ j\,=\,2\,J\\ (-1)^J\,(n\,-\,1\,-\,2\sqrt{n}\,), & \text{if}\ \ j\,=\,2\,J\,+\,1\end{cases}\ . \quad (20)$$

This produces the R part of the proposition. The I part is then obtained with the splitting

$$(n-1)\,(-1)^{\lfloor\frac{j-1}{2}\rfloor}\,+\,2\sqrt{n}\,(-1)^{\lfloor\frac{j}{2}\rfloor}\,=\,\begin{cases}(-1)^J\,(-(n\,-\,1)\,+\,2\sqrt{n}\,), & \text{if}\ \ j\,=\,2\,J\\ (-1)^J\,(n\,-\,1\,+\,2\sqrt{n}\,), & \text{if}\ \ j\,=\,2\,J\,+\,1\end{cases}\ . \quad (21)$$

$\square$

Example 3: For $n\,=\,4$ this checks with *Example* 2.

The remark made above on the ineffectiveness of these formulae for large n computations in connection with R_n and I_n applies here as well. Here the following section on angles will come to rescue.

2 Angles

a) For the outer spiral the polar angle ϕ_n of z_n is

$$\phi_n = \sum_{k=1}^{n-1} \alpha_k = \sum_{k=1}^{n-1} \arctan\left(\frac{1}{\sqrt{k}}\right), \text{ for } n = 2, 3, \ldots, \tag{22}$$

and $\phi_1 = 0$.
The recurrence is

$$\phi_{n+1} = \phi_n + \alpha_n = \phi_n + \arctan\left(\frac{1}{\sqrt{n}}\right), \quad \text{for} \quad n = 1, 2, \ldots, \quad \text{with} \quad \phi_1 = 0. \tag{23}$$

b) For the inner spiral the polar angle $\hat{\phi}_n$ of $\hat{z}_n$ is (see Figure 3)

$$\hat{\phi}_n = \sum_{k=0}^{n-1} \beta_k = \sum_{k=0}^{n-1} (2\,\alpha_{k+1} - \alpha_k) = \sum_{k=1}^{n-1} \alpha_k + 2\,\alpha_n = \phi_n + 2\,\alpha_n = \phi_{n+1} + \alpha_n\,, \tag{24}$$

where $\alpha_0 = 0$ and $n \in \mathbb{N}$.
The explicit form for these angles are obtained from those for R_n and I_n from equations (6) and (7). For this one needs the number of the sheet in the complex plane where z_k, respectively $\hat{z}_k$, lives. The point z_k lives on sheet $S_{K(k)+1}$, with $K(k) = \left\lfloor \frac{\phi_k}{2\,\pi} \right\rfloor$. The number of points z up to, and including, sheet $S_n := \rho\, e^{i\,\varphi}$, for $n \in \mathbb{N}$, with $\rho > 0$ and $\varphi \in [2\,(n-1)\,\pi,\, 2\,n\,\pi)$ is given by A072895(n) with A072895 $=$ [17, 54, 110, 186, 281, 396, 532, 686, 861, 1055, ...]. These numbers are the arguments k where the number n appears for the first time in the sequence $\{K(k)\}_{k\geq 1}$ after subtracting 1, for $n \in \mathbb{N}$.
For the numbers of points z on sheet S_n see A295338(n) with A295338 $=$ [17, 37, 56, 76, 95, 115, 136, 154, 175, 194, 214, 234, 254 ...] (first differences of A072895).
Depending on the quadrants I, II, III, and IV of sheet $S_{K(n)+1}$ (we omit the sheet index on the quadrants) the formula for the angle ϕ_n of z_n becomes in terms of real and imaginary parts R_n and I_n on this sheet

$$\phi_n = K(n)\,2\,\pi + \sigma(n)\,\pi + \arctan\left(\frac{I_n}{R_n}\right), \tag{25}$$

with

$$\sigma(n) = \begin{cases} 0 & \text{if } z_n \in I, \\ 1 & \text{if } z_n \in II \text{ or } III, \\ 2 & \text{if } z_n \in IV. \end{cases} \tag{26}$$

For example z_{19} lives in quadrant I of sheet S_2 (because $K(19) = 1$) therefore $\phi_{19} = 2\,\pi + \arctan\left(\frac{I_{19}}{R_{19}}\right)$, with $R_{19} \simeq 4.144473699$ and $I_{19} \simeq 1.350310247$ (Maple 10 digits), leading to $\phi_{19} \simeq 6.598149490$. Here we used R_n and I_n from eqs. (6) and (7).
But we can now use eq. (25) to compute R_n and I_n in terms of the reduced angle ϕ_n which we denote by φ_n:

$$\varphi_n := \phi_n - K(n)\,2\,\pi\,, \tag{27}$$

$$R_n = \sqrt{n}\,\cos(\varphi_n)\,, \quad \text{and} \quad I_n = \sqrt{n}\,\sin(\varphi_n)\,. \tag{28}$$

Note that these formulae are valid for all four quadrants. They allow fast computation of the real and imaginary part of z_n because of the simple recurrence for ϕ_n from eq. (23). This shows that there are only $n-1$ arctan terms (like in eq. (22)). For example $R_{100} \simeq 5.481207079$ which needed $0.095\,s$ for Maple13. Similarly for $I_{100} \simeq -8.363992405$, where z_{100}, with $K(100) = 2$, lives on sheet S_3 in quadrant IV, with $\varphi_{100} \simeq 5.292504709$ corresponding to about $303.2°$. This fits with $\phi_{100} = 6\,\pi + \arctan\left(\frac{I_{100}}{R_{100}}\right) \simeq 17.85887529$.

Similarly, the point $\hat{z}_k$ lives on sheet $S_{\hat{K}(k)+1}$, with $\hat{K}(k) = \left\lfloor \frac{\hat{\phi}_k}{2\,\pi} \right\rfloor$. The number of points $\hat{z}$ up to, and including, sheet S_n is given by A295339(n) with A295339 = [15, 52, 108, 184, 279, 394, 530, 684, 859, 1053, ...], and the number of points $\hat{z}$ on sheet S_n are given in A296179(n) with A296179 = [15, 37, 56, 76, 95, 115, 136, 154, 175, 194, 214, 234, 254 ...]. The apparent equality up to the first entry of this sequence with A295338 for points z will be discussed in *Section C*.

The $\hat{\phi}_n$ angle of $\hat{z}_n$ is then, depending on the quadrant on sheet $S_{\hat{K}(n)+1}$,

$$\hat{\phi}_n = \hat{K}(n)\,2\,\pi + \hat{\sigma}(n)\,\pi + \arctan\left(\frac{\hat{I}_n}{\hat{R}_n}\right), \tag{29}$$

with $\hat{\sigma}(n)$ like $\sigma(n)$ in eq. (26) with $z \to \hat{z}$.

Again the real and imaginary parts of $\hat{z}_n$ can be computed also for large n from the reduced angle $\hat{\varphi}_n$

$$\hat{\varphi}_n := \hat{\phi}_n - \hat{K}(n)\,2\,\pi\,, \tag{30}$$

with $\hat{\phi}_n$ given in terms of ϕ_{n+1} in eq. (24). They are

$$\hat{R}_n = \sqrt{n}\,\cos(\hat{\varphi}_n)\,, \quad \textit{and} \quad \hat{I}_n = \sqrt{n}\,\sin(\hat{\varphi}_n)\,. \tag{31}$$

For example, $\hat{R}_{100} \simeq 7.028904480$ is found in $0.108\,s$, and also $\hat{I}_{100} \simeq -7.112981218$. $\hat{z}_{100}$ lives on sheet S_3 ($\hat{K}(100) = 2$) in quadrant *IV* with $\hat{\varphi} \simeq 5.491842014$ corresponding to about $314.7°$. This fits with $\hat{\phi}_{100} = 6\,\pi - \arctan\left(\frac{|\,\hat{I}_{100}\,|}{\hat{R}_{100}}\right) \simeq 18.05821260$.

3 A conjecture on sequences $\hat{\mathbf{K}}$ and $\mathbf{K}$

The apparent coincidence of A95338 and A296179 up to the first entry follows from the conjecture that A295339(k) = A072895$(k) - 2$ for $k \in \mathbb{N}$. This, in turn, is equivalent to the following conjecture.

Conjecture:

$$\mathbf{C}(\mathbf{k}): \qquad \hat{K}(k-2) = K(k)\,, \quad \text{for} \quad k \geq 3\,. \tag{32}$$

This means that if the first appearance of any number m in the sequences $\hat{K}$ is at position, say p(m), then it is in sequence K at position $p(m) + 2$. Then the first differences of A072895 and A295339 coincide up to the first entry.

In order to analyze this conjecture we first use a simple consequence of the definition of $\hat{\phi}$ of eq. (24).

Lemma 1:

$$\hat{\phi}_{k-2} - \phi_k = \alpha_{k-2} - \alpha_{k-1} =: \Delta_k = \arctan\left(\frac{\sqrt{k-1} - \sqrt{k-2}}{1 + \sqrt{(k-1)\,(k-2)}}\right) > 0\,, \quad \text{for} \quad k \geq 3\,. \tag{33}$$

Proof:

By eq. (24) $\hat{\phi}_{k-2} = \phi_{k-1} + \alpha_{k-2} = (\phi_k - \alpha_{k-1}) + \alpha_{k-2}$, by eq. (23). Δ_k is obviously positive because $\alpha_k = \arctan\left(\frac{1}{\sqrt{k}}\right)$, and arctan is strictly increasing, staying below $\frac{\pi}{2}$. The explicit form results from the formula for $\arctan(x) - \arctan(y)$ if $x\,y > -1$, which is satisfied. $\square$

Consider, for $k \geq 3$, the interval $\left[\left\lfloor \frac{\phi_k}{2\pi} \right\rfloor, \left\lfloor \frac{\phi_k}{2\pi} \right\rfloor + 1\right]$ with the interior point $\frac{\phi_k}{2\pi}$. The conjecture means that $\frac{\hat{\phi}_{k-2}}{2\pi}$ is a point in the open interval $\left(\frac{\phi_k}{2\pi}, K(k) + 1\right)$ due to *Lemma 1*. That is $\Delta_k < 2\pi\left((K(k) + 1) - \frac{\phi_k}{2\pi}\right) = 2\pi\left(1 - frac\left(\frac{\phi_k}{2\pi}\right)\right)$. Alternatively this means that the conjecture is

$$\hat{\phi}_{k-2} = \phi_k + \Delta_k < 2\pi\,(K(k) + 1)\,, \quad \text{for } k \geq 3. \tag{34}$$

This is considered for the quadrants of the sheet $S_{K(k)+1}$ on which z_k lives. It turns out that the conjecture can be proved for quadrants I, II and III. For quadrant IV a stronger conjecture will be given.

Proposition 4: Proof of the conjecture for quadrants I, II and III
The conjecture, that is eq. (34), is true for points z on quadrants I, II or III of the complex plane.
Proof:
i) Quadrant I: From eq. (25), with eq. (26), $\phi(k) + \Delta_k = \left(K(k)\,2\pi + \arctan\left(\frac{I_k}{R_k}\right)\right) + \Delta_k$. Therefore the conjecture eq. (34) becomes

$$\Delta_k + \arctan\left(\frac{I_k}{R_k}\right) < 2\pi\,. \tag{35}$$

With the explicit form of Δ_k from eq. (33) the addition formula $\arctan(x) + \arctan(y)$ is applied which depends for $x = \frac{\sqrt{k-1} - \sqrt{k-2}}{1 + \sqrt{(k-1)\,(k-2)}} > 0$ on whether $x\,y < 1$ or $x\,y > 1$. If the first case should apply one will obtain arctan with the positive argument $\frac{x + y}{1 - x\,y}$ with $y = \frac{I_k}{R_k} > 0$ (for $k \geq 3$), which is $< \frac{\pi}{2} < 2\pi$, and the conjecture is true. In the second case the result is $\pi - \arctan\left(\frac{x + y}{x\,y - 1}\right)$, again with a positive argument, name it X, and eq. (34) becomes $\arctan(X) > -\pi$ which is trivially satisfied because $\arctan(X) > 0$.

ii) Quadrant II: From eq. (25), with eq. (26), $\phi(k) + \Delta_k = \left(2\pi\,K(k) + \pi - \arctan\left(\frac{I_k}{|\,R_k\,|}\right)\right) + \Delta_k$. Therefore the conjecture eq. (34) becomes $\arctan\left(\frac{I_k}{|\,R_k\,|}\right) - \Delta_k > -\pi$. The arctan difference formula applies because the product $x\,y > 0 > -1$. Whatever sign the new arctan argument may have the new arctan will be $> -\frac{\pi}{2}$, hence $> -\pi$, and the conjecture is true.

iii) Quadrant III: Here the conjecture becomes $\Delta_k + \arctan\left(\frac{|\,I_k\,|}{|\,R_k\,|}\right) < \pi$. If $x\,y < 1$, with $x = \frac{\sqrt{k-1} - \sqrt{k-2}}{1 + \sqrt{(k-1)\,(k-2)}} > 0$ and $y = \frac{|\,I_k\,|}{|\,R_k\,|} > 0$, then one obtains, after applying the formula for $\arctan(x) + \arctan(y)$, the new arctan argument, name it X, which is > 0, and the conjecture is true because $\arctan(X) < \frac{\pi}{2} < \pi$.
In the case $x\,y > 1$ one obtains $+\pi - \arctan(Y)$ with a $Y > 0$, and this satisfies the conjecture because $\arctan(Y) > 0$. □

The situation for quadrant IV is different. The conjecture eq. (34) is now, with $x > 0$ as above in case iii) and $y = \frac{|\,I_k\,|}{R_k} > 0$, $-\arctan(y) + \arctan(x) = \arctan\left(\frac{x - y}{1 + x\,y}\right) < 0$ (the arctan difference formula needed $x\,y > 0 > -1$). Because the denominator of the argument is positive the conjecture reduces to $y - x > 0$, that is,

$$\mathbf{C_{IV}(k)}: \qquad \frac{|\,I_k\,|}{R_k} - \frac{\sqrt{k-1} - \sqrt{k-2}}{1 + \sqrt{(k-1)\,(k-2)}} > 0\,, \ \text{for } k \geq 3\,. \tag{36}$$

Lemma 2: Strictly decreasing $\mathbf{x_k}$

$x_k \,:=\, \dfrac{\sqrt{k-1}\,-\,\sqrt{k-2}}{1\,+\,\sqrt{(k-1)\,(k-2)}}$ is strictly decreasing for $k\,\geq\,2$.

Proof:

i) $1\,+\,\sqrt{k\,(k-1)}\,>\,1\,+\,\sqrt{(k-1)\,(k-2)}$ because, from the strict monotonicity of $\sqrt{x}$ for $x\,\geq\,0$, this leads to $k\,(k-1)\,>\,(k-1)\,(k-2)$, *i.e.*, $k\,>\,1$, satisfied for $k\,\geq\,2$.

ii) $\sqrt{k}\,-\,\sqrt{k-1}\,<\,\sqrt{k-1}\,-\,\sqrt{k-2}$, *i.e.*, $k\,(k-2)\,<\,(k-1)^2$, *i.e.*, . $0\,<\,1$. Therefore, $x_{k+1}\,<\,x_k$ for $k\,\geq\,2$. □

From this follows that the second term on the *l.h.s.* of eq. (36), $x(k)$, takes its largest value in quadrant IV on sheet S_n for the first point $z_{\tilde{k}}\,=\,z_{\tilde{k}(n)}$ in this quadrant. The corresponding sequence $\{\tilde{k}(n)\}_{n\,\geq\,1}$ is obtained from the sequence $\{KIV(k)\}_{k\,\geq\,1}$ with

$$KIV(k)\,:=\,\left\lfloor\frac{\phi_k\,-\,\frac{3\,\pi}{2}}{2\,\pi}\right\rfloor\,+\,1\,, \tag{37}$$

by recording the positions where it becomes n for the first time. This is the sequence A296181(n), for $n\,\geq\,1$, with A296181 = [12, 44, 95, 166, 256, 367, 497, ...].

The first term on the *l.h.s.* of eq. (36), *i.e.*, $\tan(\gamma_k)\,:=\,\dfrac{|\,I_k\,|}{R_k}$, with $\gamma_k\,=\,2\,\pi\,-\,\varphi_k$, an angle in quadrant IV of sheet $S_{K(k)+1}$ counted in the negative sense from the positive real axis of the next sheet, is smallest if γ_k is smallest, *i.e.*, if z_k is nearest to this real axis. This happens on sheet S_n for $z_{k(n)}$ with $k(n)$ = A07295(n), for $n\,\geq\,1$. We now propose a stronger conjecture then $C_{IV}(n)$, *i.e.*,

$$\mathbf{CIV}_{\rm new}(\mathbf{n}):\quad \tan\left(2\,\pi\,-\,\varphi_{k(n)}\right)\,>\,x_{\tilde{k}(n)}\,,\qquad i.e.,\qquad \gamma_{k(n)}\,=\,2\,\pi\,-\,\varphi_{k(n)}\,\stackrel{!}{>}\,\arctan\left(x_{\tilde{k}(n)}\right)\,,\quad n\,\geq\,1\,, \tag{38}$$

with $k(n)$ = A072895(n) and $\tilde{k}(n)$ = A296181(n), with φ_k from eq. (27) and x_k from *Lemma 2*.

Proposition 5: $\mathbf{CIV_{new}(n)\,\Rightarrow\,CIV(k)}$

If $\mathbf{CIV}_{\rm new}(\mathbf{n})$ holds for all $n\,\in\,N$ then $\mathbf{CIV}(\mathbf{k})$ holds for all $k\,\in\,N$.

Proof:

i) For all z_k in quadrant IV of sheet S_n: $\gamma_k\,\geq\,\gamma_{k(n)}$. This is obvious from the definition of $k(n)$ as largest k value for z_k on sheet S_n (automatically in quadrant IV, nearest to the positive real axis of sheet S_{n+1}), because $\gamma_k\,=\,2\,\pi\,-\,\varphi_k$ decreases with increasing k.

ii) For all z_k on quadrant IV of sheet S_n: $x_{\tilde{k}}\,\geq\,x_k$. This is clear from *Lemma 2* and the remark following it.

Thus, for all z_k in quadrant IV of sheet S_n, for $n\,\in\,\mathbb{N}$: $\gamma_k\,\geq\,\gamma_{k(n)}\,\stackrel{!}{>}\,\arctan(x_{\tilde{k}(n)})\,\geq\,\arctan(x_k)$), because arctan is (strictly) increasing. □

Note: The largest value for x_k of *Lemma 2* with z_k in quadrant IV appears of course on sheet S_1, and for $k\,=\,12$. It is tempting to propose instead of conjecture $CIV_{\rm new}(n)$ for z_k in quadrant IV on sheet S_n the conjecture $\tan(\gamma_{k(n)})\stackrel{?}{>}x_{12}\,\simeq\,0.01343540575$ (Maple 10 digits), with $k(n)$ = A072895(n). However, this conjecture is already false for sheet S_2, because $\tan(\gamma_{54})\,\simeq\,0.004555785878$ (Maple 10 digits). But $x_{\tilde{k}(2)}\,=\,x_{44}\,\simeq\,0.001763287640$ and conjecture $CIV_{\rm new}(2)$ is true.

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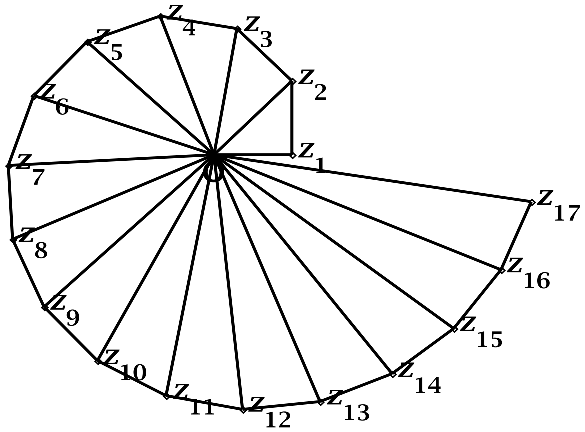

Figure 1: Outer discrete Theodorus spiral, $n = 1..17$

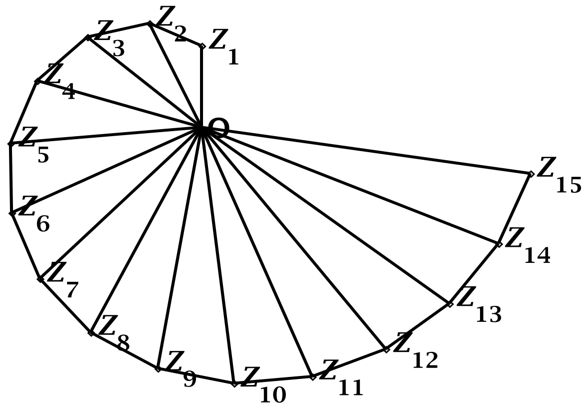

Figure 2: Inner discrete Theodorus spiral, $n = 1..15$, $(Z_n = \hat{z}_n)$

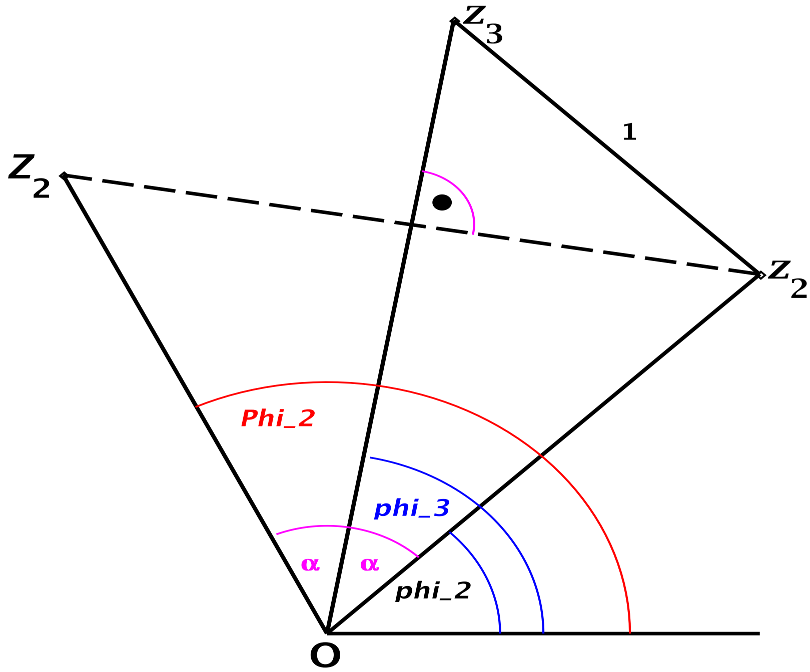

Figure 3: Inner spiral point construction
$(Z_2 = \hat{z}_2,\ \alpha = \alpha_2,\ phi\_n = \phi(n),\ Phi\_2 = \hat{\phi}(2))$