revised version, September 21 2020

The Tribonacci and ABC Representations of Numbers are Equivalent

Wolfdieter L a n g [1]
Karlsruhe
Germany
wolfdieter.lang@partner.kit.edu

Abstract

It is shown that the unique representation of positive integers in terms of tribonacci numbers and the unique representation in terms of iterated A, B and C sequences defined from the tribonacci word are equivalent. Two auxiliary representations are introduced to prove this bijection. It will be established directly on a node and edge labeled tribonacci tree as well as formally. A systematic study of the A, B and C sequences in terms of the tribonacci word is also presented.

1 Introduction

The quintessence of many applications of the tribonacci sequence $T = \{T(n)\}_{n=0}^{\infty}$ [4] A000073 [5], [6], [2] [1] is the ternary substitution sequence $2 \to 0$, $1 \to 02$ and $0 \to 01$. Starting with 2 this generates an infinite (incomplete) binary tree with ternary node labels called $TTree$. See *Fig 1* for the first 6 levels $l = 0, 1, ..., 5$ denoted by $TTree_5$. The number of nodes on level l is the tribonacci number $T(l+2)$, for $l \geq 0$. In the limit $n \to \infty$ the last level $l = n$ of $TTree_n$ becomes the infinite self-similar tribonacci word $TWord$. The nodes on level l are numbered by $N = 0, 1, ..., T(l+2) - 1$.

The left subtree, starting with 0 at level $l = 1$ will be denoted by $TTreeL$, and the right subtree, starting with 2 at level $l = 0$ is named $TTreeR$. The node 0 at level $l = 1$ belongs to both subtrees. The number of nodes on level l of the left subtree $TTreeL$ is $T(l+1)$, for $l \geq 1$; the number of nodes on level l of $TTreeR$ is 1 for $l = 0$ and $l = 1$ and $T(l+2) - T(l+1)$, for $l \geq 2$.

$TWord$ considered as ternary sequence t is given in [4] A080843 (we omit the OEIS reference henceforth if A numbers for sequences are given): $\{0, 1, 0, 2, 0, 1, 0, 0, 1, 0, 2, 0, 1, ...\}$,

[1] http://www.itp.kit.edu/~wl

starting with with $t(0) = 0$. See also *Table 1*. This is the analogue of the binary rabbit sequence A005614 in the *Fibonacci* case. Like in the *Fibonacci* case with the complementary and disjoint *Wythoff* sequences A = A000201 and B = A001950 recording the positions of 1 and 0, respectively, in the tribonacci case the sequences A = A278040, B = A278039, and C = A278041 record the positions of 1, 0, and 2, respectively. These sequences start with $A = \{1, 5, 8, 12, 14, 18, 21, 25, 29, 32, ...\}$, $B = \{0, 2, 4, 6, 7, 9, 11, 13, 15, 17, ...\}$, and $C = \{3, 10, 16, 23, 27, 34, 40, 47, 54, 60, ...\}$. The offset of all sequences is 0. See also *Table 1*.

The present work is a generalization of the theorem given in the *Fibonacci* case for the equivalence of the *Zeckendorf* and *Wythoff* representations of numbers in [3].

Note that there are other complementary and disjoint tribonacci A, B and C sequences given in OEIS. They use the same ternary sequence t = A080843 (which has offset 0), with $0 \to a$, $1 \to b$ and $2 \to c$, however with offset 1, and record the positions of a, b and c by A = A003144, B = A003145 and C = A003146, respectively. In [2] and [1] they are called a, b, and c. This tribonacci ABC-representation is given in A317206. The relation between these sequences (we call them now a, b, c) and the present one is: $a(n) = B(n-1) + 1$, $b(n) = A(n-1) + 1$, and $c(n) = C(n-1)+1$, for $n \geq 1$. We used $B(0) = 0$ in analogy to the *Wythoff* representation in the *Fibonacci* case. The ternary triboonacci sequence appears also with offset 1 and entries 1, 2 and 3 as A092782.

From the uniqueness of the ternary sequence t it is clear that the three sequences A, B and C cover the nonnegative integers $\mathbb{N}_0$ completely, and they are disjoint. In contrast to the *Fibonacci* case where the *Wythoff* sequences are *Beatty* sequences [7] for the irrational number φ =A001622, the golden section, and are given by $A(n) = \lfloor n\,\varphi \rfloor$ and $B(n) = \lfloor n\,\varphi^2 \rfloor$, for $n \in \mathbb{N}$ (with $A(0) = 0 = B(0)$), no such formulae for the complementary sequences A, B and C in the tribonacci case are considered. The definition given above in terms of $TWord$, or as sequence t, is not burdened by numerical precision problems.

Note that the irrational tribonacci constant $\tau = 1.83928675521416...$ =A058265, the real solution of characteristic cubic equation of the tribonacci recurrence $\lambda^3 - \lambda^2 - \lambda - 1 = 0$, defines, together with $\sigma = \frac{\tau}{\tau - 1} = 2.19148788395311...$ =A316711 the complementary and disjoint *Beatty* sequences $At := \lfloor n\,\tau \rfloor$ and $Bt := \lfloor n\,\sigma \rfloor$, given in A158919 and A316712, respectively.

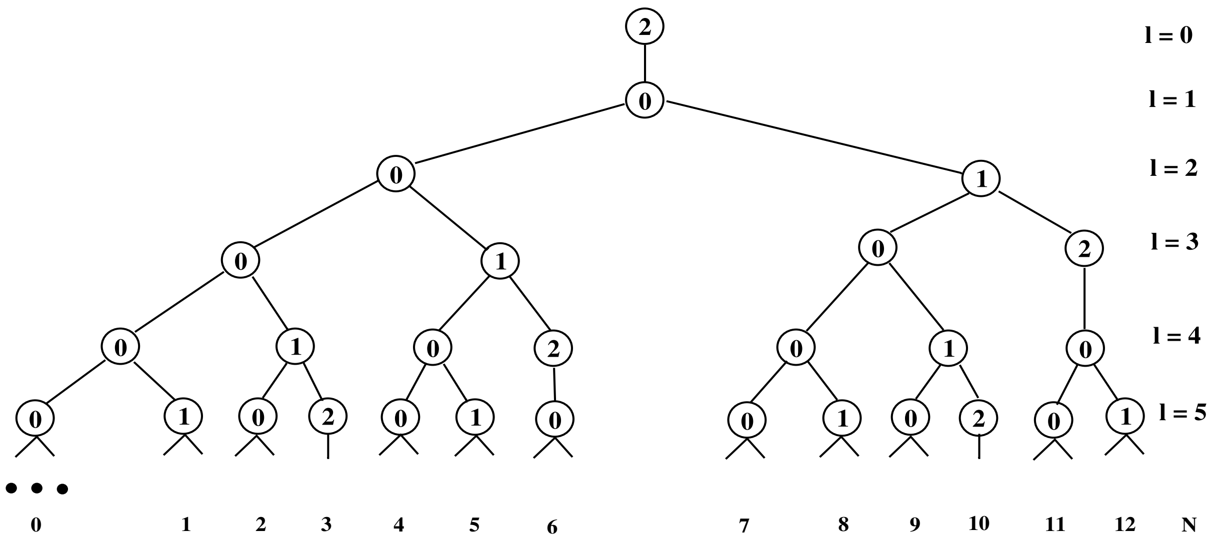

Figure 1: Tribonacci Tree TTree$_5$

The analogue of the unique *Zeckendorf* representation of positive integers is the unique tribonacci representation of these numbers.

$$(N)_T \,=\, \sum_{i=0}^{I(N)} f_i(N)\, T(i+3),\quad f_i(N) \,\in\, \{0,\,1\},\quad f_i(N)\, f_{i+1}(N)\, f_{i+2}(N) \,=\, 0,\;\, f(N)_{I(N)} \,=\, 1\,. \tag{1}$$

The sum should be ordered with falling T indices. This representation will also be denoted by (Z is used as a reminder of *Zeckendorf*)

$$\begin{aligned} ZT(N) &= \text{Ħ}_{i=0}^{I(N)}\, f(N)_{I(N)-i} \\ &= f(N)_{I(N)} f(N)_{I(N)-1} ... f(N)_0\,. \end{aligned} \tag{2}$$

The product with concatenation of symbols is here denoted by Ħ, and the concatenation symbol $\circ$ is not written. This product has to be read from the right to the left with increasing index i. This representation is given in A278038(N), for $N \,\geq\, 1$. See also *Table 2* for $ZT(N)$ for $N \,=\, 1,\, 2,\, ...,\, 100$.

E.g., $(1)_T \,=\, T(3)$, $ZT(1) \,=\, 1$; $(8)_T \,=\, T(6) \,+\, T(3)$, $ZT(8) \,=\, 1001$. The length of $ZT(N)$ is $\#ZT(N) \,=\, I(N)+1 \,=\, \{1,\,2,\,2,\,3,\,3,\,3,\,4,\,4,\,4,\,4,\,4,\,4,\,...\} \,=\, \{$A278044$(N)\}_{N\,\geq\,1}$. The number of occurrences of each $n \,\in\, \mathbb{N}$ in this sequence is given by $\{1,\,2,\,3,\,6,\,11,\,20,\,37\,...\} \,=\, \{$A001590$(n+2)\}_{n\,\geq\,1}$. These are the companion tribonacci numbers of $T \,=$A000073 with inputs 0, 1, 0 for $n \,=\, 0,\,1,\,2$, respectively.

$ZT(N)$ can be read off any finite $TTree_n$ with $T(n+2) \,\geq\, N$ after all node labels 2 have been replaced by 1. See *Figure 1* for $n \,=\, 5$ (with $2 \,\rightarrow\, 1$) and numbers $N \,=\, 1,\,2,\,...,\,12$. The branch for N is read from bottom to top, recording the labels of the nodes, ending with the last 1 label. Then the obtained binary string is reversed in order to obtain the one for $ZT(N)$. *E.g.*, $N \,=\, 9$ leads to the string 0101 which after reversion becomes $ZT(9) \,=\, 1010$.

The analogue of the *Wythoff AB* representation of nonnegative integers in the *Fibonacci* case is the tribonacci ABC representation using iterations of the sequences A, B and C.

$$(N)_{ABC} = \left(\prod_{j=1}^{J(N)} X(N)_j^{k(N)_j}\right) B(0), \text{ with } N \in \mathbb{N}_0, \ k(N)_j \in \mathbb{N}_0, \tag{3}$$

again with an ordered concatenation product. Here $X(N)_j \in \{A, B, C\}$, for $j = 1, 2, ..., J(N) - 1$, with $X(N)_j \neq X(N)_{j+1}$, and $X(N)_{J(N)} \in \{A, C\}$. Powers of $X(N)_j$ are also to be read as concatenations. Concatenation means here iteration of the sequences. The exponents can be collected in $\vec{k}(N) := (k(N)_1, ..., k(N)_{J(N)})$. For the equivalence proof only positive integers N are considered. If exponents vanish the corresponding A, B, C symbols are not present ($X(N)_j^0$ is of course not 1). If all exponents vanish, the product $\prod$ is empty, and $N = 0$ could be represented by $(0)_{ABC} = B(0) = 0$ (but this will not be used for the equivalence proof). Each ABC representation ends in a single B acting on 0 because the argument of $X \in \{A, B, C\}$ for $N = 1, 2$ and 3 is 0 written as $B(0)$, and for other N the iterative tracking of the arguments always leads back to one of these three numbers. *E.g.*, for $N = 15$ these arguments are 8 (for $X = B$), 2 (for A) and 1 (for A) and $1 = B(0)$, hence $15 = B(A(B(A(B(0)))))$, later abbreviated as 01010 (see it Table 3).

E.g., $(30)_{ABC} = (BCBA)B(0) = B(C(B(A(B(0)))))$, $J(30) = 4$, $\vec{k}(30) := (k_1, k_2, k_3, k_4) = (1, 1, 1, 1)$, $X(30)_1^{k_1} = B^1$, $X(30)_2^{k_2} = C^1$, $X(30)_3^{k_3} = B^1$, $X(30)_4^{k_4} = A^1$ (sometimes the arguments (N) are skipped).

The number of A, B and C sequences present in this representation of N is $\sum_{j=1}^{J(N)} k(N)_j + 1 =$ A316714(N), This representation is also written as

$$ABC(N) = \left(\prod_{j=1}^{J(N)} x(N)_j^{k(N)_j}\right) 0, \text{with } k(N)_j \in \mathbb{N}_0, \tag{4}$$

and $x(N)_j \in \{0, 1, 2\}$, for $j = 1, 2, ..., J(N) - 1$, with $x(N)_j \neq x(N)_{j+1}$, and $x(N)_{J(N)} \in \{1, 2\}$. Here $x = 0, 1, 2$ replaces $X = B, A, C$, respectively.

E.g., $ABC(0) = 0$, $J(0) = 0$ (empty product); $ABC(30) = 02010$.

For this ABC representation see A319195. Another version is A316713 (where for a technical reason B, A, and C are represented by 1, 2 and 3 (not 0, 1 and 2), respectively). See also *Table 3* for $ABC(N)$, for $N = 1, 2, ..., 100$.

The number of Bs, As and Cs in the ABC representation of N is given in sequences A316715, A316716 and A316717, respectively. As already mentioned the length of this representation is given in A3167174.

2 Equivalence of representations

The unique representation $(N)_T$ of $N \in \mathbb{N}$ in terms of tribonacci numbers given in eq. 1, or equivalently as a representation as binary word $ZT(N)$ in eq. 2, and the unique representation N_{ABC} as composition of A, B, C sequences given in eq. 3 will be shown to be equivalent, *i.e.*, they can be transformed into each other without computing the actual value of N. This will be done in two directions with the help of two auxiliary representations denoted by $\widehat{ZT}(N)$ and $(N)_{AB\bullet\times}$. See the *Figure 2* for the pictogram of the transformations (mappings) between these representations in the anti-clockwise and clockwise directions. The uniqueness of the two representations $ZT(N)$ and $(N)_{ABC}$ will be proved in *section 3*.

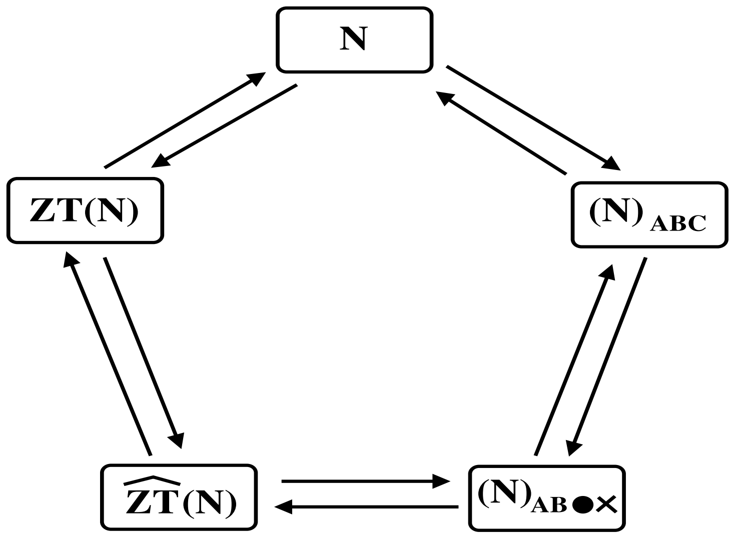

Figure 2: Equivalence of representations

I) From ZT(N) to $(\mathbf{N})_{\mathbf{ABC}}$

A) From ZT(N) to $\widehat{\mathbf{ZT}}$(N)

Definition 1. The binary word $ZT(N)$ of length A27804(N), for $N \in \mathbb{N}$, is mapped to a binary word $\widehat{ZT}(N) := 0\overline{ZT(N)}0$ of length A27804(N) + 2, where $\overline{ZT(N)}$ is the reversed word $ZT(N)$.

E.g., $\widehat{ZT}(1) = 010$ from $ZT(1) = 1$; $\widehat{ZT}(30) = 00110010$ from $ZT(30) = 100110$.

B) From $\widehat{\mathbf{ZT}}$(N) to $(\mathbf{N})_{\mathbf{AB\bullet\times}}$ The following six substitution rules for all entries of $\widehat{ZT}(N)$ except the last one are used in the following.

Definition 2. The six substitution rules

$$
\begin{array}{llcl}
(S\underline{\mathbf{0}}1) & \underline{\mathbf{0}}0 & : & \underline{\mathbf{0}} \longrightarrow B\,, \\
(S\underline{\mathbf{0}}2) & \underline{\mathbf{0}}11 & : & \underline{\mathbf{0}} \longrightarrow \times\,, \\
(S\underline{\mathbf{0}}3) & \underline{\mathbf{0}}10 & : & \underline{\mathbf{0}} \longrightarrow A\,, \\
(S\underline{\mathbf{1}}1) & \underline{\mathbf{1}}1 & : & \underline{\mathbf{1}} \longrightarrow \times\,, \\
(S\underline{\mathbf{1}}2) & \underline{\mathbf{1}}0x & : & \underline{\mathbf{1}} \longrightarrow \bullet\,, \text{ for } x \in \{0,1\}, \\
(S\underline{\mathbf{1}}3) & \underline{\mathbf{1}}0\emptyset & : & \underline{\mathbf{1}} \longrightarrow B\,.
\end{array}
$$

The boldface and underlined 0 and 1 entries of $\widehat{ZT}(N)$ are substituted depending on which entries follow. The empty word symbol $\emptyset$ marks the end of this binary word. The last entry

0 of $\widehat{ZT}(N)$ is not substituted (it disappears), therefore the resulting $(N)_{AB\bullet\times}$, a word over the alphabet of the four letters A, B, $\bullet$, $\times$, ends always in a single B.

E.g., $\widehat{ZT}(N) = 0101010011010$ with $\#\widehat{ZT}(N) = 13$ translates to $(N)_{AB\bullet\times} = A\bullet A\bullet A\bullet B\times\times\bullet AB$ with length $\#(N)_{AB\bullet\times} = 12$. The value of the number N is here not relevant.

These substitution rules suffice and are not in conflict which each other. In $\widehat{ZT}(N)$ no three consecutive 1s appear and the end is always 10. All possible entries of $\widehat{ZT}(N)$, except the last 0, can be substituted.

E.g., $1\underline{\mathbf{1}}0$ is not needed, because if this is not the end of $\widehat{ZT}(N)$ the substitution applies to $1\underline{\mathbf{1}}00$ or $1\underline{\mathbf{1}}01$, *i.e.*, $(S\underline{\mathbf{1}}2)$ in both cases. If it appears at the end substitution $(S\underline{\mathbf{1}}3)$ applies which produces $\times B$, because the leading 1 had to be substituted by $(S\underline{\mathbf{1}}1)$.

A list of the 20 triplets, called tribons, which can appear in $(N)_{AB\bullet\times}$ is given in the next lemma. The multiplicity of each tribon is shown by giving the corresponding underlined triplets of $\widehat{ZT}(N)$ and their following two entries (if also $\emptyset$ is used to signal the end). To distinguish these quintets the 1 which in the tree $TTree$ (see *Figure 1*) is actually a 2 is given in boldface and red. A vertical bar signals the end of a $(N)_{AB\bullet\times}$ representation.

Lemma 3. The 20 tribons
First member A (three tribons, multiplicities 1, 3, 1):

$$A\bullet A:\ \underline{010}10;\quad A\bullet B:\ \underline{010}00,\ \underline{010}01,\ \underline{010}0\mathbf{1};\quad A\bullet\times:\underline{010}\mathbf{1}$$

First member B (six tribons, multiplicities 3, 1, 1, 3, 1, 1):

$$BBB:\ \underline{000}00,\ \underline{000}01,\ \underline{000}0\mathbf{1};\quad BBA:\ \underline{000}10;\quad BAB|:\ \underline{001}0\emptyset,$$
$$BA\bullet:\ \underline{001}00,\ \underline{001}01,\ \underline{001}0\mathbf{1};\quad BB\times:\ \underline{00\mathbf{1}}1;\quad B\times\times:\ \underline{00\mathbf{1}}10.$$

First member • (six tribons, multiplicities 3, 1, 1 3, 1, 1):

$$\bullet A\bullet:\ \underline{101}00,\ \underline{101}01,\ \underline{101}0\mathbf{1};\quad \bullet AB|:\ \underline{101}0\emptyset;\quad \bullet BA:\ \underline{100}10,$$
$$\bullet BB:\ \underline{100}00,\ \underline{100}01,\ \underline{100}0\mathbf{1};\quad \bullet B\times:\ \underline{100}\mathbf{1}1;\quad \bullet\times\times:\ \underline{10\mathbf{1}}10.$$

First member × (five tribons, multiplicities 1, 3, 1, 3, 1):

$$\times\bullet A:\ \underline{\mathbf{1}10}10;\quad \times\bullet B:\ \underline{\mathbf{1}10}00,\ \underline{\mathbf{1}10}01,\ \underline{\mathbf{1}10}0\mathbf{1};$$
$$\times\bullet\times:\ \underline{\mathbf{1}10}\mathbf{1}1;\quad \times\times\bullet:\ \underline{0\mathbf{1}1}00,\ \underline{0\mathbf{1}1}01,\ \underline{0\mathbf{1}1}0\mathbf{1};$$
$$\times\times B|:\ \underline{0\mathbf{1}1}0\emptyset.$$

Corollary 4. *The possible* 11 *doublets in* $(N)_{AB\bullet\times}$ *are:*

$$A\bullet,\ AB|,\ BB,\ BA,\ B\times;\ \bullet A,\ \bullet B,\ \bullet\times;\ \times\bullet,\ \times\times,\ \times B|. \tag{5}$$

The following lemma collects information which is also later used for proving that the map from $\widehat{\mathbf{ZT}}(\mathbf{N})$ to $(N)_{AB\bullet\times}$ is invertible.

Lemma 5. Rules for $(\mathbf{N})_{\mathbf{AB\bullet\times}}$

1) There is no doublet $\bullet\bullet$.

2) $\times$ *always appears either as the triplet* $\times\times\bullet$ *or at the end as the triplet* $\times\times B|$. *There is no tribon* $\times\times\times$ *(see above).*

3) $\bullet$ *always appears either in the doublet* $A\bullet$ *or in the triplet* $\times\times\bullet$.

4) A always appears either as doublet $A\bullet$ *or as doublet* $AB|$ *at the end.*

Proof. 1) From the *Corollary* 4.

2) $\times$ originates either from the substitution i) $\underline{0}11$ or ii) $\underline{1}1$. Case i) continues as $\underline{0}110x$ with $x \in \{0, 1\}$ or as $\underline{0}110\emptyset$ because no three consecutive 1s appear in $\widehat{ZT}(N)$. Hence after substitution as $xx\bullet$ or as $xxB|$. Case ii) continues to the left as $0\underline{1}1$ (no three consecutive 1s), hence is substituted by $xx\bullet$ or $xxB|$ because to the right follows $0x$ or $0\emptyset$.

3) A $\bullet$ originates only from $\underline{\mathbf{1}}0x$ with $x \in \{0, 1\}$. It can be continued to the left as i) $00\underline{\mathbf{1}}0x$, ii) $10\underline{\mathbf{1}}0x$ or iii) $01\underline{\mathbf{1}}0x$ (no three consecutive 1s). After substitution i) leads to $BA\bullet$ and iii) to $\times\times\bullet$. Case ii) leads to the tribon $\bullet A\bullet$ and a continuation to the left will find either an A or a $\times\times$, or again a new $\bullet$ on the left which has to be continued, etc.

4) A originates only from $\underline{010}$ and it continues with $x \in \{0, 1\}$ if this is not the end of the representation. In the first case it becomes $A\bullet$, and in the second $AB|$ without a $\bullet$ following A. ∎

Note that whenever $\times$ appears in a tribon other than one of the two given in *2*, *i.e.*, as $\bullet\times\times$ or as $\times\bullet\times$, this means that these tribons are continued in both directions to give one of the claimed two combination. Similarly if A is not followed by $\bullet$ or the final B, or if A is preceded by a $\bullet$ the continuation to the right or left will show the claimed two proved combinations in any representation of N.

This $(N)_{AB\bullet\times}$ representation can be visualized on an edge and node labeled tribonacci tree, called $ABCTree_n$ if the bottom level is n. Such a tree can be used for the representation of $N \in \{1, 2, ..., T(n+2) - 1\}$. $N = 0$ could also be represented but in the equivalence proof only positive N is considered. The nodes are labeled like in the tribonacci tree $TTree_n$. The tree is read from the bottom level $l = n$ upwards. The edges are labeled on both sides. If the labels on both sides coincide only one label for the edge is depicted. The edges 00, 01 and 02 (always upwards directed) are labeled on the left-hand side by B and on the right-hand side by $\bullet$. An exception is the top edge 02 between levels $l = 1$ and $l = 0$ which has only a B label for both sides. The edge 10 has the labels A and $\times$ on the left- and right-hand side, respectively. The outermost left branch has also only edge label B for both sides.

The labeling is done in accordance with the rules of the $(N)_{AB\bullet\times}$ word given above. One has to start for the N representation at the bottom level $l = n$ always with the left-hand label of the first edge. For the next edges going out from from a node, the choice is fixed from the direction from which the previous edge reached the node. If it reached the node from the right-hand side, the label on the right-hand side of the outgoing edge has to be chosen, and similarly for the left-hand side. Because the representation ends always in a single B if an N belongs to the left sub-tree $TTreeL_n$ the path stops after the first edge labeled B on the leftmost branch has been reached. This explains why also the edge between levels $l = 2$ and

$l = 1$ has to belong to $TTreeL_n$, hence the node 0 at level $l = 1$ belongs to both sub-trees. For N belonging to the right sub-tree $TTreeR_n$ the path goes all the way up to level $l = 0$ with last edge B. See *Figure 3* for $ABCTree_5$ for numbers $N = (0), 1, 2, ..., 12$.

E.g., $N = 6$: $B \times\times B$, starting with the left-hand side label B, then the second $\times$ has to be chosen because the previous edge came from the right-hand side. $N = 9$: $BA \bullet AB$.

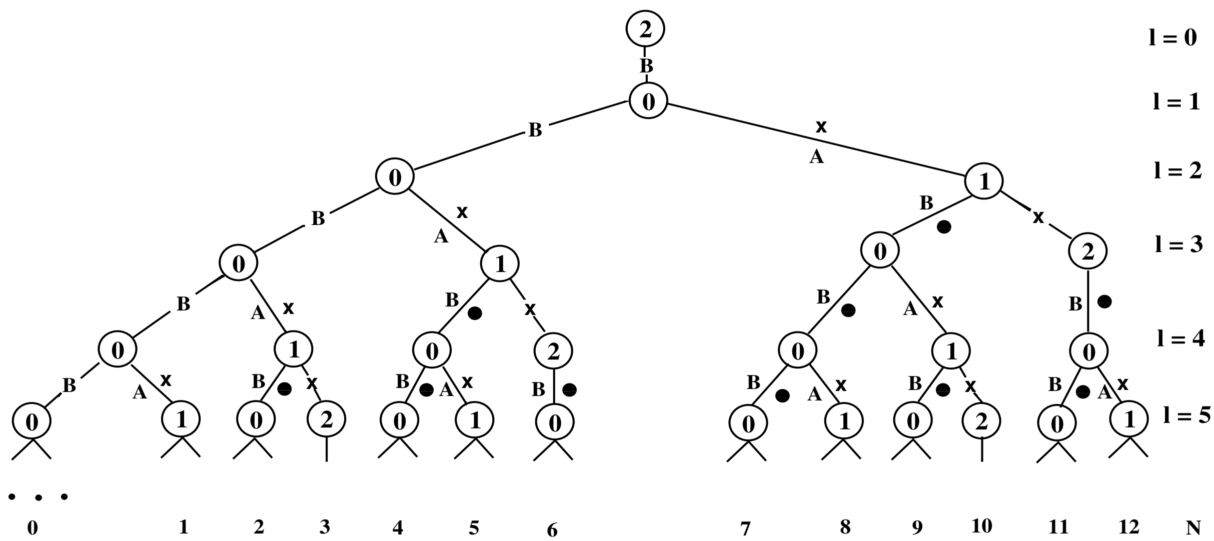

Figure 3: ABC-representation with the $\mathbf{ABCTree_5}$

C) From $\mathbf{(N)_{AB\bullet\times}}$ to $\mathbf{(N)_{ABC}}$

This step is simply performed by the following replacements. It is understood that compositions are meant.

$$A\bullet \rightarrow A, \text{ and at the end } AB \rightarrow AB(0),$$
$$\times\times\bullet \rightarrow C, \text{and at the end } \times\times B \rightarrow CB(0),$$
$$B \rightarrow B.$$

E.g., $N = 6$: $B \times\times B \rightarrow BCB(0)$, actually $(6)_{ABC} = B(C(B(0)))$. In *Table 3* this is denoted as $ABC(6) = 020$, denoting A, B, C as $1, 0, 2$.
$N = 9$: $BA \bullet AB \rightarrow BAAB(0)$, actually $(9)_{ABC} = B(A(A(B(0))))$. Or $ABC(9) = 0110$.

These transformations (mappings) between the representations are now proved to be invertible by reversing the steps.

II) From $\mathbf{(N)_{ABC}}$ to $\mathbf{ZT(N)}$

$\overline{\mathbf{C}}$) From $\mathbf{(N)_{ABC}}$ to $\mathbf{(N)_{AB\bullet\times}}$

$$\text{a final } AB(0) \rightarrow AB, \text{ and } A \rightarrow A\bullet,$$
$$\text{a final } CB(0) \rightarrow CB, \text{ and } C \rightarrow \times\times\bullet,$$
$$B \rightarrow B.$$

E.g., $(N)_{ABC} \;=\; A^3BCAB(0) \;\rightarrow\; (N)_{AB\bullet\times} \;=\; A\bullet A\bullet A\bullet B\times\times\bullet AB$.

$\overline{\mathbf{B}}$) From $(\mathbf{N})_{\mathbf{AB}\bullet\times}$ to $\widehat{\mathbf{ZT}}(\mathbf{N})$

Two equivalent replacements can be given based on the inversion of the substitution rules of *Definition* 2 taking into account the structure of $(N)_{AB\bullet\times}$.

Version 1)

$$\begin{array}{llll} \text{at the end} & AB| \;\rightarrow\; 010, & A\bullet \;\rightarrow\; 01\,, \\ \text{at the end} & \times\times\, B| \;\rightarrow\; 0110\,, & \times\times\bullet \;\rightarrow\; 011, \\ B \;\rightarrow\; 0\,. \end{array}$$

Version 2)

Begin with a 0 and replace

$$\begin{array}{llll} \text{at the end} & AB| \;\rightarrow\; 10, & A\bullet \;\rightarrow\; 10\,, \\ \text{at the end} & \times\times\, B| \;\rightarrow\; 110\,, & \times\times\bullet \;\rightarrow\; 110, \\ B \;\rightarrow\; 0\,. \end{array}$$

This version can be stated as: Begin with a 0 and replace A and $\times$ by 1, and B and $\bullet$ by 0.

E.g., Version 1) The substitution rules are

$$\begin{array}{rcl} (N)_{AB\bullet\times} &=& A\bullet A\bullet A\bullet B\;\times\times\;\bullet AB\,, \\ \widehat{ZT}(N) &=& 01\;01\;01\,0\;\;0\;1\;\;1\,0\,1\,0\,. \end{array}$$

This coincides with the stated replacements.

In version 2) one shifts $(N)_{AB\bullet\times}$ by one position to the right, producing the leading 0 in $\widehat{ZT}(N)$.

$$\begin{array}{rcl} (N)_{AB\bullet\times} &=& A\bullet A\bullet A\bullet B\;\times\times\;\bullet AB\,, \\ \widehat{ZT}(N) &=& 010\;10\,10\;0\;1\;\;1\;\;\;0\,1\,0\,. \end{array}$$

$\overline{\mathbf{A}}$) From $\widehat{\mathbf{ZT}}(\mathbf{N})$ to $\mathbf{ZT}(\mathbf{N})$

Omit the two zeros at the beginning and end, and reverse the binary string to obtain $ZT(N)$.

3 Equivalence of representations ZT(N) and ABC(N)

First the uniqueness of the tribonacci-representation $ZT(N)$ of eq. 2 is considered.

It is clear that every binary sequence starting with 1, without three consecutive 1s, represents some $N \in \mathbb{N}$. An algorithm for finding such a representation for every $N \in \mathbb{N}$ is given to prove the following lemma.

Lemma 6. *The tribonacci-representation $ZT(N)$ of eq.2 is unique.*

Proof:

The recurrence of the tribonacci sequence $T := \{T(l)\}_{l=3}^{\infty}$, with inputs $T(3) = 1$, $T(4) = 2$ and $T(5) = 4$, shows that this sequence is strictly increasing. Define the floor function $floor(T;\, n)$, for $n \in \mathbb{N}$, giving the largest member of T smaller or equal to n. The corresponding index of T will then be called $Ind(floor(T;\, n))$. Define the finite sequence $Nseq := \{N_j\}_{j=1}^{j_{\max}}$ recursively by

$$N_j = N_{j-1} - floor(T;\, N_{j-1})\,, \quad \text{for } j = 1,\, 2,\, ...,\, j_{\max}\,, \tag{6}$$

with $N_0 = N$ and $N_{j_{\max}} = 0$, but$N_{j_{\max}-1} \neq 0$.

It is clear that this recurrence reaches always 0. Define the finite sequences $fTN := \{floor(T;\, N_j)\}_{j=0}^{j_{\max}-1}$ and $IfTN := \{Ind(fTN_j)\}_{j=0}^{j_{\max}-1}$. Then $I(N)$ in eq. 2 is given by $I(N) = IFTN_0$ and the finite sequence $fseq = \{f_{I(N)-k}\}_{k=0}^{I(N)}$ is given by

$$f_{I(N)-k} = \begin{cases} 1 & \text{if } I(N) - k \in IfTN\,, \\ \\ 0 & \text{otherwise .} \end{cases} \tag{7}$$

□

Example 3 $N = 263$. $Nseq = \{263,\, 144,\, 33,\, 9\, 2,\, 0\}$, $fTN = \{149,\, 81,\, 24,\, 7,\, 2\}$, $IfTN = \{8,\, 7,\, 5,\, 3,\, 1\}$, $I(N) = 8$, $fseq = \{1,\, 1,\, 0,\, 1,\, 0,\, 1,\, 0,\, 1,\, 0\}$.

Next follows the lemma on the uniqueness of the ABC representation given in eq. 3 or eq. 4.

Lemma 7. *The tribonacci ABC representation $(N)_{ABC}$ of eq.3, for $N \in \mathbb{N}_0$, is unique.*

Proof:

From the definition of the A, B and Csequences (each with offset 0) based on the value 1, 0 and 2, respectively, of $t(n)$, for $n \in \mathbb{N}_0$, it is clear that these sequences are disjoint and $\mathbb{N}_0$-complementary. 0 is represented by $B(0)$. Therefore the n-fold iteration $B^{[n]}(0)$ (written as $B^n(0)$) is allowed only for $n = 1$, and any representation ends in $B(0)$. Iterations acting on 0 are encoded by words over the alphabet $\{A,\, B,\, C\}$, and n-fold repetition of a letter X is written as X^n, named $X-$block, where $n = 0$ means that no such $X-$block is present. Then any word consisting of consecutive different non-vanishing $X-$blocks ending in the $B-$block B^1 represents a number $N \in \mathbb{N}_0$.

In order to prove that with such representations every $N \in \mathbb{N}_0$ is reached the following algorithm is used. Replace any number $n \in \mathbb{N}_0$, which is $n = X_n(k(n))$ with $X_n \in \{A,\, B,\, C\}$ and $k(n) \in \mathbb{N}_0$, by the 2-list $L(n) = [L(n)_1,\, L(n)_2] := [X_n,\, k(n)]$. Define the recurrence

$$L(j) = [L(L(j-1)_2)_1,\, L(L(j-1)_2)_2], \quad \text{for } j = 1,\, 2,\, ...,\, j_{\max}\,, \tag{8}$$

with input $L(0) = [X_N,\, k(N)]$, and $j_{\max}$ is defined by $L(j_{\max}) = [B,\, 0]$.

Then the word is $w(N) = \bigsqcup_{j=0}^{j_{\max}} L(j)_1$ (a concatenation product), and read as iterations acting on 0 this becomes the representation $(N)_{ABC}$. The length of the word $w(N)$ is $j_{\max} + 1$.

□

Example 4 $N = 38$. $L(0) = [A, 11]$, $L(1) = [B, 6]$, $L(2) = [B, 3]$, $L(3) = [C, 0]$, and $L(4) = [B, 0]$, hence $j_{\max}(38) = 4$, $w(38) = ABBCB$, and $(38)_{ABC} = ABBCB(0)$, to be read as $A(B(B(C(B(0)))))$.

After these preliminaries the main theorem can be stated.

Theorem. *The tribonacci-representation $ZT(N)$ of eq. 1, is equivalent to the tribonacci ABC-representation $(N)_{ABC}$ eq. 3, for $N \in \mathbb{N}$. I.e., given any binary word $ZT(N)$ of eq. 1, beginning with 1 and no three consecutive 1s, which represents a unique N one finds the unique representation of this N given by $(N)_{ABC}$ without knowing the actual value of N; and vice versa, given a word $(N)_{ABC}$, representing uniquely a number N one finds the representation of this N as a binary word $ZT(N)$.*

Proof:
Part **I)**: The proof of the map $ZT(N) \to (N)_{ABC}$ is performed in three steps:

$$\begin{aligned} \text{Step A)}: &\quad ZT(N) \to \widehat{ZT}(N) := 0(ZT(N)_{\text{reverse}})0\,, \\ \text{Step B)}: &\quad \widehat{ZT}(N) \to (N)_{AB\bullet\times}\,, \\ \text{Step C)}: &\quad (N)_{AB\bullet\times} \to (N)_{ABC}\,. \end{aligned}$$

These steps have been explained in *section 2*, part I) in detail. Step A) is clear. In step B) the six substitution rules of *Definition* 2 are important. They lead to the result that in $(N)_{AB\bullet\times}$ only certain combinations like $A\bullet$, $\times\times\bullet$, and at the end of the representation AB or $\times\times B$ can occur, besides any powers of B, but only one B at the end. This $(N)_{AB\bullet\times}$ words can also be read off from the $ABCTree_n$, given for $n = 5$ in *Figure 3*, read from bottom to top with certain rules to choose the edge labels given in *section 2*. Step C) is again simple, because only the mentioned combinations can appear in $(N)_{AB\bullet\times}$.
Part **II)**: The proof of the map $(N)_{ABC} \to ZT(N)$ is performed in three steps:

$$\begin{aligned} \text{Step } \overline{C}): &\quad (N)_{ABC} \to (N)_{AB\bullet\times}\,, \\ \text{Step } \overline{B}): &\quad (N)_{AB\bullet\times} \to \widehat{ZT}(N)\,, \\ \text{Step } \overline{A}): &\quad \widehat{ZT}(N) \to ZT(N)\,. \end{aligned}$$

Step $\overline{C}$) uses the specific occurrences of $A\bullet$, $\times\times\bullet$, and special endings mentioned above. Step $\overline{B}$) is given in two equivalent versions. This is based on the six substitution rules 2 in the reverse direction, using again the special combinations in which A and $\times\times$ appear. Step $\overline{C}$) is again trivial. □

4 Investigation of the A, B and C sequences

In this section a detailed investigation of the A, B and C sequences is presented. Some of these results can be found in references [2], [1], in [4] and elsewhere, but here the emphasis is on a derivation based on the infinite ternary tribonacci word $TWord$ written as a sequence $t =$ A080843 (see also *Table 1*). Its self-similarity leads to the following *Definition*, *Lemma* and six *Propositions*.

Definition 8. The tribonacci words $tw(l)$ over the alphabet $\{0, 1, 2\}$ of length $\#tw(l) = T(l+2)$ are defined recursively by concatenations (we omit the concatenation symbol $\circ$) as

$$tw(l) = tw(l-1)\, tw(l-2)\, tw(l-3), \quad \text{with } tw(1) = 0,\, tw(2) = 01,\, tw(3) = 0102\,. \tag{9}$$

Also $tw(0) = 2$ is used.
The substitution map acting on tribonacci words and other strings with characters $\{0, 1, 2\}$ is defined as a concatenation homomorphism by $\sigma : 0 \mapsto 01,\ 1 \mapsto 02,\ 2 \mapsto 0$. The inverse map is $\sigma^{[-1]}$ (One replaces first each 01 and 02 then the left over 0). With σ the words $tw(l)$ are generated iteratively from $tw(0) = 2$. $\sigma(tw(l)) = tw(l+1)$, for $l \in \mathbb{N}_0$, and $\lim_{l\to\infty} \sigma^{[l]}(0) = TWord$. Self-similarity of $TWord$ means $\sigma(TWord) = TWord$.
Substrings of $TWord$ of length n, starting with the first letter (number) $t(0) = 0$, are denoted by $s_n := \text{Ħ}_{j=0}^{n-1} t(n)$. If $n = T(l+2)$, for $l \in \mathbb{N}_0$, then $s_n = tw(l)$ (the string becomes a tribonacci word), and the numbers of s_n map to the node labels of the last level of $TTree_l$ read from the left-hand side.
Also substrings of $TWord$ not starting with $t(0)$ are used, like $\hat{s}_2 = 02 = \sigma(1)$, starting with $t(2)$.

In the following *Lemma* four definition of infinite words t_i, for $i = 1, 2, ..., 4$, based on certain substrings s_n (not all of them are tribonacci words) are given which are proved to be different partitions of $TWord$. This result will be used in the proofs of the following first *Propositions* 10, 11 and 13.

Lemma 9.

A) *With* $s_{13} = 0102010010201 = tw(5)$, $s_{11} = 01020100102$ *and* $s_7 = 0102010 = tw(4)$ *define*

$$t_1 = s_{13}s_{11}s_{13}s_7s_{13}s_{11}s_{13}s_{13}s_{11}s_{13}s_7s_{13}\ldots = \text{Ħ}_{j=0}^{\infty}\, s_{\varepsilon(t(j))}, \tag{10}$$

where $\varepsilon(0) = 13$, $\varepsilon(1) = 11$ *and* $\varepsilon(2) = 7$.
B) *With* $s_7 = 0102010 = tw(4)$, $s_6 = 010201$ *and* $s_4 = 0102 = tw(3)$ *define*

$$t_2 = s_7s_6s_7s_4s_7s_6s_7s_7s_6s_7s_4s_7\ldots = \text{Ħ}_{j=0}^{\infty}\, s_{\pi(t(j))}, \tag{11}$$

where $\pi(0) = 7$, $\pi(1) = 6$ *and* $\pi(2) = 4$.
C) *With* $s_4 = 0102 = tw(3)$, $s_3 = 010$ *and* $s_2 = 01 = tw(2) = \sigma(0)$ *define*

$$t_3 = s_4s_3s_4s_2s_4s_3s_4s_4s_3s_4s_2s_4\ldots = \text{Ħ}_{j=0}^{\infty}\, s_{\tau(t(j))}, \tag{12}$$

where $\tau(0) = 4$, $\tau(1) = 3$ *and* $\tau(2) = 2$.
D) *With* $s_2 = 01$, $\hat{s}_2 = 02$ *and* $s_1 = 0 = tw(1) = \sigma(2)$ *define*

$$t_4 = s_2\hat{s}_2s_2s_1s_2\hat{s}_2s_2s_2\hat{s}_2s_2s_2s_1 \ldots \tag{13}$$

Here the string follows t *with* s_2, $\hat{s}_2$ *and* s_1 *playing the role of* 0, 1 *and* 2, *respectively. Then*

$$t_1 = t_2 = t_3 = t_4 = TWord\,. \tag{14}$$

Proof:
For t_4 from **D**: The definition of $\sigma^{[-1]}$ shows that $\sigma^{[-1]}(t_4) = TWord$. Hence $t_4 = \sigma(TWord) = TWord$.
For t_3 from **C**: Because $\sigma(s_2) = s_4$, $\sigma(\hat{s}_2) = s_3$ and $\sigma(s_1) = s_2$ it follows that $t_3 = \sigma(t_4) = TWord$.
For t_2 from **B**: Because $\sigma(s_4) = s_7$, $\sigma(s_3) = s_6$ and $\sigma(s_2) = s_4$ it follows that $t_2 = \sigma(t_3) = TWord$.
For t_1 from **A**: Because $\sigma(s_7) = s_{13}$, $\sigma(s_6) = s_{11}$ and $\sigma(s_4) = s_7$ it follows that $t_1 = \sigma(t_2) = TWord$.
□

Using eq.12 a formula for sequence entry $A(n) =$ A278040(n) in terms of $z(n) := \sum_{j=0}^{n} t(j)$ is derived. This sequence $\{z(j)\}_{j=0}^{\infty}$ is given in A319198.

Proposition 10.

$$A(n) = 4\,n + 1 - z(n-1), \text{ for } n \in \mathbb{N}_0, \text{ with } z(-1) = 0\,. \tag{15}$$

Proof:
Define $\triangle A(k+1) := A(k+1) - A(k)$. Consider the word t_3 of eq. 12. The distances between the 1s in the pairs s_4s_3, s_3s_4, s_4s_2, s_2s_4 and s_4s_4 are 4, 3, 4, 2, 4. Therefore, the sequence of these distances is 4, 3, 4, 2, 4, 3, 4, 4, 3, 4, 2, ... Thus, because the s-string t_3 follows the pattern of t, *i.e.*, of $TWord$,

$$\triangle A(k+1) = 4 - t(k)\,, \quad \text{for} \quad k = 0,\, 1,\, \ldots\,. \tag{16}$$

Then the telescopic sum produces the assertion, using $A(0) = 1$.

$$A(n) = A(0) + \sum_{k=0}^{n-1} \triangle A(k+1) = 1 + 4\,n - z(n-1), \quad \text{with} \quad z(-1) = 0. \tag{17}$$

□

The B numbers A278039, giving the increasing indices k with $t(k) = 0$, come in three types: $B0$ numbers form the sequence of increasing indices k of sequence t with $t(k) = 0 = t(k+1)$. Similarly the $B1$ sequence lists the increasing indices k with $t(k) = 0$, $t(k+1) = 1$ and for the $B2$ sequence the indices k are such that $t(k) = 0$, $t(k+1) = 2$.
These numbers $B0(n)$, $B1(n)$ and $B2(n)$ are given by A319968$(n+1)$, A278040$(n) - 1$, and A278041$(n) - 1$, respectively.

Before giving proofs we define the counting sequences $z_A(n)$, $z_B(n)$ and $z_C(n)$ to be the numbers of A, B and C numbers not exceeding $n \in \mathbb{N}$, respectively. If these counting functions appear for $n = -1$ they are set to 0.

These sequences are given by A276797$(n+1)$, A276796$(n+1)$ and A276798$(n+1) - 1$ for $n \geq -1$.

Obviously,

$$z(n) = 1\, z_A(n) + 0\, z_B(n) + 2\, z_C(n) = z_A(n) + 2\, z_C(n), \quad \text{for} \quad n = -1, 0, 1, \ldots . \tag{18}$$

These counting functions are obtained by partial sums of the corresponding characteristic sequences for the A, B and C numbers (or 0, 1, and 2numbers in t), called k_A, k_B and k_C, respectively.

$$z_X(n) = \sum_{k=0}^{n} k_X(k), \quad \text{for} \quad X \in \{A, B, C\} . \tag{19}$$

The characteristic sequences members $k_A(n)$, $k_B(n)$ and $k_C(n)$ are given in A276794$(n+1)$, A276793$(n+1)$ and A276791$(n+1)$, for $n \in \mathbb{N}_0$, and they are, in terms of t, obviously given by

$$k_A(n) = t(n)\,(2 - t(n)), \tag{20}$$

$$k_B(n) = \frac{1}{2}\,(t(n) - 1)\,(t(n) - 2), \tag{21}$$

$$k_C(n) = \frac{1}{2}\,t(n)\,(t(n) - 1). \tag{22}$$

By definition it is trivial that (note the offset 0 of the A, B, C sequences)

$$z_X(X(k)) = k + 1, \quad \text{for} \quad X \in \{A, B, C\} \quad \text{and} \quad k \in \mathbb{N}. \tag{23}$$

Proposition 11.

For $n \in \mathbb{N}_0$:

$$\textbf{B0})\; B0(n) = 13\,n + 6 - 2\,[z_A(n-1) + 3\,z_C(n-1)] = 2\,C(n) - n, \tag{24}$$

$$\textbf{B1})\; B1(n) = 4\,n - z(n-1) = 4\,n - [z_A(n-1) + 2\,z_C(n-1)] = A(n) - 1, \tag{25}$$

$$\textbf{B2})\; B2(n) = 7\,n + 2 - [z_A(n-1) + 3\,z_C(n-1)] = \frac{1}{2}\,(B0(n) + n - 2) = C(n) - 1, \tag{26}$$

$$\textbf{B})\; B(n) = 2\,n - z_C(n-1)\,. \tag{27}$$

For $n = 0$ *empty sums in* z_A, z_C *and* z *are set to 0.*

Proof:

B0: Part 1: Define $\triangle B0(k+1) := B0(k+1) - B0(k)$ and consider the word t_1 of eq. 10. The distances between pairs of 00 in $s_{13}s_{11}$, $s_{11}s_{13}$, $s_{13}s_7$, s_7s_{13} and $s_{13}s_{13}$ are 13, 11, 13, 7, 13. Note that s_7 has no substring 00, however because s_7 is always followed by s_{13} the last

0 of s_7 and the first of s_{13} build the 00 pair. Similarly, in the $s_{13}s_7$ case the last 0 of s_7 is counted as a beginning of a 00 pair. Therefore, the sequence of these distances is 13, 11, 13, 7, 13, 11, 13, 13, 11, 13, 7, ... Because the s-string t_1 follows the pattern of t the defect from 13 is 0, -2, -6 if $t(k) = 0, 1, 2$, hence

$$\triangle B0(k+1) \,=\, 13 \,-\, t(k)\,(t(k) \,+\, 1)\,, \quad \text{for} \quad k \,\in\, \mathbb{N}_0\,. \tag{28}$$

The telescopic sum gives for $n \in \mathbb{N}$, with $B0(0) = 6$,

$$\begin{aligned} B0(n) \;&=\; B0(0) \,+\, \sum_{k=0}^{n-1} \triangle\, B0(k+1) \\ &=\; 6 \,+\, 13\,n \,-\, 2\,(z_A(n-1) \,+\, 3\,z_C(n-1))\,. \end{aligned} \tag{29}$$

In the last step eqs. 19, 20 and 22 have been used. This proves the first part of **B0**. The proof of part 2 follows later from **B2**.

B1: With $\triangle B1(k+1) := B1(k+1) - B1(k)$ and t_3 of eq. 12 one finds for the distances between occurrences of 01s similar to the above argument

$$\triangle B1(k+1) \,=\, 4 \,-\, t(k)\,, \quad \text{for} \quad k \,\in\, \mathbb{N}_0\,. \tag{30}$$

The telescopic sum gives, for $n \in \mathbb{N}$, with $B1(0) = 0$,

$$B1(n) \,=\, 4\,n \,-\, z(n-1),$$

the first part of **B1**, which shows, with eq 15, also the third one. The second part uses eq. 18. Note that $B1(n) = A(n) - 1$ is trivial because 1 in the tribonacci word $TWord$ can only come from the substitution $\sigma(0) = 01$, and $TWord$ (and t) starts with 0. Therefore, one could directly prove **B1** from eqs. 15 and 18 without first computing $\triangle B1(k+1)$.

B2: Because 2 in $TWord$ appears only from $\sigma(1) = 02$, it is clear that $B2(n) = C(n) - 1$. Now one finds a formula for C by looking first at $\triangle\, C(k+1) := C(k+1) - C(k)$ using t_2 of eq. 11. The distances between consecutive 2s in the five pairs s_7s_6, s_6s_7, s_7s_4, s_4s_7 and s_7s_7 is 7, 6, 7, 4, 7, respectively, and

$$\triangle\, C(k+1) \,=\, 7 \,-\, \frac{1}{2}\,t(k)\,(t(k) \,+\, 1)\,, \quad \text{for} \quad k \,\in\, \mathbb{N}_0\,. \tag{31}$$

The telescopic sum leads here, using $C(0) = 3$, $z(n-1)$ from eq. 18 to

$$C(n) \,=\, 7\,n \,+\, 3 \,-\, [z_A(n-1) \,+\, 3\,z_C(n-1)]\,, \quad \text{for} \quad k \,\in\, \mathbb{N}_0\,. \tag{32}$$

This proves **B2**, and also the second part of **B0**.

B): Here t_4 of eq. 13 can be used. The differences of 0s in the five pairs $s_2\hat{s}_2$, $\hat{s}_2s_2$, s_2s_1, s_1s_2 and s_2s_2 is 2, 2, 2, 1, 2. Thus

$$\triangle\, B(k+1) \,:=\, B(k+1) - B(k) \,=\, 2 - \frac{1}{2}\,t(k)\,(t(k) \,-\, 1) \,=\, 2 - k_C(n)\,, \quad \text{for} \quad k \,\in\, \mathbb{N}_0\,. \tag{33}$$

In the last expression eq. 22 has been used. By telescoping, using $B(0) = 0$ and the definition of $z_C(n-1)$ from 19 proves the assertion. $\square$

Eqs. 31 and 33 show that $\triangle C(k+1) - \triangle B(k+1) = 5 - t(k)$, for $k \in \mathbb{N}_0$. Telescoping leads to the result, obtained directly from eqs. 32 and 27, with eq. 18,

$$C(n) - B(n) = 5\,n + 3 - z(n-1)\,, \text{ for } k \in \mathbb{N}_0\,, \tag{34}$$

and with A from eq. 15 this becomes

$$C(n) - (A(n) + B(n)) = n + 2\,, \text{ for } k \in \mathbb{N}_0\,. \tag{35}$$

This equation can be used to eliminate C from the equations.
Next the formulae for z_X for $X \in \{A, B, C\}$ are listed, valid for $n = -1, 0, 1, ...$

Proposition 12.

$$z_A(n) = 2\,B(n+1) - A(n+1) + 1\,, \tag{36}$$
$$z_B(n) = A(n+1) - B(n+1) - (n + 2), \tag{37}$$
$$z_C(n) = 2\,(n + 1) - B(n+1)\,. \tag{38}$$

Proof: Version 1. The inputs are $z_X(-1) = 0$, for $X \in \{A, B, C\}$, by definition, and are satisfied due to $B(0) = 0$ and $A(0) = 1$. Therefore $z_X(n) = \sum_{k=0}^{n} \triangle z_X(k)$, with $z_X(k) := z_X(k) - z_X(k-1)$. The claimed formula and the known $\triangle A(k+1)$ and $\triangle B(k+1)$ from eqs. 16 and 33, respectively, produce the results $k_X(k)$ given in eqs. 20 to 22. Therefore $z_X(n)$ coincides with eq. 19.
Version 2. Besides eq. 18 the trivial formula

$$z_A(n) + z_B(n) + z_C(n) = n + 1 \tag{39}$$

can be used.
$z_A(n-1)$ is computed from the difference of $3\,(z_A(n-1) + 2\,z_C(n-1))$ from eq. 25 and $2\,(z_A(n-1) + 3\,z_C(n-1))$ from eq. 26, with $C(n)$ from eq. 35. This difference leads to the claim eq. 36 with $n \to n+1$.
The $z_C(n)$ formula is eq. 27 with $n \to n+1$.
$z_B(n)$ can then be computed from eq. 39. $\square$

Next all formulae for compositions of the types $X(Y(k)+1)$ and $X(Y(k))$, for $X, Y \in \{A, B, C\}$ and $k \in \mathbb{N}_0$ shall be given. They are of interest in connection with the tribonacci ABC representation given in the preceding section. For this, one needs first the results for the compositions $z(X(k))$. The formulae will be given in terms of A and B (with C eliminated by eq. 35).

Proposition 13.

$$z(A(k)) = 2\,(A(k) - B(k)) - k - 1, \tag{40}$$
$$z(B(k)) = -A(k) + 3\,B(k) - k + 1, \tag{41}$$
$$z(C(k)) = B(k) + 2\,k + 3. \tag{42}$$

Proof: $z(X(k))$ will be found from the self-similarity properties given in eqs. 12, 13 and 11, for $X = A$, B and C, respectively. These strings t_3, t_4 and t_2 are chosen because the relevant numbers 1, 0 and 2, respectively, appear precisely once in all $s-$substrings. For $z(X(k)) = \sum_{j=0}^{X(k)} t(j)$ one has to sum all the numbers of the first k substrings s but in the last one only the numbers up to the number standing for X are summed.
A) In the t_3 substrings $s_4 = 0102$, $s_3 = 010$ and $s_2 = 01$ the number 1 appears just once. In all three substrings the sum up to the relevant number 1 (for A) is $0 + 1 = 1$, so for the last s one has always to add 1. Because s_4, s_3 and s_2, with sums 3, 1 and 1, play the role of 0, 1 and 2, respectively, in t_3 one obtains $z(A(k)) = 3\, z_B(k-1) + 1\,(z_A(k-1) + z_C(k-1)) + 1$. With the identity eq. 39 this becomes $2\, z_B(k-1) + k + 1$, and with the z_B formula eq. 37 this leads to the claim eq. 40.

B) In t_4 the sums of the substrings s_2, $\hat{s}_2$, s_1 are 1, 2, 0,respectively, and because all three begin with the relevant number 0 nothing to be summed for the last s. Thus $z(B(k)) = 1\, z_B(k-1) + 2\, z_A(k-1) + 0 + 0$. Using eqs. 37 and 36 this becomes the claim.

C) In t_2 the sums are 4 for s_7, s_6 and 3 for s_4. The sums up to the relevant number 2 are 3 for each case. Therefore $z(C(k)) = 4\,(z_B(k-1) + z_A(k-1)) + 3\, z_C(k-1) + 3 = z_B(k-1) + z_A(k-1) + 3\,k + 3 = B(k) + 2\,k + 3$, with eqs. 39, 37 and 36. □

Proposition 14.

$$
\begin{aligned}
A(A(k)+1) &= 2\,(A(k) + B(k)) + k + 6\,, & A(A(k)) &= A(A(k)+1) - 3\,, && (43)\\
A(B(k)+1) &= A(k) + B(k) + k + 4\,, & A(B(k)) &= A(B(k)+1) - 4\,, && (44)\\
A(C(k)+1) &= 4\,A(k) + 3\,B(k) + 2\,(k + 5)\,, & A(C(k)) &= A(C(k)+1) - 2\,. && (45)
\end{aligned}
$$

$$
\begin{aligned}
B(A(k)+1) &= A(k) + B(k) + k + 3\,, & B(A(k)) &= B(A(k)+1) - 2\,, && (46)\\
B(B(k)+1) &= A(k) + 1\,, & B(B(k)) &= B(B(k)+1) - 2\,, && (47)\\
B(C(k)+1) &= 2\,(A(k) + B(k)) + k + 5\,, & B(C(k)) &= B(C(k)+1) - 1\,. && (48)
\end{aligned}
$$

$$
\begin{aligned}
C(A(k)+1) &= 4\,A(k) + 3\,B(k) + 2\,(k + 6)\,, & C(A(k)) &= C(A(k)+1) - 6\,, && (49)\\
C(B(k)+1) &= 2\,(A(k) + B(k)) + k + 8\,, & C(B(k)) &= C(B(k)+1) - 7\,, && (50)\\
C(C(k)+1) &= 7\,A(k) + 6\,B(k) + 4\,(k + 5)\,, & C(C(k)) &= C(C(k)+1) - 4\,. && (51)
\end{aligned}
$$

Proof:
The two versions are related by $\triangle X(n+1) = X(n+1) - X(n)$ given in eqs. 16, 33 and 31, for $X \in \{A, B, C\}$, respectively, and n replaced by $Y(k)$ with $Y \in \{A, B, C\}$. For $C(n)$ eq. 35 is always used.
A) This follows from $A(n+1)$, given in eq. 15 with $n \to Y(k)$, $z(Y(k))$ from eqs. 40, 41, and 42.

B) One proves that $B(A(k)) = A(k) + B(k) + k + 1$ from which $B(A(k)+1)$ follows. With eq. 35 this means that one has to prove

$$B(A(k)) \stackrel{!}{=} C(k) - 1 = B2(k).$$

The second equality is 26. After applying z_B on both sides, using eq. 23, this is equivalent to

$$A(k) + 1 \stackrel{!}{=} z_B(C(k) - 1)) = z_B(C(k)).$$

The second equality is trivial. This assertion is now proved. From eqs. 37, 15 and 38 follows $z_B(n) = n + 1 - z(n) + z_C(n)$. Hence $z_B(C(k)) = C(k) + 1 - z(C(k)) + (k + 1)$, with eq. 23. This equals $C(k) - k - 1 - B(k)$ from eq. 42, and replacing $C(k)$ by eq. 35 gives $A(k) + 1$.

One proves $B(B(k)) = A(k) - 1$ or, after application of z_B with eq. 37 on both sides, $B(k) + 1 \stackrel{!}{=} z_B(A(k) - 1) = z_B(A(k))$, where the second equality is trivial. But from eqs. 39 and 23 follows $z_B(A(k)) = A(k) + 1 - (k + 1) - z_C(A(k))$. Applying eq. 38 for $z_C(A(k))$, and the just proven eq. 46 for $B(A(k)+1)$ shows that

$$z_B(A(k)) = B(k) + 1. \tag{52}$$

The $B(C(k))$ claim can be written in terms of C from eqs. 35 and 24 as

$$B(C(k)) \stackrel{!}{=} 2\,C(k) - k = B0(k)\,,$$

where eq. 24 has been repeated.

Indeed, eqs. 27 and 23 imply for $B(C(k)) = 2\,C(k) - z_C(C(k)-1) = 2\,C(k) - (z_C(C(k)) - 1) = 2\,C(k) - k$. The second equality is trivial.

C) These claims follow with $C(n+1)$ from eq. 35 after replacement $n \to Y(k)$, and the already proved formulae for $A(Y(k)+1)$ and $B(Y(k)+1)$.

□

The collection of the results for $z_X(Y(k))$ is, for $k \in \mathbb{N}_0$:

Proposition 15.

$$\begin{aligned} z_A(A(k)) &= k + 1, \\ z_A(B(k)) &= A(k) - B(k) - (k + 1) = z_C(A(k)), \\ z_A(C(k)) &= B(k) + 1. \end{aligned} \tag{53}$$

$$\begin{aligned} z_B(A(k)) &= B(k) + 1 = z_A(C(k)), \\ z_B(B(k)) &= k + 1, \\ z_B(C(k)) &= A(k) + 1. \end{aligned} \tag{54}$$

$$\begin{aligned} z_C(A(k)) &= A(k) - B(k) - (k + 1) = z_A(B(k)), \\ z_C(B(k) &= 2\,B(k) - A(k) + 1, \\ z_C(C(k)) &= k + 1. \end{aligned} \tag{55}$$

Proof:

That $z_X(X(k)) = k + 1$ has been noted already in eq. 23.

The other claims follow from the $z_X(n)$ results in *Proposition* 12 after replacing n by $Y(k) \neq X(k)$, and application of the formulae from *Proposition* 14. $\square$

As above mentioned many of the formulae of this section appear in [2] and [1] with the above mentioned translation between their sequences a, b, and c to our B, A, and C. For example, *Theorem 13* of [2], p. 57, for the nine twofold iterations (in our notation $X(Y(k))$ of *Proposition* 14) can be checked.

Acknowledgment: Thanks go to *Neil Sloane* for an e-mail motivating this study. The author is grateful to an unknown referee of the first version who gave plenty of comments, suggestions and some questions concerning the proof of the theorem which led to this revision.

References

[1] Elena Barcucci, Luc Blanger, and Srecko Brlek, On Tribonacci Sequences, Fib. Q., 42 (2004), 314-320.

[2] L. Carlitz, Richard Scoville, and V. E. Hoggatt, Jr., Fibonacci Representations of Higher Orders, *The Fibonacci Quarterly* 10 (1972) 43–69. http://www.fq.math.ca/Scanned/10-1/carlitz3-a.pdf

[3] Wolfdieter Lang, The Wythoff and the Zeckendorf representations of numbers are equivalent, in G. E. Bergum et al. (eds.) Application of Fibonacci numbers vol. 6, Kluwer, Dordrecht, 1996, pp. 319-337. A scanned copy with corrections is https://oeis.org/A317208/a317208.pdf (see also A189921).

[4] The On-Line Encyclopedia of Integer Sequences (2010), published electronically at http://oeis.org.

[5] Eric Weisstein's World of Mathematics, Tribonacci Numbers,
http://mathworld.wolfram.com/TribonacciNumber.html.

[6] Wikipedia,Generalizations of Fibonacci numbers ,
https://en.wikipedia.org/wiki/Generalizations_of_Fibonacci_numbers.

[7] Wikipedia, Beatty Sequence, https://en.wikipedia.org/wiki/Beatty_sequence

---

2010 Mathematics Subject Classification: Primary 11Y55; Secondary 32H50.

Keywords: Tribonacci numbers, tribonacci constant, tribonacci word, tribonacci tree, tribonacci ABC-sequences, tribonacci ABC-tree.

Concerned with OEIS [4] sequences: A0000073, A000201, A001622, A001590, A001950, A003144, A003145, A003146, A005614, A058265, A080843, A092782 A158919, A189921, A276791, A276793, A276794, A276796, A276797, A276798, A278038, A278039, A278040, A278041, A278044, A316174, A316711, A316712, A316713, A316714, A316715, A316716, A316717, A317206, A319195, A319198, A319968.

Table 1: Sequences t, A, B, C, for n = 0, 1, ..., 79

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
t	0	1	0	2	0	1	0	0	1	0	2	0	1	0	1	0	2	0	1	0
A	1	5	8	12	14	18	21	25	29	32	36	38	42	45	49	52	56	58	62	65
B	0	2	4	6	7	9	11	13	15	17	19	20	22	24	26	28	30	31	33	35
C	3	10	16	23	27	34	40	47	54	60	67	71	78	84	91	97	104	108	115	121
n	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39
t	0	1	0	2	0	1	0	2	0	1	0	0	1	0	2	0	1	0	1	0
A	69	73	76	80	82	86	89	93	95	99	102	106	110	113	117	119	123	126	130	133
B	37	39	41	43	44	46	48	50	51	53	55	57	59	61	63	64	66	68	70	72
C	128	135	141	148	152	159	165	172	176	183	189	196	203	209	216	220	227	233	240	246
n	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59
t	2	0	1	0	0	1	0	2	0	1	0	0	1	0	2	0	1	0	1	0
A	137	139	143	146	150	154	157	161	163	167	170	174	178	181	185	187	191	194	198	201
B	74	75	77	79	81	83	85	87	88	90	92	94	96	98	100	101	103	105	107	109
C	253	257	264	270	277	284	290	297	301	308	314	321	328	334	341	345	352	358	365	371
n	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79
t	2	0	1	0	0	1	0	2	0	1	0	2	0	1	0	0	1	0	2	0
A	205	207	211	214	218	222	225	229	231	235	238	242	244	248	251	255	259	262	266	268
B	111	112	114	116	118	120	122	124	125	127	129	131	132	134	136	138	140	142	144	145
C	378	382	389	395	402	409	415	422	426	433	439	446	450	457	463	470	477	483	490	494

Table 2: ZT(N), for N = 1, 2, ..., 100

N	ZT(N)	N	ZT(N)	N	ZT(N)	N	ZT(N)	N	ZT(N)
1	1	**21**	11001	**41**	110100	**61**	1010100	**81**	10000000
2	10	**22**	11010	**42**	110101	**62**	1010101	**82**	10000001
3	11	**23**	11011	**43**	110110	**63**	1010110	**83**	10000010
4	100	**24**	100000	**44**	1000000	**64**	1011000	**84**	10000011
5	101	**25**	100001	**45**	1000001	**65**	1011001	**85**	10000100
6	110	**26**	100010	**46**	1000010	**66**	1011010	**86**	10000101
7	1000	**27**	100011	**47**	1000011	**67**	1011011	**87**	10000110
8	1001	**28**	100100	**48**	1000100	**68**	1100000	**88**	10001000
9	1010	**29**	100101	**49**	1000101	**69**	1100001	**89**	10001001
10	1011	**30**	100110	**50**	1000110	**70**	1100010	**90**	10001010
11	1100	**31**	101000	**51**	1001000	**71**	1100011	**91**	10001011
12	1101	**32**	101001	**52**	1001001	**72**	1100100	**92**	10001100
13	10000	**33**	101010	**53**	1001010	**73**	1100101	**93**	10001101
14	10001	**34**	101011	**54**	1001011	**74**	1100110	**94**	10010000
15	10010	**35**	101100	**55**	1001100	**75**	1101000	**95**	10010001
16	10011	**36**	101101	**56**	1001101	**76**	1101001	**96**	10010010
17	10100	**37**	110000	**57**	1010000	**77**	1101010	**97**	10010011
18	10101	**38**	110001	**58**	1010001	**78**	1101011	**98**	10010100
19	10110	**39**	110010	**59**	1010010	**79**	1101100	**99**	10010101
20	11000	**40**	110011	**60**	1010011	**80**	1101101	**100**	10010110

Table 3: ABC(N), for N = 1, 2, ..., 100

N	ABC(N)	N	ABC(N)	N	ABC(N)	N	ABC(N)	N	ABC(N)
1	10	**21**	1020	**41**	00120	**61**	001110	**81**	000000010
2	010	**22**	0120	**42**	1120	**62**	11110	**82**	10000010
3	20	**23**	220	**43**	0220	**63**	02110	**83**	01000010
4	0010	**24**	0000010	**44**	00000010	**64**	000210	**84**	2000010
5	110	**25**	100010	**45**	1000010	**65**	10210	**85**	00100010
6	020	**26**	010010	**46**	0100010	**66**	01210	**86**	1100010
7	00010	**27**	20010	**47**	200010	**67**	2210	**87**	0200010
8	1010	**28**	001010	**48**	0010010	**68**	0000020	**88**	00010010
9	0110	**29**	11010	**49**	110010	**69**	100020	**89**	1010010
10	210	**30**	02010	**50**	020010	**70**	010020	**90**	0110010
11	0020	**31**	000110	**51**	0001010	**71**	20020	**91**	210010
12	120	**32**	10110	**52**	101010	**72**	001020	**92**	0020010
13	000010	**33**	01110	**53**	011010	**73**	11020	**93**	120010
14	10010	**34**	2110	**54**	21010	**74**	02020	**94**	00001010
15	01010	**35**	00210	**55**	002010	**75**	000120	**95**	1001010
16	2010	**36**	1210	**56**	12010	**76**	10120	**96**	0101010
17	00110	**37**	000020	**57**	0000110	**77**	01120	**97**	201010
18	1110	**38**	10020	**58**	100110	**78**	2120	**98**	0011010
19	0210	**39**	01020	**59**	010110	**79**	00220	**99**	111010
20	00020	**40**	2020	**60**	20110	**80**	1220	**100**	021010

Here $0, 1$ and 2 stand for B, A and C, respectively. *E.g.*, $ABC(6) = 020$ to be read as $BCB = B(C(B(0))) = (6)_{ABC}$.