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Improved Formula for the Multi-Section of the Linear Three-Term Recurrence Sequence

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Abstract

The standard formula for the multi-section of the general linear three-term recurrence relation is simplified in terms of Chebyshev S-polynomials.

1 Introduction

The $m-$section (multi- or modular-section) of an integer sequence consists of set of m sequences which carry as indices the equivalence classes modulo m.
The general decomposition of the ordinary generating function (o.g.f.) of the sequence into the m o.g.fs of the members of the set of $m-$sections is given in terms of a special $m \times m$ *Vandermonde* matrix. The inverse of this matrix gives the o.g.fs of these members in terms of the o.g.f. of the sequence. The computation which brings these m fractions into one, either by hand (tedious) or by computer, does not give an insight into the structure of this final rational fraction.
For the general sequence satisfying a linear three-term recurrence relation (called *Horadam*-sequence) it is shown that the result for the o.g.fs of the $m-$section sequences can be given in terms of *Chebyshev-S* (and -R polynomials, the monic T-polynomials, which are a difference of two S-polynomials).
This is achieved by a proposal for the m-section of the *Horadam* sequence, first a conjecture for the first element of this section by one of the authors (*G. D.*), then generalized for all elements, and later proved by the second author.

The first section summarizes the standard treatment of the m-section of a sequence and the o.g.fs. The second section is a reminder of some elementary properties of the *Horadam* sequence. In the third section the conjectures for the m-section of this sequence are formulated, and the last section gives the proof of these conjectures.
The proof uses a lemma a (known) alternative bisection of the *Chebyshev-S* polynomials (not the one obtained for improved $m = 2$ case).

2 Multi-Section of a sequence

This section is a reminder of the standard treatment of the $m-$section of a sequence.
The ordinary generating function (o.g.f.) $G(m, l, x) = \sum_{n=0}^{\infty} a(m\,n + l)\,x^n$ of the lth part of the m-section of a sequence $\{a(n)\}_{n>=0}$ with o.g.f. $G(x) = \sum_{n=0}^{\infty} a(n)\,x^n$, for integer $m \geq 2$ and $l \in \{0,\, 1,\, ...,\, m-1\}$, satisfies

$$G(x) = \sum_{l=0}^{m-1} G(m, l, x^m)\,x^l\,. \tag{1}$$

For the solution of $G(m, l, x)$ for given $G(x)$ one uses the roots of the polynomial $x^m - 1$, that is $w(m,\, k) = e^{2\pi k/m}$, for $k \in \{0,\, 1,\, ...,\, m-1\}$, and considers the inhomogeneous system of m equations,

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for $k \in \{0,\, 1,\, ...,\, m-1\}$, for the m unknowns $\{G(m,l,x)\}_{l=0}^{m-1}$,using a *Vandermonde* matrix $V_m(x)$ with elements

$$[V_m(x)]_{k,l} \;=\; (w(m,\,k)\,x)^l, \tag{2}$$

as

$$\sum_{l=0}^{m-1} [V_m(x)]_{k,l}\, G(m,l,x^m) \;=\; G(w(m,\,k)\,x) \tag{3}$$

Note that $(w(m,\,k)\,x)^m \;=\; x^m$ has been used.
The inverse of a general *Vandermonde* matrix is known. *e.g.*, [5], and for the present case its elements become

$$[V_m^{-1}(x)]_{l,j} \;=\; N(m,l,j,x)/DN(m,j,x), \tag{4}$$

with denominator

$$DN(m,j,x) \;=\; x^{m-1} \prod_{0\leq\, <i\neq j\leq m-1} (w(m,\,i) \;-\; w(m,\,j)), \tag{5}$$

and numerator

$$N(m,l,j,x) \;=\; (-1)^l\, x^{m-1-l} \sum_{n=1}^{\#Ch(m,l,j)} \prod_{k=1}^{m-1-l} (Ch(m,l,j)[n])[k], \tag{6}$$

where the list of lists (order respected, and the kth elements of a list L is denoted by $L[k]$)

$$Ch(m,l,j) \;=\; choose(P(m,\,j), m-1-l), \tag{7}$$

with the list

$$P(m,\,j) \;=\; [w(m,\,0), ..., w(m,\,j-1), w(m,\,j+1),\; ...,\; w(m,\,m-1)]\;. \tag{8}$$

The length of list $Ch(m,l,j)$ is $\#Ch(m,l,j) \;=\; \binom{m-1}{l}$ and the length of the lists of $Ch(m,l,j)$ is $m-1-l$ with $\#P(m,\,j) \;=\; m-1$.
Thus, using new arguments $x \to x^{1/m}$, one obtains, for $l \in \{0,\,1,\,...,\,m-1\}$

$$G(m,l,x) \;=\; \sum_{j=0}^{m-1} [V_m^{-1}(x^{1/m})]_{l,j}\, G(w(m,j)\,x^{1/m}). \tag{9}$$

Example 1: m = 3
With $w(3,\,0) \;=\; 1$, $w(3,\,1) \;=\; w \;=\; \frac{1}{2}(-1 \,+\, \sqrt{3}\,i)$ and $w(3,2) \;=\; \overline{w} \;=\; -\frac{1}{2}(1 \,+\, \sqrt{3}\,i)$ one finds $[V_3^{-1}(x)]_{1,2} \;=\; w/(3\,x)$, because $DN(3,2,x) = x^2\,(1 \,-\, \overline{w})\,(w \,-\, \overline{w}) \;=\; x^2\,\frac{1}{2}\,(3 \,+\, i\,\sqrt{3})\,i\,\sqrt{3} \;=\; 3\,w\,x^2$, and from $P(3,2) \;=\; [1,\,w]$, and $Ch(3,1,2) \;=\; [[1],\,[w]]$ one obtains $N(m,l,j,x) \;=\; (-1)^1\,x\,(1 \,+\, w) \;=\; x\,\overline{w}$. Indeed, $[V_3^{-1}(x)]_{1,2} \;=\; \overline{w}/(3\,w) \;=\; w/(3\,x)$, due to $\overline{w}^2 \;=\; w$.

$$V_3^{-1}(x) \;=\; \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1/x & \overline{w}/x & w/x \\ 1/x^2 & w/x^2 & \overline{w}/x^2 \end{pmatrix}. \tag{10}$$

Therefore the standard trisection of $G(x)$ is

$$G(3,0,x) \;=\; \frac{1}{3}\left(G(x^{1/3}) \,+\, G(w\,x^{1/3}) \,+\, G(\overline{w}\,x^{1/3})\right), \tag{11}$$

$$G(3,1,x) \;=\; \frac{1}{3\,x}\left(G(x^{1/3}) \,+\, \overline{w}\,G(w\,x^{1/3}) \,+\, w\,G(\overline{w}\,x^{1/3})\right), \tag{12}$$

$$G(3,2,x) \;=\; \frac{1}{3\,x^2}\left(G(x^{1/3}) \,+\, w\,G(w\,x^{1/3}) \,+\, \overline{w}\,G(\overline{w}\,x^{1/3})\right). \tag{13}$$

This should then be simplified for given $G(x)$, by finding the rational function $P(x)/Q(x)$ which can become tedious in the general m-section case (the computer will help).

The topic of this paper is to give for the general linear three-term recurrence relation the coefficients of these polynomials P and Q in terms of well known polynomials which are functions of the signature of this recurrence.

3 General linear three term recurrence

This section is a review of basic formulas for the considered recurrence relation.
The sequence $\{H(p,q;r,s;n)\}_{n=0}^{\infty}$ satisfies the following linear three- term (also called second order) recurrence relation of signature $(r,\, s)$, with integer numbers r and s, both non-vanishing, and initial conditions (seeds or inputs) $(p,\, q)$, with integer numbers p and q. Only integer sequences are considered. In the following these domains for $p,\, q,\, r,\, s$ will not be repeated in the formulas.
The letter H is used because this sequence has been studied by *A. F. Horadam* in many publications. See *e.g.*, [2], [3],[4], and also [9].

$$H(p,q;r,s;\,n) \;=\; r\,H(p,q;r,s;\,n-1) \,+\, s\,H(p,q;r,s;\,n-2),\ \text{for } n \,\geq\, 2,\ \text{and} \tag{14}$$

$$H(p,q;r,s;\,0) \;=\; p,\ \ H(p,q;r,s;\,1) \,=\, q. \tag{15}$$

It is sufficient to consider the seeds $(p,\, q) \,=\, (0,\, 1)$, naming the sequence $\{H01(r,s;\, n)\}_{n=0}^{\infty}$, because

$$H(p,q;r,s;\,n) \;=\; q\,H01(r,s;\,n) \,+\, p\,s\,H01(r,s;\,n-1). \tag{16}$$

Also $H01(r,s;-1) \,=\, 1/s$ and $H01(r,s;-2) \,=\, -r/s^2$ will be used.
One can also extend this sequence to all negative integer n, by

$$H01(r,s;\,n) \;=\; -(-s)^n\,H01(r,s;\,-n), \tag{17}$$

which implies the result for negative indices for sequence H.
The *Binet - de Moivre* formula is

$$H01(r,s;n) \;=\; \frac{\lambda(r,s)^n \,-\, (-s/\lambda(r,s))^n}{\lambda(r,s) \,-\, (-s/\lambda(r,s))},\ \text{where } \lambda(r,\, s) \;=\; \frac{1}{2}\left(r \,-\, \sqrt{r^2 \,+\, 4\,s}\right). \tag{18}$$

The transfer matrix, also called $\mathbf{Q}$ matrix, for the $(r\,s)$ recurrence relation is

$$\mathbf{Q} \;=\; \begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix}. \tag{19}$$

The powers of this 2×2 matrix with trace $Tr\,\mathbf{Q} \,=\, r$ and determinant $Det\,\mathbf{Q} \,=\, -s \neq 0$, can be found with the help of the *Cayley-Hamilton* Theorem in terms of Chebyshev S-polynomials by

$$\mathbf{Q}^n(r,s) \;=\; (\sqrt{-s})^n\left[S\left(n,\, \frac{r}{\sqrt{-s}}\right)\mathbf{1} \,+\, S\left(n-1,\, \frac{r}{\sqrt{-s}}\right)\frac{1}{\sqrt{-s}}\,(\mathbf{Q}(r,s) \,-\, r\,\mathbf{1})\right]. \tag{20}$$

For the *Chebyshev* S-polynomials see OEIS [7] A049310 for their coefficients, their properties, and references, *e.g.*, [1], [8]. OEIS A-number links will henceforth be used without citation.

$$S(n,\, x) \;:=\; H(1,x;x,-1;\,n),\ \text{for } n \,\geq\, 0. \tag{21}$$

For negative n one finds $S(-1,\, x) \,=\, 0$, and $S(n,\, x) \,=\, -S(-n-2,\, x)$, for $n \,\leq\, -2$.
This produces the matrix

$$\mathbf{Q}^n(r,\, s) \;=\; (\sqrt{-s})^n\begin{pmatrix} S\left(n,\, \frac{r}{\sqrt{-s}}\right) & \frac{s}{\sqrt{-s}}\,S\left(n-1,\, \frac{r}{\sqrt{-s}}\right) \\ \frac{1}{\sqrt{-s}}\,S\left(n-1,\, \frac{r}{\sqrt{-s}}\right) & -S\left(n-2,\, \frac{r}{\sqrt{-s}}\right) \end{pmatrix}. \tag{22}$$

The (generalized) *Chebyshev* T-polynomials are defined from the trace as

$$T\left(n,\, \frac{r}{2\sqrt{-s}}\right) \;:=\; \frac{1}{2}\,Tr\,\mathbf{Q}^n(r,\,s) \;=\; \frac{1}{2}\,\left(S\left(n,\,\frac{r}{\sqrt{-s}}\right) - S\left(n-2,\,\frac{r}{\sqrt{-s}}\right)\right)\,. \tag{23}$$

For $(r,s) \;=\; (x,-1)$ these are the usual *Chebyshev* T-polynomials: $T(n,\,x/2) \;=\; \frac{1}{2}\,(S(n,x) - S(n{-}2,\,x))$. Later $R(n,\,x) \;=\; T(n,\,x/2)/2$ will be used.
Because $Det\,\mathbf{Q}(s,\,r) \;=\; -s$ one has $Det\,\mathbf{Q}^n(s,\,r) \;=\; (-s)^n$, by the product theorem for determinants, and this leads to the *Cassini-Simson* identity in n and (r,s) (with $n \to n+1$)

$$S(n,\,y)^2 \;-\; S(n-1,\,y)\,S(n+1,\,y) \;=\; 1, \tag{24}$$

where r and s only enter *via* $\;y \;=\; y(r,\,s) \;=\; \frac{r}{\sqrt{-s}}$.

A further reduction of the $H01$ sequence, important for the main part of this paper, is possible in terms of the usual *Chebyshev* S-polynomials by

$$H01(r,s;n) \;=\; (\sqrt{-s}\,)^{n-1}\,S\left(n-1,\,\frac{r}{\sqrt{-s}}\right)\,. \tag{25}$$

This follows from comparing the recurrence and the seeds.

The ordinary generating functions (o.g.f) of $\{H01(r,s;n\}_{n=0}^{\infty}$ is

$$GH01(r,s;x) \;=\; \frac{x}{1\,-\,r\,x\,-\,s\,x^2}\,. \tag{26}$$

The o.g.f. of $\{H(p,q;r,s;n\}_{n=0}^{\infty}$ in terms of $GH01(r,s;x)$ is

$$GH(p,q;r,s;x) \;=\; p + (q\,+\,p\,s\,x)\,GH01(r,s;x), \tag{27}$$

$$= \frac{p\,-\,(p\,r\,-\,q)\,x}{1\,-\,r\,x\,-\,s\,x^2}\,. \tag{28}$$

The o.g.f. of $\{S(n,\,y\}_{n=0}^{\infty}$ is

$$GS(x) \;=\; \frac{1}{1\,-\,y\,x\,+\,x^2}\,. \tag{29}$$

4 Conjecture for improved formulas for the m-section of the linear three -term recurrence sequences

This section contains conjectures for simplified formulas for the m-section or the special sequences H, $H01$ and S. In the next section these conjectures will be proved.
One of the authors ($G.\ D.$) heuristically found a formula for the sequence $\{H(p,q;r,s;\,m\,n)\}_{n=0}^{\infty}$, for $m\,\geq\,0$, that identifies it as an H sequence with different input $(p,\,q')$ and signature $(r',\,s')$. See his comment in A034807 where $p,\ q,\ r,\ s$ are denoted as $a,\ b,\ c,\ d$, respectively.
The second author generalized this conjecture to the m-section of the sequence H and their o.g.f.s. He also proved a conjecture for the sequence $H01$ which implies the one for H. In the next section the proof will be given for the conjecture for the m section of *Chebyshev* S-polynomials and the o.g.f.s., that will lead to the other two conjectures.

Conjecture for H

For n $\geq$ 0, m $\geq$ (1), 2 and l $\in$ {0, 1, ..., m $-$ 1}

$$H(p,q;r,s;\,m\,n+l) \;=\; H(H(p,q;r,s;\,l),H(p,q;r,s;\,m+l);SUM(r,s;\,m),-(-s)^m;\,n)\,. \tag{30}$$

with

$$SUM(r,s;\,m) \;=\; r^m \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \text{A034807}(m,\,k)\,(s/r^2)^k\,. \tag{31}$$

Therefore $SUM(r,s;\,m)$ is the polynomial $P(m,\,x)$ of row m of this irregular triangle evaluated at $x\,=\,s/r^2$ and scaled by r^m.

Note that the symmetry between n and m for the left-hand side is not obvious for the right-hand side, but true because the later proof can be done with interchanged n and m. This symmetry holds for all versions of the conjecture given later.

The recurrence relation for the triangle $T =$ A034807 (given there by *Michael Somos*, given here without proof) is

$$\begin{aligned} T(n,\,k) &= T(n-1,\,k) + T(n-2,\,k-1),\ \text{for n} \,\geq\, 2,\ \text{and} \\ T(0,\,k) &= 2,\ \text{for}\ k\,=\,0,\ \text{otherwise}\ 0, \\ T(1,\,k) &= 1,\ \text{for}\ k\,=\,0,\ \text{otherwise}\ 0. \end{aligned} \tag{32}$$

The explicit form (given in A034807 by *Alexander Elkins*, here also given without proof) is

$$T(n,\,k) \;=\; \frac{n\,(n-1-k)!}{k!\,(n\,-\,2\,k)!},\ \text{for n} \,\geq\, 1,\ \text{k} \,=\, 0,\,1,\,...,\,\lfloor \text{n}/2 \rfloor\ \ \text{and}\ \text{T}(0,\,0)\,=\,2\,. \tag{33}$$

The o.g.f. for the row polynomials $\{P(n,\,y)\}$ of T (the o.g.f. of the triangle) (given there by *Vladeta Jovovic*, here given also without proof) is

$$\frac{2\,-\,x}{1\,-\,x\,-\,y\,x^2}\,. \tag{34}$$

Lemma 1

$$\begin{aligned} \textbf{i)}\ SUM(r,s;\,m) &= H(2,r;r,s;\,m), && (35)\\ \textbf{ii)}\ &= s\,H01(r,s;\,m-1)\,+\,H01(r,s;\,m+1), && (36)\\ \textbf{iii)}\ &= (\sqrt{-s}\,)^m\,R\left(m,\,\frac{r}{\sqrt{-s}}\right),\ \text{with R(n, x)} := \text{S(n, x)} - \text{S(n}-2,\,\text{x)}\,. && (37) \end{aligned}$$

The polynomials R are the monic *Chebyshev* T-polynomials. See A127672 for their coefficients and properties.

Proof

i) This follows from the definition of SUM in Eq. 31 and the recurrence of the irregular triangle T given in Eq. 32, leading to the signature $(r,\,s)$, and the inputs $SUM(r,s;\,0)\,=\,2$ and $SUM(r,s;\,1)\,=\,r$.

ii) Results after replacing the sequence H by $H01$ according to Eq. 16.

ii) Uses the replacement of the sequence $H01$ by the *Chebyshev* S-polynomials , Eq. 25, evaluated at $x\,=\,r/\sqrt{-s}$. □

Example 2: Fibonacci trisection

$F(n)\,=\,H(0,1;1,1;\,n)\,=$ A000045(n), for $n\,\geq\,0$. The first part of the trisection $(m\,=\,3,\,l\,=\,0)$ is $F(3\,n)\,=$ A014445$(n)\,=\,\{0,\,2,\,8,\,34,\,144,\,...\}$. The conjecture leads to
$F(3\,n)\,=\,H(0,\,F(3);\,-i\,R(3,\,1/\sqrt{-1}\,),-(-1)^3;\,n)$, where $R(3,\,-i)\,=\,4\,i$. $F(3\,n)\,=\,H(0,2;4,1;\,n)$.
In terms of $H01$ this becomes $F(3\,n)\,=\,2\,H01(4,1;\,n)$.
In terms of S one finally finds $F(3\,n)\,=\,2\,i^{n-1}\,S(n-1,-4\,i)$.

The other parts of the trisection $F(3n + 1) =$ A033887(n) and $F(3n + 2) =$ A015448$(n + 1)$, for $n \geq 0$,follow similarly, and the results are

$$\begin{aligned} F(3n) &= H(0,2;4,1;\, n) = 2\,H01(4,1;\, n) = 2\,i^{n-1}\,S(n-1,-4\,i), &(38)\\ F(3n+1) &= H(1,3;4,1;\, n) = H01(4,1;\, n+1) - H01(4,1;\, n) &(39)\\ &= -i^n\,(S(n,\,-4\,i) + i\,S(n-1,\,-4\,i)), \\ F(3n+2) &= H(1,5;4,1;\, n) = H01(4,1;\, n+1) + H01(4,1;\, n) &(40)\\ &= i^n\,(S(n,\,-4\,i) - i\,S(n-1,\,-4\,i))\,. \end{aligned}$$

The above conjecture for H is equivalent to the one for its o.g.f.

$$GHml(p,q;r,s;m,l;\, x) := \sum_{n=0}^{\infty} H(p,q;r,s;\, m\,n + l)\,x^n\,. \qquad (41)$$

Conjecture for GHml

$$GHml(p,q;r,s;m,l;\, x) = \frac{H(p,q;r,s;\, l) - (H(p,q;r,s;\, l)\,SUM(r,s;\, m) - H(p,q;r,s;\, m+l))\,x}{1 - SUM(r,s;\, m)\,x + (-s)^m\,x^2}\,. \qquad (42)$$

Proof: This equivalence of conjectures is clear from the o.g.f. of the sequence H given in Eq. 28. One has just to insert the conjectured values for the inputs and signature from Eq. 30 □

Example 3: O.g.f. Fibonacci trisection

For $m = 3$, $(p,\, q) = (0,\, 1)$ and $(r,\, s) = (1,\, 1)$ the denominator of $GF3l(x) := \sum_{n=0}^{\infty} F(3\,n + l)\,x^n$ is $1 - (-i\,R(3,\, 1/i))\,x + (-1)^3\,x^2 = 1 - 4\,x - x^2$, for $l = 0$, 1 and 2. The numerators are $F(3)\,x = 2\,x$, $1 - (1 \cdot 4 - 3)\,x = 1 - x$, and $1 - (1 \cdot 4 - 5)\,x = 1 + x$, for these l values, respectively.

The conjecture for the m-section of H implies the one for $H01$, and the corresponding o.g.fs, are obtained setting $(p,\, q) = (0,\, 1)$, and then rewriting H in terms of $H01$ using Eq. 16 with the new parameters. In Example 2 this second step has been used for $m = 3$ and $(r,\, s) = (1,\, 1)$.

Conjecture for H01

$$\begin{aligned} &H01(r,s;\, m\,n + l) = q'\,H01(r',s';\, n) + p'\,s'\,H0(r',s';\, n-1)\,, \text{ where}\\ &p' = p'(r,s;\, l) = H01(r,s;\, l),\;\; q' = p'(r,s;\, m+l) = H01(r,s;\, m+l),\\ &r' = r'(r,s;\, m) = (\sqrt{-s}\,)^m\,R(m,\, r/\sqrt{-s}\,),\;\; s' = s'(s,\, m) = -(-s)^m. \end{aligned} \qquad (43)$$

The part $l = 0$ simplifies to

$$H01(r,s;\, m\,n) = H01(r,s;\, m)\,H01((\sqrt{-s}\,)^m\,R(m,\, r/\sqrt{-s}\,),\, -(-s)^m;\, n)\,. \qquad (44)$$

Conjecture for GH01ml

The conjecture for the o.g.f. $GH01ml(r,s;m,l;\, x) := \sum_{n=0}^{\infty} H01(r,s;m\,n{+}l)\,x^n$ is obtained from $GHml(x)$ in Eq. 42, and is given with $y = y(r,\, s)/\sqrt{-s}$ as

$$GH01ml(r,s;m,l;\, x) = \frac{H01(r,s;\, l) - ((\sqrt{-s}\,)^m\,H01(r,s;\, l)\,R(m,\, y) - H01(r,s;\, m+l))\,x}{1 - (\sqrt{-s}\,)^m\,R(m,\, y)\,x + (-s)^m\,x^2}\,. \qquad (45)$$

Because the sequences H and $H01$ are determined by the *Chebyshev* polynomials $\{S(n,\, y = r/\sqrt{-s}\,)\}$ the conjecture for $S(m\,n + l,\, r/\sqrt{-s}\,)$ is fundamental.

Conjecture for S

For $n \geq 0$, $m \geq (1), 2$ and $l \in \{0, 1, ..., m-1\}$:

$$S(m\,n + l, y) = c(s, m)^{n-1}\,\{S(m + l, y)\,S(n-1, c(s, m)\,R(m, y)) - c(s, m)\,S(l, y)\,S(n-2, c(s, m)\,R(m, y))\}\,, \tag{46}$$

$$\text{with } \mathrm{y} = \mathrm{r}/\sqrt{-\mathrm{s}},\ \mathrm{c(s, m)} := \frac{(\sqrt{-\mathrm{s}}\,)^{\mathrm{m}}}{\sqrt{(-\mathrm{s})^{\mathrm{m}}}},\ \mathrm{S}(-2,\mathrm{y}) = -1,\ \text{and } \mathrm{S}(-1,\mathrm{y}) = 0\,.$$

The part $l = 0$ simplifies, using $c(s, m)^2 = 1$, the recurrence relation of S and then the definition of R, to

$$S(m\,n, r/\sqrt{-s}) = (c(s, m))^n \cdot \cdot\{S(n, c(s, m)\,R(m, r/\sqrt{-s})) + c(s, m)\,S(m-2, r/\sqrt{-s}\,)\,S(n-1, c(s, m)\,R(m, r/\sqrt{-s}\,))\}. \tag{47}$$

The following proof that this conjecture is equivalent to the conjecture for $H01$ uses $y = r/\sqrt{-s}$ and Eq. 25.

Proof of the equivalence between the conjectures H01 and S

With $y = r/\sqrt{-s}$ and Eq. 25 $S(m\,n + l, y) = (1/\sqrt{-s}\,)^{m\,n+l})\,H01(r, s; m\,n + l + 1)$. With the conjecture for $H01$ from above this becomes in terms of S, again using Eq. 25,

$$(\sqrt{-s}\,)^{m\,n+l}\,S(m\,n + l, y) = \widehat{q'}\,(\sqrt{-s'}\,)^{n-1}\,S(n-1, r'/\sqrt{-s'}\,) + \widehat{p'}\,s'\,(\sqrt{-s'}\,)^{n-2}\,S(n-2, r'/\sqrt{-s'}\,)\,,$$

with r', s', p' and q' from Eq. 43, and q' and p' are written in terms of S as $\widehat{q'} = \sqrt{-s}^{\,m+l}\,S(m + l, y)$ and $\widehat{p'} = \sqrt{-s}^{\,l}\,S(l, y)$. Also $r'/\sqrt{-s'} = c(s, m)\,R(m, y)$.

Dividing both sides by $(\sqrt{-s}\,)^{\,m+l}\,(\sqrt{(-s)^m}\,)^{\,n-1}$ produces

$$c(s, m)^{n-1}\,S(m\,n + l, y) = \{S(m + l, y)\,S(n-1\,c(s, m)\,R(m\,y)) - (1/c(s, m))\,S(l, y)\,S(n-2, c(s, m)\,R(m, y))\}\,. \tag{48}$$

Because $c(s, m)^2 = 1$ one replaces $1/c(s, m)$ by $c(s, m)$, giving the final result. □

Note that $c(s, m)$ has only values from $\{+1, -1\}$. $c(-1, m) = 1$, for $m \geq 1$, and $\{c(1, m)\}_{m \geq 1} =$ repeat $\{1, -1, -1, 1\}$ = A087960 with offset 1.

Example 4: Trisection of Chebyshev S-polynomials

m = 3, r = y, s = -1. Note that y is now an indeterminate.

$\mathbf{l = 0}$: $S(m\,n, y) = S(n, R(3, y)) + y\,S(n-1, R(3, y))$, with $R(3, y) = y\,(y^2 - 3)$.

$\mathbf{l = 1}$: $S(m\,n + 1, y) = S(4, y)\,S(n-1, R(3, y)) - y\,S(n-2, R(3, y)))$, with $S(4, y) = 1 - 3\,y^2 + y^4$.

$\mathbf{l = 2}$: $S(m\,n + 2, y) = S(2, y)\,(R(3, y)\,S(n-1, R(3, y)) - S(n-2, R(3, y)) = S(2, y)\,S(n, R(3, y))$, because $S(5, y) = S(2, y)\,R(3, y)$, and $S(2, y) = y^2 - 1$.

The conjecture for the o.g.f. $GSml(r, s; m, l; x) := \sum_{n=0}^{\infty} S(m\,n + l, y = r/\sqrt{-s}\,)\,x^n$ is obtained from the one for $GHml(x)$ given above.

Conjecture for GSml

With $y = \frac{r}{\sqrt{-s}}$:

$$GSml(r, s; m, l; x) = \frac{1}{(\sqrt{-s}\,)^l}\,GH01ml\left(r, s; m, l + 1; \frac{x}{(\sqrt{-s}\,)^m}\right), = \frac{S(l, y) - (S(l, y)\,R(m, y) - S(m + l, y))\,x}{1 - R(m, y)\,x + x^2}\,. \tag{49}$$

Note that the advantage of working with the o.g.fs instead of the sequences is that the $(r,\, s)$ dependence appears only in y (not like in Eq. 46 also in $c(s,\, m)$).

Exercise

In order to appreciate these formulas on should compare them with the standard computation according to *Section 2*. Done either by hand or by computer the result will not be expressed in terms of *Chebyshev* polynomials.

Proof of the equivalence between GSml and GH01ml

This uses the relation between $S(n,\, y)$ and $H01(r, s;\, n+1)$ obtained from Eq. 25, for $n \,\to\, m\,n + l$. This leads to the relation between the o.g.fs. Then in Eq. 45 the $H01$ sequences are rewritten in terms of S, with $y \,=\, r/\sqrt{-s}$. □

Example 5: O.g.f.s for the trisection of Chebyshev S polynomials

$m \,=\, 3,\ r \,=\, y,\ s \,=\, -1$. Note that y is now an indeterminate.
$\mathbf{l = 0}$: $GS30(y,\, x) \,=\, (1 - (R(3,\, y) \,-\, S(3,\, y))\,x)/(1 \,-\, R(3,\, y)\,x \,+\, x^2)$, With $R(3,\, y)$ from above in Example 4, and $S(3,\, y) \,=\, y\,R(2,\, x) \,=\, y\,(y^2 \,-\, 2)$ one obtains $R(3,\, y) \,-\, S(3,\, y) \,=\, -y$, hence $GS30(y,\, x) \,=\, (1 \,+\, y\,x)/(1 \,-\, y\,(y^2 \,-\, 3)\,x \,+\, x^2)$.
$\mathbf{l = 1}$: In the numerator appears $y\,R(3,\, y) \,-\, S(4,\, y) \,=\, -1$. Hence
$GS31(y,\, x) \,=\, (y \,+\, x)/(1 \,-\, y\,(y^2 \,-\, 3)\,x \,+\, x^2)$.
$\mathbf{l = 2}$: In the numerator appears $S(2,\, y)\,R(3,\, y) \,-\, S(5,\, y) \,=\, 0$ (see Example 4). Hence
$GS32(y,\, x) \,=\, (y^2 \,-\, 1)/(1 \,-\, y\,(y^2 \,-\, 3)\,x \,+\, x^2)$.

5 Proof of the conjectures

The proof is given for the conjectured o.g.fs, equivalent to the conjectures for the corresponding sequences. Here the proof for the conjecture of the o.g.f. of the sequence S, *i.e.*, $GSml$ of Eq. 49, is given which is equivalent to the o.g.f. of sequence $H01$, *i.e.*, $GH01ml$ of Eq. 45.
The conjecture for the o.g.f. of the sequence H, *i.e.*, $GHml$ of Eq. 42, follows from the conjecture of $GSml$ by

$$\begin{aligned} GHml(p,q;r,s;m,l;\,x) \;&=\; q\,(\sqrt{-s}\,)^{l-1}\,GSml\left(r,s;m,l-1;\,(\sqrt{-s}\,)^m\,x\right) \\ &\quad + p\,s\,(\sqrt{-s}\,)^{l-2}\,GSml\left(r,s;m,l-2;\,(\sqrt{-s}\,)^m\,x\right)\,. \end{aligned} \tag{50}$$

Note that for $m \,\geq\, 2$ and $l \,=\, 0$ and 1 the sequences H, $H01$ and S appear also with negative indices $n \,=\, -1$ and -2, namely $H01(r,s;-1) \,=\, 1/s$, $S(-2,\, x) \,=\, -1$ and $S(-1,\, x) \,=\, 0$.

This $GHml$ formula coincides with the original one of Eq. 42, after sequence H is replaced by sequence $H01$, and then by sequence S.

Theorem: *The conjecture for the o.g.f.* $GSml$ *of* $\{S(m\,n + l,\, r/\sqrt{-s}\,)\}_{n=0}^{\infty}$ *is true.*

Proof:

One proves that the o.g.f. $GS(y,\, x) \,=\, 1/(1 \,-\, y\,x \,+\, x^2)$ for the *Chebyshev* polynomials $\{S(n,\, y)\}_{n=0}^{\infty}$, with $y \,=\, r/\sqrt{-s}$, satisfies the $m-$section formula according to Eq. 1 in terms of the conjectured part l o.g.fs $GSml$ from Eq. 49. This can be rewritten, by bringing the identical (l-independent) denominators of $GSml$ to the left hind side, and the denominator of $GS(y,\, x)$ to the right-hand side as

$$\begin{aligned} LHS(m,l;\,y,\,x) \;&:=\; 1 \,-\, R(m,\, y)\,x^m \,+\, x^{2\,m}, \\ RHS(m,l;\,y,\,x) \;&:=\; (1 \,-\, y\,x \,+\, x^2)\,\sum_{l=0}^{m-1} x^l\,N(m,l;\,y,\,x^m), \\ \text{with} \qquad & N(m,l;\,y,\,x^m) \,=\, S(l,\, y) \,-\, (S(l,\, y)\,R(m,\, y) \,-\, S(m+l,\, y))\,x^m\,. \end{aligned} \tag{51}$$

Remember that by working with o.g.fs instead of sequences the $(r,\, s)$ dependence appears only y. Therefore the proof will be given for the indeterminate y.

Because the *Vandermonde* matrix has an inverse (see Eq. 4) the proof will automatically hold also for $GSml$ in terms of GS like in Eq. 9.

All powers of x will be compared on both sides in order to prove that $LHS\,=\,RHS$.
In RHS all powers $x^0,\, x^1,\, ...,\, x^m,\, ...,\, x^{2\,m},\, x^{2\,m+1}$ appear. In LHS only $x^0,\, x^m,\, x^{2\,m}$ are present.
It will turn out that the proof for the two highest powers $x^{2\,m+1}$ and $x^{2\,m}$ differs from the one for the other powers. Usually the recurrence of the *Chebyshev S* polynomials will show directly that $RHS - LHS\,=\,0$ but for the two highest powers one has to use results from the bisection of these polynomials.
For the other powers the contribution of the R terms in the numerator N of $GSml$ will be considered separately from the remainder (pure S terms). For the powers x^m to $x^{2\,m-1}$ it will turn out that the R terms are multiplied by factors which vanish because of the recurrence of the S-polynomials (the structure of R will thus be irrelevant).

One starts with the two highest powers.
For $x^{2\,m+1}$ only RHS is present, namely $-(S(m-1,\, y)\, R(m,\, y)\,-\,S(2\,m\,-\,1))$. It vanishes if

$$S(2\,m\,-\,1,\, y)\;=\;S(m-1,\, y)\, R(m,\, y)\,. \tag{52}$$

For $x^{2\,m}$ the RHS becomes (-y) {-(S(m-1, y) R(m,y) - S(2 m - 1, y))} + (+1) {-(S(m-2, y) R(m,y) - S(2 m - 2, y))}, and $RHS\,=\,1$. The first term vanishes if the $x^{2\,m+1}$ power contribution vanishes, and then for this x-power $RHS\,-\,LHS\,=\,0$ if

$$S(2\,(m-1),\, y)\;=\;1\,+\,S(m-2,\, y)\, R(m,\, y)\,. \tag{53}$$

Lemma 2: *Eqs. (52) and (53) are satisfied for all* $m\,\geq\,0$.
Proof:
These two equations are found in [6], written for Chebyshev T and U-polynomials.
Using a proof of the standard bisection will not help here. The proof is done by induction on m on both equations simultaneously, employing the *Cassini-Simson* identity from Eq. 24.
For $m\,=\,0$ the first equation is fulfilled because $S(-1,\, y)\,=\,0$, and the second one because $S(-2,\, y)\,=\,-1$ and $R(0,\, y)\,=\,2$.
Assume that both equations hold for $m'\,=\,1,\, 2,\, ...,\, m$. First, Eq. 53 will be proved for $m\,\rightarrow\,m+1$. Multiplying Eq. 52 by y and subtracting Eq. 53 yields, after using recurrence relations,

$$S(2\,m,\, y)\;=\;S(m,\, y)\, R(m,\, y)\,-\,1\,. \tag{54}$$

For Eq. 53 one wants to prove $S(2\,m,\, y)\,=\,1+S(m-1,\, y)\, R(m+1,\, y)$, *i.e.*, $0\,=\,1+S(m-1,\, y)\, R(m+1,\, y) - (S(m,\, y)\, R(m,\, y)\,-\,1)$, *i.e.*, $0\,=\,2\,-\,S(m,\, y)\, R(m,\, y)\,+\,S(m-1,\, y)\, R(m+1,\, y)$. Replacing R in terms of S, using $S(m\,-\,1,\, y)\, S(m\,+\,1,\, y)\,=\,S(m,\, y)^2\,-\,1$ (*Cassini-Simson*) leads to a cancellation of $S(m,\, y)^2$, leaving $0\,=\,1\,-\,S(m\,-\,1,\, y)^2\,+\,S(m,\, y)\, S(m\,-\,2,\, y)$, which is again a *Cassini-Simson* identity. Thus Eq. 53 is proved.

For Eq. 52 one wants to prove $S(2\,m\,+\,1,\, y)\,=\,S(m,\, y)\, R(m\,+\,1,\, y)$. By recurrence $S(2\,m\,+\,1,\, y)\,=\,y\, S(2\,m,\, y)\,-\,S(2\,m\,-\,1,\, y)$ which becomes, with the induction assumptions Eq. 54 and Eq. 52 (for m), $S(2\,m+1,\, y)\,=\,-y+y\, S(m,\, y)\, R(m,\, y) - S(m-1,\, y)\, R(m,\, y)$. This is by recurrence $S(2\,m+1,\, y)\,=\,-y\,+\,S(m\,+\,1,\, y)\, R(m,\, y)$. One wants now to prove $S(2\,m\,+\,1,\, y)\,=\,S(m,\, y)\, R(m+1,\, y)\,=\,-y\,+\,S(m+1,\, y)\, R(m,\, y)$. Replacing R in terms of S gives, after cancellation of $S(m,\, y)\, S(m+1,\, y)$, $S(m+1,\, y)\, S(m-2,\, y) - S(m,\, y)\, S(m-1,\, y)\,=\,-y$. Replacing $S(m-2,\, y)$ by $y\, S(m-1,\, y) - S(m,\, y)$, and then $S(m\,+\,1,\, y)\, S(m\,-\,1,\, y)\,=\,-1\,+\,S(m,\, y)^2$ (*Cassini-Simson*) leads to $0\,=\,S(m,\, y)\,(y\, S(m,\, y)\,-\,S(m-1,\, y)) - S(m+1,\, y)\, S(m,\, y)$, which holds because of the recurrence relation. □

Consider the R term contributions in the numerator N together with the pre-factor with powers x^i for $i\,\in\,\{0,\, 1\,, 2\}$. The powers are x^{i+l+m}, for $m\,\geq\,2$ and $l\,\in\,\{0,\, 1,\, ...,\, m-1\}$, but only for $e\,:=$

$i + l + m <= 2m - 1$, because the powers x^m and x^{2m+1} have just been treated separately. R terms appear only for the exponents $e = 2m - 1, ..., m$.
In RHS the general l term contributes, for $i = 0, 1, 2$, with $\widehat{R}(l) := -R(m, y)\, S(l, y)$ to x^{m+l+i}. Therefore one considers an m rows and three columns array AR with entries $AR(l, i) = \widehat{R}(l)\, x^{m+l+i}$. The anti-diagonals of AR have identical powers of x.
In the last row, $l = m-1$, the last two entries, and in the row $l = m-2$ the last entry are not relevant because the powers are x^{2m} and x^{2m+1}, already treated.
A special case is $AR(0, 0) = \widehat{R}(0)\, x^m$ because here the LHS has entry $-R(m, y)\, x^m$, and to this power also the later discussed array AS contributes with three terms $S(m, y)$, $-y\, S(m-1, y)$ and $S(m-2, y)$ from the second terms of $AS(0, 0)$, the first terms of $AS(m-1, 1)$ and $AS(m-2, 2)$, respectively. Then $RHS - LHS$ vanishes for x^m because $-R(m, y) + (S(m, y) - y\, S(m-1, y) + S(m-2, y)) - (-R(m, y)) = 0$, by cancellation of R and the recurrence relation of the S polynomials. As announced, the R term is irrelevant, only the S recurrence enters.
Another case where the length of the anti-diagonal in AR is not 3 is $AR(1, 0) = \widehat{R}(1)$ and $AR(0, 1) = \widehat{R}(0)$. This produces for the power x^{m+1} only $-R(m, y)(\, S(1, y) - y\, S(0, y)) = 0$, because $S(-1, y) = 0$. There will be no contribution from the array AS for this power.
All other anti-diagonals of AR, *i.e.*, those with powers x^{m+j}, for $j \in \{2, 3, ..., m-1\}$, have length 3, and their contributions $R(m, y)\, (S(j, y) - y\, S(j-1, y) + S(j-2, y))$ vanish because of the recurrence for the S-polynomials, independently of R.

For the pure S terms of RHS one considers the companion array AS with two term entries $AS(l, i) = S(l, y)\, x^{l+i} + S(m+l, y)\, x^{m+l+i}$.
The anti-diagonals with length 3, *i.e.*, those for powers $\{x^j, x^{m+j}),$ for $j \in \{2, 3, ..., m-1\}$ give vanishing contributions to RHS because for both S-polynomial terms their recurrence relation appears. The first term of the entry $AS(0, 0) = S(0, y)\, x^0 + S(m, y)\, x^m$ is identical with $1 \cdot x^0$ of LHS, and the second term has been needed above (among others) in the proof of the vanishing of the contribution to x^m.
The second anti-diagonal $A(1, 0)$ and $A(0, 1)$ contributes to x^1 with $S(1, y) - y\, S(0, y) = 0$ ($S(-1, y) = 0$), and to x^{m+1} with $S(m+1, y) - y\, S(m, y)$ which is needed, together with the first term of the last entry $AS(m-1, 2) = S(m-1, y)\, x^{m+1}$, to prove the vanishing for the contribution of AS to x^{m+1}. For the corresponding vanishing of the AR contribution see above. There is no such LHS contribution.

The second term of $AS(m-1, 2) = S(2m-1, y)\, x^{2m+1}$ has been used in the treatment of the highest power above.
Finally, the second terms of the two anti-diagonal entries $AS(m-1, 1)$ and $AS(m-2, 2)$ contribute $-y\, S(2m-1, y) + S(2m-2, y)$. They have been treated above together with the AR contribution to the second highest power above.
All RHS entries of AR and AS have been considered and shown to contribute only to the three powers x^1, x^m and x^{2m} giving the LHS, which ends the proof. □

To end this work the results of the alternative bisection formulas Eq. 52 and Eq. 53 are given for the $H01$ and H sequences. There derivation is done with the help of eqs. (25) and (16).

$$
\begin{aligned}
H01(r,s;\,2\,m\,+\,1) &= (\sqrt{-s}\,)^{m+1}\,H01(r,s;\,m)\,R(m+1,\,r/\sqrt{-s}\,)\,+\,(-s)^m\,,\\
&= (-s)^m\,\left(S(m-1,\,r/\sqrt{-s}\,)\,R(m+1,\,r/\sqrt{-s}\,)\,+\,1\right)\,, \qquad (55)\\
H01(r,s;\,2\,m) &= (\sqrt{-s}\,)^m\,H01(r,s;\,m)\,R(m,\,r/\sqrt{-s}\,)\,,\\
&= (\sqrt{-s})^{2\,m-1}\,S(m-1,\,r/\sqrt{-s}\,)\,R(m,\,r/\sqrt{-s}\,). \qquad (56)\\
H(p,q;r,s;\,2\,m\,+\,1) &= (\sqrt{-s}\,)^m\,\left\{H01(r,s;\,m)\,(\sqrt{-s}\,q\,R(m\,+\,1,\,r/\sqrt{-s})+p\,s\,R(m,\,r/\sqrt{-s}))\right.\\
&\qquad \left.+(\sqrt{-s}\,)^m\,q\right\}\\
&= (-s)^m\,\{S(m\,-\,1,\,r/\sqrt{-s}\,)\,(q\,R(m\,+\,1,\,r/\sqrt{-s}\,)\,-\,\sqrt{-s}\,p\,R(m,\,r/\sqrt{-s}))\\
&\qquad +q\}\,, \qquad (57)\\
H(p,q;r,s;\,2\,m) &= (\sqrt{-s}\,)^m\,\{R(m,\,r/\sqrt{-s}\,)\,(q\,H01(r,s;\,m)\,+\,s\,p\,H01(r,s;\,m\,-\,1))\\
&\qquad -(\sqrt{-s}\,)^m\,p\}\,,\\
&= (-s)^{m-1}\,\{R(m,\,r/\sqrt{-s}\,)\,(\sqrt{-s}\,q\,S(m\,-\,1,\,r/\sqrt{-s}\,)\,+\,s\,p\,S(m\,-\,2,\,r/\sqrt{-s}\,))\\
&\qquad +s\,p\}\,. \qquad (58)
\end{aligned}
$$

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