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A Fibonacci-Fractal: a Bicolored Self-Similar Multifractal

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ABSTRACT

An example of a self-similar fractal based on the Fibonacci substitution rule $1 \rightarrow 1, 0$, $0 \rightarrow 1$, starting with 1, is analyzed. The inverse of the function $\tau(q) = (q-1)D_q$, with the generalized dimensions D_q can be computed analytically. A parametric equation for the singularity spectrum function $f(\alpha)$, the Legendre transform of $-\tau(q)$, is also given exactly.

Besides uniform and nonuniform Cantor sets this example (and obvious generalizations thereof) can serve as further touchstone for approximation methods used in real or computer experiments to extract D_q or $f(\alpha)$ curves.

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1. Introduction and Summary

Since the concept of a multifractal (or multiscale fractal) has been introduced within the theory of fully developed fluid turbulence [10,4,12,13] it has found widespread applications in the description of fractal geometry in nature [9,2]. In physics multifractals emerged also in the study of chaotic dynamics, disordered systems, critical phenomena and pattern growth [5,14,15]. Whereas in homogeneous fractals the scaling laws of quantities like density of points in boxes of D dimensional space or phase space and probability distributions are characterized by a single exponent, or dimension $D_0 < D$, a continuous range for such exponents is necessary to describe multifractals. Generalized dimensions D_q defined for real q enter the stage [6].

Typical examples for self-similar fractals are Cantor sets. The standard middle thirds set in $D = 1$ with equal probability distribution, $p = 1/2$, is homogeneous (or uniform) with fractal dimension $D_0 = ln\ 2/ln\ 3 \approx .631$. If in the construction of this set the generating process splits the probability unevenly, $p \neq 1/2$, a multifractal will result with $D_q = ln\ (p^q + (1-p)^q)/((1-q)(ln\ 3))$. In the literature, *e.g.* [5,12], a two scale Cantor set is usually quoted as simple example of a multifractal. Such self-similar multifractals have the property that a *single* generator is applied iteratively to produce a cascade of intervals.

In this paper a self-similar multifractal in dimension $D = 1$ is proposed which needs *two* generating transformations for the intervals depending on their color, white and black. A simple example of such a fractal is the $D = 1$ middle thirds Cantor set where odd numbered intervals (say, white ones) on each level of the cascade are replaced by a white and a black one with probability split $p_1, 1-p_1$, and even numbered intervals (black ones) are replaced by a white and black one with probability split $p_2, 1-p_2$, with $p_1 \neq p_2$. This bicolored fractal is based on the odd, even substitution rule $1 \rightarrow 1, 0$ and $0 \rightarrow 1, 0$. For $p_1 = p_2 =: p$ one recovers the middle thirds Cantor set with probability p which we mentioned above. Our example uses the more subtle Fibonacci substitution rule $1 \rightarrow 1, 0$, and $0 \rightarrow 1$. The positioning of the intervals will be very special. One may use other cascades as long as the principal substitution rule is the same, the recursion method used in this paper will work for all such fractals. Our interest in such Fibonacci cascades stems from the spectra of Fibonacci chains [7] which are special periodic linear chains of atoms of two types (AB- chains). They have nearest neighbor harmonic interaction and the order of the atoms in the elementary unit consisting of N atoms follows the first N entries of the binary Fibonacci sequence $1, 0, 1, 1, 0, 1, 0, ...$ (in this paper called $\{h(n)\}$). The frequency spectra of such chains with a Fibonacci number of atoms $N = F_{l+2}$ in the unit cell can be considered as a cascade (level number l) of white and black intervals. However this cascade is not strictly self-similar. In this paper an approximation to these spectra is studied. One salient feature of these spectra is the range increase when a white parent interval is split into a white and a black one (in our model by a factor of $3/2$). This is in contrast to Cantor type constructions.

The paper is organized as follows. In *section 2* we define the generators for the self-similar Fibonacci-fractal. Interval lengths of the cascade will be computed based on recursion relations. The starting points, as well as the end points, of the intervals at each level are computed. The fractal $\mathcal{A}$ is defined in the limit of infinite level number. In *section 3* the recursion formula for the probabilities is given. A multifractal measure is thus defined on the interval sets. *Section 4* is devoted to the analysis of this multifractal. Here we follow the method of ref.[5]. Multifractals have been discussed earlier in refs.[10,4]. Based on the recursion relations for the interval lengths and probabilities, the partition function is computed for given level number l. The requirement of finite, non- vanishing limit $l \to \infty$ leads to an implicitly defined curve $\tau(q)$, where τ and q are the scaling exponents for the inverse interval lengths and the probabilities. The generalized dimensions [6] are given by $D_q = \tau(q)/(q-1)$ [5,12]. It turns out that for a certain input probability, related to the golden section $\varphi = \frac{1}{2}(1+\sqrt{5}\ )$, one can compute the function $q(\tau)$ explicitly. An explicit result can also be found in the more trivial case when all intervals at a given level have the same probability. These $q(\tau)$ functions are the main results of this paper. Besides various special Cantor sets this provides a new example of a multifractal where the q, τ relationship can be found analytically. Therefore one can test approximation methods which are used in real or computer experiments to extract the $\tau(q)$, or D_q, curve, also with this Fibonacci model.

A convenient way to characterize a multifractal is to plot the spectrum curve $f(\alpha)$, technically the Legendre transform of $-\tau(q)$, which pictures the dimensions of the intermingled fractal sets with a given interval $[\alpha, \alpha + d\alpha]$ of scaling exponents. A parametric form of this spectrum curve can also be computed from the inverse function $q(\tau)$. *Fig. 6* shows this spectrum for the Fibonacci-fractal with the two different probability assignments mentioned above.

In *Appendix A* the entropy (in the sense of information theory) for each level of the Fibonacci cascade is computed. Of course, only the probability assignments (not the interval length) enter this quantity. For the non-trivial probability distribution for which an explicit result for $q(\tau)$ was found, the entropy divided by $ln\ \varphi$ is an integer in the quadratic number field $\mathbf{Q}(\sqrt{5})$ for all level numbers l. The entropy diverges linearly in the limit $l \to \infty$.

In *Appendix B* the partition function for the case of uniformly distributed probabilities is discussed. The corresponding spectrum curve $f_{u.p.}(\alpha)$, shown in *Fig.6*, is much narrower (*i.e.* nearer to a homogeneous fractal) than the one for the non-trivial probability distribution.

2. Construction of the fractal set $\mathcal{A}$

In this section a generation of a self-similar cascade of closed intervals of two colors, white (W) and black (B), is given which is based on the Fibonacci substitution rule $1 \to 1, 0$ and $0 \to 1$. The levels of the cascade will be numbered by $l \in \mathbf{N_0}$ and the union of all disjoint intervals at each level defines a set $\mathcal{A}^{(l)}$. In the limit $l \to \infty$ the self-similar set $\mathcal{A}$ is obtained.

The $1 \to 1, 0$ part of the substitution rule is represented by a generation of a white and a black interval from a white one. Similarly, the rule $0 \to 1$ means to transform a black to a white interval. At the lowest level, $l = 0$, one starts with a white interval. In order to be specific we use the following generators with the notation $[a,c]_W$ for white intervals, I, and $[b,d]_B$ for black ones, J.

$$g_W: \qquad [a,c]_W \to [a, \frac{1}{2}(a+c)]_W \ \cup \ [\frac{1}{4}(a+3c), \frac{1}{2}(3c-a)]_B \quad , \qquad (2.1a)$$

$$g_B: \qquad [b,d]_B \to [\frac{1}{2}(b+d), \frac{1}{4}(b+3d)]_W \quad . \qquad (2.1b)$$

In other words, the length of the white interval, $|I|$, generated by g_W is $1/2$ of the length of the original white one, and it starts at the same point a. The length of the black interval generated by g_W is $3/4$ of the length of the original white one, and it starts after $3/4$ of this white interval. The white interval generated by g_B has length $1/4$ of the original black one, and it starts at the middle of this black parent interval. The first interval at level number $l = 0$, is taken as a white one $I_1^{(0)} := [0, \frac{2}{3}]$. Then g_W and g_B are applied to generate iteratively the higher level sets of bicolored intervals $\mathcal{A}^{(l)}$, $l = 1, 2, ...$ *Fig.1* shows the first steps of this construction, and in *Fig.2* one sees the first five levels. (We have used black and white bars to color the intervals, but the intervals have, of course, zero thickness.) Note that the g_W application expands the range of the interval by a factor of $3/2$. However, for all $l \in \mathbf{N}$: $\mathcal{A}^{(l)} \subset [0,1]$.

We now list some *Lemmas* which give the data for the sets $\mathcal{A}^{(l)}$.

Lemma 1: Number and order of the intervals

a) At level $l \in \mathbf{N}_0$ there are F_{l+1} white intervals $I_i^{(l)}$, $i = 1, 2, ..., F_{l+1}$, and F_l black ones $J_j^{(l)}$, $j = 1, 2, ..., F_l$. Here $\{F_n\}$ are the Fibonacci numbers.

b) The sequence of the white and black intervals at level l follows the binary Fibonacci sequence $\{h(n)\}_1^{F_{l+2}}$ which is defined by

$$h(n) \ := \ A(n+1) \ - \ A(n) \ - \ 1 \quad , \quad A(n) \ := \ \lfloor n\varphi \rfloor \ , \ \ \varphi \ := \ \frac{1}{2}(1+\sqrt{5}) \ , \qquad (2.2)$$

with 1 for white and 0 for black.

Proof: *a*) This is a simple consequence of the Fibonacci substitution rule $1 \to 1, 0$ and $0 \to 1$. Starting at level $l = 0$ with 1, this rule produces at level l a word over the 'alphabet' $\{0, 1\}$ with F_{l+2} letters, namely F_{l+1} $1'$s and F_l $0'$s.

b) This follows from the substitution rule for $l \in \mathbf{N}$

$$g_W \ : \qquad I_i^{(l)} \ \to \ I_{A(i)}^{(l+1)} \cup J_i^{(l+1)} \quad , \quad i = 1, 2, ..., F_{l+1} \qquad (2.3a)$$

$$g_B \ : \qquad J_j^{(l)} \to I_{B(j)}^{(l+1)} \qquad\qquad , \quad j = 1, 2, ..., F_l \qquad (2.3b)$$

with $B(n) := \lfloor n\varphi^2 \rfloor \ = \ n + A(n)$ which is the Wythoff partner of $A(n)$. An updated list of references on the rabbit sequence $\{h(n)\}$ and Wythoff pairs $\{A(n), B(n)\}$ can be found in refs.[1,8]. Here we use the basic fact $h(A(n)) = 1, \ h(B(n)) = 0$.

Lemma 2: Interval lengths

For $l \in \mathbf{N}$, $i = 1, 2, ..., F_{l+1}$ and $j = 1, 2, ..., F_l$ let $s_i^{(l)} = |I_i^{(l)}|$, resp. $t_j^{(l)} = |J_j^{(l)}|$, be the length of the white, resp. black, intervals. Put $s_1^{(0)} = \frac{2}{3}$.
Then the following recursion formulae hold:

$$l \in \mathbf{N}: \qquad s_{A(k)}^{(l)} = \frac{1}{2}\, s_k^{(l-1)} \quad , \quad k = 1, 2, ..., F_l \qquad (2.4a)$$

$$l \in \{2, 3, ...\}: \qquad s_{B(k')}^{(l)} = \frac{1}{4}\, t_{k'}^{(l-1)} \quad , \quad k' = 1, 2, ..., F_{l-1} \qquad (2.4b)$$

$$l \in \mathbf{N}: \qquad t_j^{(l)} = \frac{3}{4}\, s_j^{(l-1)} \quad , \quad j = 1, 2, ..., F_l \quad . \qquad (2.4c)$$

The input is: $s_1^{(0)} = 2/3$.

Proof: This follows from the rules (2.1) which produce the sequence of intervals given in *Lemma 1b)*. From $(2.4b, c)$ one has $s_{B(k')}^{(l)} = \frac{3}{16}\, s_{k'}^{(l-2)}$.

Definition 1: An edge labeled Fibonacci tree $(\widetilde{WZ})_{l+1}$

The length of each interval at a given level l can be found by using an edge labeled Fibonacci tree T_{l+1} with F_{l+2} final branches. There are three types of branches in this tree labeled L, R, and V for left, right, and vertical. An example of such a T_5 $(l = 4)$ tree is depicted in *Fig.3*. It has root $(*)$ at level $l = 0$ and $F_{4+2} = 8$ terminating branches at level $l = 4$. Because these labeled trees T_{l+1} coincide with the Wythoff-Zeckendorf trees $(WZ)_{l+1}$ used in ref.[8], after renaming the vertical edges labels from $L\bullet$ to V and the left edges labels from $L\bullet$ or L^2 to L, we shall call these trees $(\widetilde{WZ})_{l+1}$.

Definition 2: Algorithm for interval lengths

At level number $l \in \mathbf{N}$ the interval lengths $\{s_i^{(l)}\}_1^{F_{l+1}}$ and $\{t_j^{(l)}\}_1^{F_l}$ can be computed from the $(\widetilde{WZ})_{l+1}$ tree defined above as follows. One assignes a value v to each edge label, *viz* $v(L) = \frac{1}{2}$, $v(R) = \frac{3}{4}$, and $v(V) = \frac{1}{2}$. The F_{l+2} intervals at level l are symbolized by the endpoints of the final branches of the tree $(\widetilde{WZ})_{l+1}$ which are numerated from left to right with $1, 2, ..., F_{l+2}$. Each of these numbers k has a unique (reversed) Wythoff code obtained by collecting the labels of the monotone path in the tree starting with the root $(*)$ at the top and ending with the branch at level l which is numbered k. See *Fig.4* for the example $l = 4$ and the (reversed) Wythoff codes of $1, 2, ..., 8$ in the second row of the table.(The first row gives the original reversed Wythoff code used in ref.[8].) We now associate to each of the F_{l+2} final nodes of the tree $(\widetilde{WZ})_{l+1}$ a value v. If the node is numbered with k its value $v(k)$ is the product of all the values of the labels of the Wythoff code of k. For example: $v(5) = v(L)v(R)v(V)v(R) = \frac{1}{2}\frac{3}{4}\frac{1}{4}\frac{3}{4} = \frac{9}{128}$.

Proposition 1: Interval lengths

At level $l \in \mathbf{N}$ the interval lengths are given by $s_i^{(l)} = \frac{2}{3}v(A_i)$ and $t_j^{(l)} = \frac{2}{3}v(B_j)$, where $v(k)$ is the value of the $k-$th terminating node of the $(\widetilde{WZ})_{l+1}$ tree. $v(k)$ has been given in *Definition 2*.

Examples: $s_3^{(4)} = \frac{2}{3}v(A(3) = 4) = \frac{2}{3}v(LRVL) = \frac{2}{3}\frac{1}{2}\frac{3}{4}\frac{1}{4}\frac{1}{2} = \frac{1}{32}$,

$t_3^{(4)} = \frac{2}{3}v(B(3)=7) = \frac{2}{3}v(RVLR) = \frac{2}{3}\frac{3}{4}\frac{1}{4}\frac{1}{2}\frac{3}{4} = \frac{3}{64}$.

Proof: The recursion relations (2.4) up to level l are visualized with the help of of the $(\widetilde{WZ})_{l+1}$ tree. (2.4a), (2.4b) and (2.4c) correspond to the L, V and R branches, respectively. This is because the generation of the underlying Fibonacci tree is based on the $1 \to 1, 0$ and $0 \to 1$ substitution rule, as is the generation of white and black intervals. The values $\frac{1}{2}, \frac{3}{4}, \frac{1}{4}$ for L, R, V are the shrinking factors for the intervals. The input value for the root $(*)$ at level $l = 0$ is $\frac{2}{3}$, the length of the input interval $I_1^{(0)}$.

From the construction of the sets of intervals $\mathcal{A}^{(l)}$ it is clear that they are for any l non-overlapping, *i.e.* disjoint. Also, the cascade of sets $\mathcal{A}^{(l)}$, $l = 0, 1, ...,$ is self-similar. This means that at any level l a white interval together with its descendant intervals can be inflated (in the horizontal direction only) such that it fits to any other white interval at lower level $l' < l$ with all its descendants. The same is true for black intervals and their descendants. By the same token deflation (scaling in the horizontal direction with a factor < 1) can be used to map intervals with their descendants to those of the same color at higher levels. This self-similarity reflects the invariance of the rabbit sequence $\{h(n)\}$ under the $1 \to 1, 0$ and $0 \to 1$ substitution. The generating elements of the self-similarity are g_W and g_B of (2.1).

Lemma 3: Interval boundaries

The lefthand side endpoints of the white intervals at level $l \in \mathbf{N}$, $a_i^{(l)}$, $i = 1, 2, ..., F_{l+1}$, and the black ones , $b_j^{(l)}$, $j = 1, 2, ..., F_l$, obey the following recursion relation:

$$l \in \mathbf{N}: \qquad a_{A(k)}^{(l)} = a_k^{(l-1)} \tag{2.5a}$$

$$l \in \{2, 3, ...\}: \qquad a_{B(k')}^{(l)} = a_{k'}^{(l-2)} + \frac{9}{8}\, s_{k'}^{(l-2)} \ , \tag{2.5b}$$

$$l \in \mathbf{N}: \qquad b_j^{(l)} = a_j^{(l-1)} + \frac{3}{4}\, s_j^{(l-1)} \ . \tag{2.5c}$$

The input is $a_1^{(0)} = 0$, and $\{s_i^{(l)}\}$ obey eqs.(2.4).

Example: $l = 3$, $a_1^{(3)} = a_1^{(1)} = 0$, $a_3^{(3)} = 0 + \frac{9}{8}\frac{2}{3} = \frac{3}{4}$; $a_2^{(3)} = 0 + \frac{9}{8}\frac{1}{2}\frac{2}{3} = \frac{3}{8}$;
$b_1^{(3)} = 0 + \frac{3}{4}\frac{1}{2}\frac{1}{2}\frac{2}{3} = \frac{1}{8}$, $b_2^{(3)} = 0 + \frac{9}{8}\frac{2}{3} + \frac{3}{16}\frac{3}{4}\frac{2}{3} = \frac{27}{32}$.

Proof: This follows from the action of the generators g_W and g_B defined in (2.1) and the structure of the binary $\{h(n)\}$ sequence (2.2).

The righthand side interval endpoints are, of course, $c_i^{(l)} = a_i^{(l)} + s_i^{(l)}$ and $d_j^{(l)} = b_j^{(l)} + t_j^{(l)}$.

Definition 3: The set $\mathcal{A}$

The set $\mathcal{A}$ is the union of the black and white intervals generated in the limit of level number $l \to \infty$:

$$\mathcal{A} \ := \ lim_{l\to\infty}\{\cup_{i=1}^{F_{l+1}} I_i^{(l)} \ \cup_{j=1}^{F_l} J_j^{(l)} \ \} \ . \tag{2.6}$$

Note that $\mathcal{A}$ is *not* the section of all $\mathcal{A}^{(l)}$ sets like in the usual Cantor set construction. Therefore it not clear whether $\mathcal{A}$ is closed. It is not empty because $0 \in \mathcal{A}$. Later it will be seen that $|\mathcal{A}| = 0$, thus it is totally disconnected, *i.e.* contains no intervals, only isolated points.

Lemma 4: The set $\mathcal{A}$ is a fixed point under the combined generating transformations g_W and g_B defined in eqs.(2.1).

Proof: This follows from $\mathcal{A}^{(l+1)} = g_W \; g_B \mathcal{A}^{(l)}$ and the definition of $\mathcal{A}$ in the limit $l \to \infty$.

Lemma 5: Upper boundaries at level l

$$c^{(2k)} := c^{(2k)}_{F_{2k+1}} = a^{(2k)}_{F_{2k+1}} + s^{(2k)}_{F_{2k+1}} = \frac{3}{4}\frac{1-\frac{5}{18}(\frac{3}{16})^k}{1-\frac{3}{16}} \;, \tag{2.7a}$$

$$d^{(2k+1)} := d^{(2k+1)}_{F_{2k+1}} = b^{(2k+1)}_{F_{2k+1}} + t^{(2k+1)}_{F_{2k+1}} = \frac{3}{4}\frac{1+\frac{1}{12}(\frac{3}{16})^k}{1-\frac{3}{16}} \;. \tag{2.7b}$$

Therefore: $c := \lim_{k\to\infty} \; c^{(2k)} = \frac{12}{13} = \lim_{k\to\infty} \; d^{(2k+1)} =: d$ and $\mathcal{A} \subseteq [0, 12/13]$.

Proof: We use the Wythoff representation of odd indexed Fibonacci numbers (*cf.* [8]) $F_{2k+1} = B^k(1)$, with k iterations of the Wythoff sequence $\{B(n)\}$ defined after eqs.(2.3). The $a^{(2k)}_{F_{2k+1}} = a^{(2k)}_{B(B^{k-1}(1))}$ recursion can then be applied, together with the $s-$ recursions (2.4b). Iteration leads to

$$a^{(2k)}_{B^k(1)} = a^{(2)}_2 + \frac{9}{8}\frac{1}{16^{k-2}}(16^{k-2} + 3(16^{k-3} + 3(... + 3(16^1+3)...) \; s^{(2)}_2 \;,$$

which produces

$$a^{(2k)}_{F_{2k+1}} = \frac{3}{4}\frac{1-(\frac{3}{16})^k}{1-\frac{3}{16}} \;, \quad k \in \mathbf{N}_0 \;. \tag{2.8a}$$

Similarly one finds $s^{(2k)}_{F_{2k+1}} = \frac{2}{3}(\frac{3}{16})^k$. Together this is (2.7a). In the same manner one finds

$$b^{(2k+1)}_{F_{2k+1}} = \frac{3}{4}\frac{1-\frac{11}{24}(\frac{3}{16})^k}{1-\frac{3}{16}} \;, \quad k \in \mathbf{N}_0 \;. \tag{2.8b}$$

and $t^{(2k+1)}_{F_{2k+1}} = \frac{1}{2}(\frac{3}{16})^k$, which yields (2.7b).

3. Construction of the fractal measure μ

In this section first a probability measure $\mu^{(l)}$ is defined on the sets of $\mathcal{A}^{(l)}$ of white and black intervals at a given level l. In the limit $l \to \infty$ this will then define the measure μ on the fractal set $\mathcal{A}$.

We think of 'mass' distributed along the intervals of $\mathcal{A}^{(l)}$, for $l \in \mathbf{N}_0$. At level $l = 0$ the probability is normalized to 1: $p^{(0)}_1 = 1$. At the first level we split the probabilities into $p^{(1)}_1 = p$, with $0 < p < 1$ and $q^{(1)}_1 = 1-p$, and we shall use the notations $p^{(l)}_i$, $i = 1, ..., F_{l+1}$, resp. $q^{(l)}_j$, $j = 1, ..., F_l$, for the probabilities of white, resp. black intervals. These probabilities will be distributed in a self-similar way on the cascade $\{\mathcal{A}^{(l)}\}$. Therefore it is sufficient to define the probability assignments for the generating elements g_W and g_B of (2.1). In the $0 \to 1$ step (2.3b) the probability is carried over from the black interval, *i.e.* $p_{B(j)}^{(l+1)} = q_j^{(l)}$. In the $1 \to 1, 0$ step the probablity for a white interval $I_i^{(l)}$ is split into $p\, p_i^{(l)}$, resp. $(1-p)\, p_i^{(l)}$ for the generated intervals $I_{A(i)}^{(l+1)}$, resp. $J_i^{(l+1)}$. Thus we have the recursion rules for $l \in \mathbf{N}$:

$$p_{A(k)}^{(l)} = p\, p_k^{(l-1)} \quad , \qquad k = 1, ..., F_l \quad , \tag{3.1a}$$

$$p_{B(k')}^{(l)} = q_{k'}^{(l-1)} \quad , \qquad k' = 1, ..., F_{l-1} \quad , \tag{3.1b}$$

$$q_j^{(l)} = (1-p)\, p_j^{(l-1)} \quad , \qquad j = 1, ..., F_l \quad . \tag{3.1c}$$

The input is $p_1^{(0)} = 1$.

Probability conservation is evident: $\sum_{i=1}^{F_{l+1}} p_i^{(l)} + \sum_{j=1}^{F_l} q_j^{(l)} = 1$ for all l. In *Fig.2* the probabilities for the first four levels are written above the intervals for the example $p = 1/\varphi^2 = 2 - \varphi$, with the golden section φ, whence $1 - p = 1/\varphi = \varphi - 1$. It is clear that the same algorithm, based on the labeled $(\widetilde{WZ})_{l+1}$ Fibonacci tree, which produces the interval lengths can be used to compute the probabilities at a given level l as well. The values to be used for the branch labels L, R, V are $p, 1, 1-p$, respectively. Again the product over all values of the labels occuring in the relevant code $\widetilde{W}(k)_{rev}$ for the interval number k has to be taken. *E.g.* $p_3^{(4)}$ is obtained from the $l = 4$ code for $A(3) = 4$, which is $LRVL$. Therefore it is $p(1-p)1p = p^2(1-p)$, which, for the case $p = 1/\varphi^2$, becomes φ^{-5}.

The probability measure $\mu^{(l)}$ at level $l \in \mathbf{N}$ is defined by $\mu^{(l)}(I_i^{(l)}) = p_i^{(l)}$ and $\mu^{(l)}(J_j^{(l)}) = q_j^{(l)}$, satisfying recursions (3.1). $\mu^{(l)}(\mathcal{A}^{(l)}) = 1$, for all l. Therefore, also $\mu(\mathcal{A}) := \lim_{l\to\infty} \mu^{(l)}(\mathcal{A}^{(l)}) = 1$.

Note that a uniform probability distribution would have at level l: $p_i^{(l)} = q_j^{(l)} = 1/F_{l+2}$, independent of i and j. See *Appendix B* for a discussion of this case.

4. Multifractal analysis

In this section the cascade of sets $\mathcal{A}^{(l)}$ together with their probability measure $\mu(\mathcal{A}^{(l)})$ are analysed using the multifractal formalism proposed in [5]. This method starts with the partition function at level $l \in \mathbf{N}_0$ which in our case is

$$\Gamma^{(l)}(q, \tau) := \sum_{i=1}^{F_{l+1}} \frac{(p_i^{(l)})^q}{(s_i^{(l)})^\tau} + \sum_{j=1}^{F_l} \frac{(q_j^{(l)})^q}{(t_j^{(l)})^\tau} \quad . \tag{4.1}$$

This function of the $q-$th moment of the probabilities and the $\tau-$ th moment of the inverse interval lengths provides a convenient quantity to study the probability distribution $\mu^{(l)}$ defined in *sec.3* on the sets $\mathcal{A}^{(l)}$ constructed in *sect.2*. We shall also use $P^{(l)}(q) := \Gamma^{(l)}(q, \tau = 0)$, the moment function of the probabilities. Obviously, $\mu^{(l)}(\mathcal{A}^{(l)}) = P^{(l)}(1) = 1$. The total length of the intervals at level l is $\Sigma^{(l)} = \Gamma^{(l)}(0, -1)$.

It is clear that $\Gamma^{(l)}(q, \tau)$ does not encode all information on the sets $\mathcal{A}^{(l)}$; for example, the position of the intervals in $[0, 1]$ does not enter, only their length matters. However

$\Gamma^{(l)}(q,\tau)$ can be used to find the generalized (or Rényi) dimensions [6] D_q for the fractal measure μ defined on the set $\mathcal{A}$. To this end a functional relation $q = q(\tau)$, or $\tau = \tau(q)$, is found from the requirement that in the limit $l \to \infty$ $\Gamma^{(l)}(q,\tau)$ stays finite and does not vanish. The relation of this exponent $\tau(q)$ to D_q is $(q-1)D_q = \tau(q)$. For $q = 0$ one has $D_0 = -\tau(0)$, the fractal dimension of the set $\mathcal{A} = \lim_{l\to\infty} \mathcal{A}^{(l)}$. This coincides with its Hausdorff dimension D_H. Other D_q of special interest are the *information dimension* $D_1 = \tau'(q)|_{q=1}$ (L'Hospital's rule applies because $\tau(1) = 0$ due to $\Gamma^{(l)}(1,0) = P^{(l)}(1) = 1$, for all l), and the *correlation dimension* $D_2 = \tau(2)$.

We now compute $\Gamma^{(l)}(q,t)$ and then find the functional dependence $q = q(\tau)$. It will turn out that this function can be given analytically for the probabilities (3.1) with $p = 1/\varphi^2$.

Proposition 2: Partition function at level l

For $l \in \mathbf{N}$, $0 < p < 1$, $\tau, q \in \mathbf{R}$, $-\infty < q,\tau < +\infty$:

$$\Gamma^{(l)}(q,\tau) = 3^\tau p^q \Big(1 + (\frac{2}{3})^\tau (\frac{1-p}{p})^q\Big) \Big[\, \mathcal{U}_l(q,\tau) + \frac{(\frac{1-p}{p})^q}{(\frac{3}{8})^\tau (1 + (\frac{2}{3})^\tau (\frac{1-p}{p})^q)}\, \mathcal{U}_{l-1}(q,\tau) \,\Big] \ , \tag{4.2a}$$

$$\mathcal{U}_l(q,\tau) \ := \ \big(\lambda_+^l(q,\tau) \ - \ \lambda_-^l(q,\tau)\big)/(\lambda_+(q,\tau) \ - \ \lambda_-(q,\tau)) \ , \tag{4.2b}$$

$$\lambda_\pm(q,\tau) \ := \ 2^{\tau-1}\ p^q \Big(1 \ \pm \sqrt{1 + 4(\frac{4}{3})^\tau (1-p)^q\ p^{-2q}}\,\Big) \quad . \tag{4.2c}$$

Note that the p dependence of the symbols is implicit.

Corollary 1: The polynomials $\{\mathcal{U}_l\}$ (4.2b) satisfy the generalized Fibonacci recursion relation

$$\mathcal{U}_l(q,\tau) \ = \ x(q,\tau)\, \mathcal{U}_{l-1}(q,\tau) \ + \ y(q,\tau)\, \mathcal{U}_{l-2}(q,\tau) \tag{4.3a}$$

$$x(q,\tau) \ := \ p^q\ 2^\tau \qquad , \qquad y(q,\tau) \ := \ (1-p)^q\ (\frac{16}{3})^\tau \ , \tag{4.3b}$$

with the inputs $\mathcal{U}_0(q,\tau) \equiv 0$, $\mathcal{U}_1(q,\tau) \equiv 1$. Thus $y(q,\tau)\, \mathcal{U}_{-1}(q,\tau) \equiv 1$.

Corollary 2: For $p = 1/\varphi^2$, $\varphi = \frac{1}{2}(1+\sqrt{5})$ one has

$$\Gamma^{(l)}(q,\tau) \ = \ 3^\tau \varphi^{-2q} (1 + (\frac{2}{3})^\tau \varphi^q) \Big[\, \mathcal{U}_l(q,\tau) \ + \ \frac{\varphi^q}{(\frac{3}{8})^\tau (1 + (\frac{2}{3})^\tau \varphi^q)}\, \mathcal{U}_{l-1}(q,\tau) \Big], \tag{4.4a}$$

$$\lambda_\pm(q,\tau) \ = \ 2^{\tau-1} \varphi^{-2q} \Big(1 \ \pm \sqrt{1 \ + \ 4(\frac{4}{3})^\tau \varphi^{3q}}\,\Big) \ . \tag{4.4b}$$

For the *Proof* of Proposition 2 we define

$$\Gamma_W^{(l)}(q,\tau) \ := \ \sum_{i=1}^{F_{l+1}} \frac{(p_i^{(l)})^q}{(s_i^{(l)})^\tau} \quad , \quad \Gamma_B^{(l)}(q,\tau) \ := \ \sum_{j=1}^{F_l} \frac{(q_j^{(l)})^q}{(t_j^{(l)})^\tau} \ . \tag{4.5}$$

Thus $\Gamma^{(l)}(q,\tau) \;=\; \Gamma^{(l)}_W(q,\tau) \;+\; \Gamma^{(l)}_B(q,\tau)$. Because the Wythoff sequences $\{A(n)\}$ and $\{B(n)\}$ cover the integers disjointly we can write

$$\Gamma^{(l+1)}_W(q,\tau) \;=\; \sum_{i=1}^{F_{l+2}} \frac{(p_i^{(l+1)})^q}{(s_i^{(l+1)})^\tau} \;=\; \sum_{k=1}^{F_{l+1}} \frac{(p_{A(k)}^{(l+1)})^q}{(s_{A(k)}^{(l+1)})^\tau} \;+\; \sum_{j=1}^{F_l} \frac{(p_{B(j)}^{(l+1)})^q}{(s_{B(j)}^{(l+1)})^\tau} \;\;. \tag{4.6}$$

The recursion formula (2.4) for the interval lengths and the one for the probabilities (3.1) is now employed to yield

$$\Gamma^{(l+1)}_W(q,\tau) \;=\; p^q 2^\tau \;\Gamma^{(l)}_W(q,\tau) \;+\; 4^\tau \Gamma^{(l)}_B(q,\tau) \;\;. \tag{4.7}$$

In the same way

$$\Gamma^{(l+1)}_B(q,\tau) \;=\; (\frac{4}{3})^\tau \;(1-p)^q \;\Gamma^{(l)}_W(q,\tau) \;\;. \tag{4.8}$$

Inserting (4.8) into (4.7) produces the Γ_W recursion relation

$$\Gamma^{(l+1)}_W(q,\tau) \;=\; x(q,\tau)\;\Gamma^{(l)}_W(q,\tau) \;+\; y(q,\tau)\;\Gamma^{(l-1)}_W(q,\tau) \;\;, \tag{4.9}$$

with the coefficients given in (4.3b). This is a generalized Fibonacci recursion with inputs

$$\Gamma^{(0)}_W(q,\tau) \;=\; (\frac{3}{2})^\tau \quad , \quad \Gamma^{(1)}_W(q,\tau) \;=\; 3^\tau \;p^q \;=\; x(q,\tau)\;\Gamma^{(0)}_W \;, \tag{4.10}$$

$\Gamma^{(l)}_W(q,\tau)$ can be written as linear combination of $\mathcal{U}_l$ and $y\,\mathcal{U}_{l-1}$ with the $\{\mathcal{U}_l\}$ polynomials defined in (4.3) with the standard input $\mathcal{U}_0 = 0$, $\mathcal{U}_1 = 1$. For general $\Gamma^{(0)}_W$ and $\Gamma^{(1)}_W$ input one has $\Gamma^{(l)}_W \;=\; \Gamma^{(1)}_W\,\mathcal{U}_l \;+\; \Gamma^{(0)}_W\;y\,\mathcal{U}_{l-1}$ or with (4.10) which are found from data of the lowest levels.

$$\Gamma^{(l)}_W(q,\tau) \;=\; 3^\tau \;p^q \Big[\,\mathcal{U}_l(q,\tau) \;+\; (\frac{1-p}{p})^q (\frac{8}{3})^\tau \;\mathcal{U}_{l-1}(q,\tau)\,\Big] \;=\; (\frac{3}{2})^\tau \;\mathcal{U}_{l+1}(q,\tau) \,. \tag{4.11}$$

The characteristic roots of the recursion relation (4.9) (or (4.3)) are, for given q and τ,

$$\lambda_\pm(q,\tau) \;=\; \frac{1}{2}\big(x(q,\tau) \;\pm\; \sqrt{x^2(q,\tau) \;+\; 4y(q,\tau)}\;\big) \;\;, \tag{4.12}$$

which is just (4.2c). In the case $\lambda_+ \neq \lambda_-$ (which is always satisfied for $x(q,\tau) \neq 0$ or $\tau \neq -\infty$, $q \neq +\infty$) the Binet form of the $\mathcal{U}$-polynomials (4.3) is given by (4.2b), which proves *Corollary 1.* (4.8) becomes with (4.11) for $l \in \mathbf{N}$

$$\Gamma^{(l)}_B(q,\tau) \;=\; (1-p)^q \;2^\tau \;\mathcal{U}_l(q,\tau) \;\;. \tag{4.13}$$

The sum of (4.13) and (4.11) produces (4.2) which proves *Proposition 2.*

Lemma 6: For large l one has

$$\Gamma^{(l)}(q,\tau) \sim c(q,\tau)\;\lambda_+^l(q,\tau) \;\;. \tag{4.14}$$

One could write down $c(q,\tau)$ but it will not be of interest later on.
Proof: This follows from (4.2a) observing that $|\lambda_-/\lambda_+| < 1$ and $\mathcal{U}_l(q,\tau) \sim \lambda_+^l/(\lambda_+ - \lambda_-)$ for large l.
In order to find a non-vanishing and finite $\Gamma(q,\tau)$ in the limit $l \to \infty$ one therefore has to impose

$$\lambda_+(q,\tau) \;=\; 1 \quad , \tag{4.15}$$

which yields the implicit q versus τ relation $y(q,\tau) \;+\; x(q,\tau) \;=\; 1$, or

$$(\frac{1-p}{p})^q\,(\frac{4}{3})^\tau \;-\; (\frac{1}{p})^q(\frac{1}{4})^\tau \;+\; (\frac{1}{2})^\tau \;=\; 0 \;, \tag{4.16}$$

for the general $0 < p < 1$ case. This is only a necessary condition for (4.15), in addition one has the constraint $0 < x(q,\tau) \leq 2$, or $q(\tau) \geq ln\;2\;(\tau \;-\; 1)/ln(1/p)$. (4.16) can be solved explicitely if $p = 1/\varphi^2$, with the golden mean $\varphi = (1+\sqrt{5})/2$. One obtains

$$(\varphi^q)^2 \;-\; (\frac{16}{3})^\tau\;\varphi^q \;-\; 2^\tau \;=\; 0 \;. \tag{4.17}$$

The positive solution of this quadratic eq. for φ^q leads to

$$q(\tau) \;=\; \Big(\tau\;ln\frac{16}{3} \;+\; ln\Big[\frac{1}{2}\big(1 \;+\; \sqrt{5 \;+\; 4\big((3\sqrt{2}/16)^{2\tau}-1\big)}\;\big)\Big]\Big)/ln\;\varphi \;. \tag{4.18}$$

A plot of this monotone $q(\tau)$ curve is shown in *Fig.5.* $q(\tau)$ satisfies the above found constraint. Note that $q(0) \;=\; 1$ for general p because the constraint $\lambda_+(q,0) \;=\; 1$, *viz.* $p^q + (1-p)^q \;=\; 1$, is satisfied for $q = 1$. The fractal dimension of the set $\mathcal{A} \;:=\; \lim_{l\to\infty} \mathcal{A}^{(l)}$ (which is p independent) is obtained from $D_0 \;=\; -\tau(0)$ which obeys

$$(\frac{3}{16})^{D_0} \;+\; (\frac{1}{2})^{D_0} \;=\; 1 \;. \tag{4.19}$$

One can find a numerical value from a close up of *Fig.5* : $D_0 \;\approx\; .62469$, or solve for τ in (4.18) with $q \;=\; 0$, or solve (4.19), numerically. The information dimension D_1 is p dependent and can be given explicitly because $D_1 \;=\; \tau'(q)|_{q=1}$ and $\tau(1) = 0$ (for all p). The result is from (4.16)

$$D_1 \;=\; \frac{(1-p)\;ln(1-p) \;+\; p\;ln\;p}{(1-p)\;ln\;3 \;-\; (4-3p)\;ln\;2} \quad , \tag{4.20}$$

which for $p = 1/\varphi^2$ becomes

$$D_1 \;=\; \frac{(2+\varphi)\;ln\;\varphi}{ln\;2 \;+\; \varphi\;ln\frac{16}{3}} \approx .51181 \;. \tag{4.21}$$

Due to a general theorem [6] $D_1 \leq D_0$. The fact that in our case $D_1 < D_0$ shows that we are dealing with an inhomogeneous or multifractal.

Next we compute the total lengths of white, resp. black, resp. all intervals on a given level l, denoted by $\Sigma_W^{(l)}$, resp. $\Sigma_B^{(l)}$, resp. $\Sigma^{(l)}$. These quantities follow from (4.12), (4.13) and (4.2a) with $q = 0$ and $\tau = -1$.

Lemma 7: Concerning $\Sigma_W^{(l)}$, $\Sigma_B^{(l)}$, $\Sigma^{(l)} \; = \; \Sigma_W^{(l)} \; + \; \Sigma_B^{(l)}$
The total interval length are expressed in terms of

$$u_l \; = \; \frac{1}{4}\big(3^l \; - \; (-1)^l\big) \quad , \quad l \in \mathbf{N}_0 \quad , \tag{4.22a}$$

and are

$$\begin{aligned}
\Sigma_W^{(l)} \; &= \; u_{l+1}/(3 \cdot 2^{2l-1}) \ , && (4.22b)\\
\Sigma_B^{(l)} \; &= \; u_l/2^{2l-1} \quad , && (4.22c)\\
\Sigma^{(l)} \; &= \; (u_{l+2} \; - \; u_{l+1})/(3 \cdot 2^{2l-1}) \quad . && (4.22d)
\end{aligned}$$

Example: The sequence $\{u_l\}_0^\infty \; = \; \{0, 1, 2, 7, 20, 61, 182, 547, 1640, 4921, ...\}$ leads to $\Sigma_W^{(3)} = 20/(3 \cdot 2^5) = 5/24$, $\Sigma_B^{(3)} = 7/32$, $\Sigma^{(3)} = 41/96$.
Proof: It is convenient to define $u_l \; := \; 2^{2(l-1)} \; \mathcal{U}_l(q = 0, \tau = -1)$, $l \in \mathbf{N}_0$, with the $\mathcal{U}_l$ polynomials defined in (4.3a) with $x = 1/2$ and $y = 3/16$. Then the new polynomials satisfy the recursion relation

$$u_l \; = \; 2u_{l-1} \; + \; 3u_{l-2} \quad , \quad u_0 = 0 \ , \ u_1 = 1 \quad , \tag{4.23}$$

and their Binet form is given by (4.22a). With these $\{u_l\}$ one finds from (4.12) $\Gamma_W^{(l)}(0, -1) \; = \; (2 \; u_l \; + \; 3 \; u_{l-1})/(3 \cdot 2^{2l-1})$ which becomes (4.22b) after use of the recursion (4.23). Similarly (4.22c) follows from (4.13) for $q = 0$, $\tau = -1$. Finally adding both results together yields (4.22d), again using the recursion relation.

Corollary 3: Large l behavior of $\Sigma_W^{(l)}$, $\Sigma_B^{(l)}$, $\Sigma^{(l)}$

$$\Sigma_W \; := \; \lim_{l\to\infty} \; \Sigma_W^{(l)} \; = \; 0 \quad , \quad \Sigma_B \; := \; \lim_{l\to\infty} \; \Sigma_B^{(l)} \; = \; 0 \quad , \quad \Sigma \; := \; \lim_{l\to\infty} \; \Sigma^{(l)} \; = \; 0 \ . \tag{4.24}$$

This follows immediately from (4.22$a - d$). Thus the fractal $\mathcal{A}$ has length or Lebesgue measure zero.

On can also compute various ratios, *e.g.*

$$\lim_{l\to\infty} \; \Sigma_B^{(l)}/\Sigma_B^{(l-1)} \; = \; \lim_{l\to\infty} \; \Sigma_W^{(l)}/\Sigma_W^{(l-1)} \; = \; \lim_{l\to\infty} \; \Sigma^{(l)}/\Sigma^{(l-1)} \; = \; \frac{3}{4} \ .$$

Or

$$\lim_{l\to\infty} \; \Sigma_B^{(l)}/\Sigma_W^{(l)} \; = \; 1 \; = \; \frac{1}{2}\lim_{l\to\infty} \; \Sigma^{(l)}/\Sigma_B^{(l)} \quad .$$

We omit the proofs which are elementary.

Another quantity of interest is the 'entropy' at level l which is defined by

$$S^{(l)} \;:=\; -\Big(\sum_{i=1}^{F_{l+1}} p_i^{(l)}\; ln\; p_i^{(l)} \;+\; \sum_{j=1}^{F_l} q_j^{(l)}\; ln\; q_j^{(l)}\Big) \;=\; -\frac{d}{dq}\; P^{(l)}(q)|_{q=1}\;, \tag{4.25}$$

where the last step follows with $P^{(l)}(q) \;:=\; \Gamma^{(l)}(q,0)$ and the definition (4.1). In the *Appendix* we give the computation of this entropy for the case $p = 1/\varphi^2$. There we also find a bin length $B^{(l)}$ which can be used for large level numbers l. It is found from the alternative definition of $D_1 \;=\; -\lim_{l\to\infty}\; S^{(l)}/ln\; B^{(l)}$.

We conclude this section with a discussion of the multifractal spectrum $f(\alpha)$ [3,4,5], the Legendre transform of $-\tau(q)$, defined for the general case in implicit form by (4.16).

$$f(\alpha) \;=\; -\tau(q) \;+\; \alpha q \quad \text{with} \quad \alpha(\mathrm{q}) \;=\; \tau'(\mathrm{q}) \quad . \tag{4.26}$$

If $\tau'(q)$ is invertible (which is the case for $p = 1/\varphi^2$ where we know $q(\tau)$) one finds from (4.26) $q(\alpha) \;=\; f'(\alpha)$. The Legendre transform can be considered as parametric equation for $f(\alpha)$ if one writes

$$\tilde{f}(q) \;=\; (q\frac{d}{dq} \;-\; 1)\;\tau(q) \quad , \quad \alpha(q) \;=\; \tau'(q) \;\; . \tag{4.27}$$

and $f(\alpha) \;=\; \tilde{f}(q(\tau))$. We have to express the Legendre transform of $-\tau(q)$ through the inverse function $q(\tau)$ which is known explicitely for the $p = 1/\varphi^2$ case. Therefore we use the following parametric representation of $f(\alpha)$

$$\hat{f}(\tau) \;:=\; \tilde{f}(q(\tau)) \;=\; -\tau \;+\; q(\tau)/q'(\tau) \quad , \quad \hat{\alpha}(\tau) \;:=\; \alpha(q(\tau)) \;=\; 1/q'(\tau) \quad , \tag{4.28}$$

with $f(\hat{\alpha}) \;=\; \hat{f}(\tau(\hat{\alpha}))$. $q'(\tau)$ can be computed from (4.18) but we refrain from quoting the result. Instead we show in *Fig.6* the spectrum curve $f(\alpha)$ computed from (4.28) with (4.18). Some well-known features of spectrum curves $f(\alpha)$, originating from their definition as Legendre transforms, are (see *e.g.* [12])

i) $f(\alpha)$ is convex ,

ii) $-\tau(q)$ is the intercept of the slope q tangent to the $f(\alpha)$ curve with the f axis.
For example: The maximum of f corresponds to parameter $q = 0$ and $-\tau(0) = D_0 = f_{max}$. Because $\tau(1) = 0$ the main bisector of the f and α axes touches the curve at the point with parameter $q = 1$. For $q = \pm\infty$ one has vertical tangents to $f(\alpha)$ and $\alpha(\pm\infty) = \tau'(\pm\infty) = D_{\pm\infty}$ (L'Hospital's rule).

iii) Related to *ii)* one finds the dimension D_q in the $f(\alpha)$ curve as the ordinate of the intersection point of the slope q tangent with the main bisector of the axes.
For example: For slope $q = 0$ $\;D_0 = f(\alpha_{max})$; for $q = 1$ $\;D_1 = \alpha(1) = f(\alpha(1))$.

Finally, some open questions:

- Is the set $\mathcal{A}$ a Cantor set?
- Is it possible to find a code for points belonging to $\mathcal{A}$?

Appendix A: Computation of the entropy

The entropy at level l has been defined in eq.(4.25)

Lemma: For level number $l \in \mathbf{N}$ and $p = 1/\varphi^2$

$$S^{(l)} \;=\; (ln\ \varphi)\big(\ C_l \;-\; D_l\ \varphi\ \big)\ \ , \tag{A.1a}$$

$$C_l \;:=\; 7F_{l+2} \;+\; 4F_{l+1} \;-\; 11 \;-\; 4l\ \ ,\quad D_l \;:=\; C_l \;+\; 4 \;+\; l \;-\; L_{l+3}\ \ , \tag{A.1b}$$

and the Fibonacci and Lucas numbers $\{F_l\}$ and $\{L_l\}$.

Remark: The sequences are $\{C_l\}_0^\infty \;=\; \{0, 3, 10, 24, 49, 92, 164, 283, 478, 796, ...\}$ and $\{D_l\}_0^\infty = \{0, 1, 5, 13, 28, 54, 98, 171, 291, 487, ...\}$.

Proof: One has to compute $-\frac{d}{dq}\Gamma^{(l)}(q,0)|_{q=1}$, using $\Gamma^{(l)}(q,0)$ which follows from (4.4) by putting $\tau = 0$. For this computation one needs

$$\mathcal{U}_l(1,0) \;=\; (1-(-1+p)^l)/(2-p) \;=\; (F_{l+2}-1) \;-\; (F_{l+1}-1)\ \varphi\ \ , \tag{A.2}$$

where the first step follows from $(4.2b,c)$ with $\lambda_+(1,0) = 1$, $\lambda_-(1,0) = p-1$, and in the second step $p = 1/\varphi^2$ was used. One also needs the derivative $\frac{d}{dq}\mathcal{U}_l(q,0)|_{q=1}$ which follows from $(4.2b)$ with $\frac{d}{dq}\lambda_\pm(q,0)|_{q=1}$. This latter expressions are $(4-3\varphi)ln\ \varphi$ and $(5\varphi\ -\ 8)ln\ \varphi$ for the $+$ and $-$ sign. It is also possible to find the desired derivative from differentiation of the recursion relation (4.3). Both ways lead to $\frac{d}{dq}\mathcal{U}_l(q,0)|_{q=1} \;=\; (ln\ \varphi)\ v_l$, where v_l satisfies the inhomogeneous recursion relation

$$v_l \;=\; (2-\varphi)\ v_{l-1} \;+\; (\varphi-1)\ v_{l-2} \;+\; \gamma_l\ \ , \tag{A.3a}$$

$$\gamma_l \;:=\; (F_{l+3} \;-\; 3)\varphi \;-\; (4F_{l+1} \;+\; F_{l-2}-4)\ \ , \tag{A.3b}$$

with inputs $v_0 = v_1 \;=\; 0$. The solution could be given but for the $S^{(l)}$ computation only the combination $w_l \;:=\; v_l \;-\; (1-\varphi)v_{l-1}$ is of interest. $(A.3)$ implies $w_l \;=\; w_{l-1} \;+\; \gamma_l$ with $w_0 = 1\ \ (v_{-1} = \varphi)$. The solution is for $l \in \mathbf{N}$

$$w_l \;=\; \sum_{k=2}^{l} \gamma_k \;=\; (F_{l+5}-5-3l)\varphi \;-\; (F_l + 4F_{l+3} - 9 - 4l)\ . \tag{A.4}$$

The final result $(A.1)$ then follows from $S^{(l)} \;=\; ln\ \varphi\big\{(2-\varphi)\ \mathcal{U}_l(1,0) + 1 - w_l\big\}$ with $(A.2)$ and $(A.4)$.

Corollary: For large l:

$$S^{(l)}/ln\ \varphi \sim (3\varphi-4)l \approx .411\ l\ \ . \tag{A.5}$$

This follows from $F_l \sim \varphi^l/\sqrt{5}$ for large l.

It is interesting to define an average interval length $B^{(l)}$ at level l such that in the $l \to \infty$ limit the information dimension D_1, which is known exactly (for the $p = 1/\varphi^2$ case see (4.21)), coincides with the alternative definition

$$D_1 \;=\; -\lim_{l\to\infty}\ S^{(l)}/ln\ B^{(l)}\ \ . \tag{A.6}$$

This interval length has to scale for large l like

$$B^{(l)} \sim e^{-\beta l} \quad , \quad \beta \;=\; (2\varphi-3)(ln\ 2 \;+\; (ln\ \frac{16}{3})\ \varphi) \;\approx .803 \quad . \tag{A.7}$$

With this $B^{(l)}$ for high levels l one may be interested in the alternative definition of generalized dimensions, $\tilde{D}_q$, given by

$$\tilde{D}_q \;:=\; \lim_{l\to\infty} \frac{1}{q-1}\big(ln\ P^{(l)}(q)/ln\ B^{(l)}\big) \quad , \tag{A.8}$$

with $P^{(l)}(q) \;:=\; \Gamma^{(l)}(q,0)$. $\tilde{D}_1 = D_1$ for $p = 1/\varphi^2$ by definition of $B^{(l)}$. For example, $\tilde{D}_0 \;=\; ln\ \varphi/((2\varphi-3)(ln\ 2 \;+\; \varphi\ ln\frac{16}{3})) \approx .599$, which should be compared with the value $D_0 \approx .625$. In the case $p = 1/\varphi^2$ one finds

$$\tilde{\tau}(q) \;=\; (q-1)\tilde{D}_q \;=\; \tilde{D}_0\big(ln\ 2 \;+\; 2q ln\ \varphi \;-\; ln(1+\sqrt{1+4\varphi^{3q}}\ )\big)/ln\ \varphi \quad , \tag{A.9}$$

which should be compared with the inverse of the exact result $q(\tau)$ found in (4.18).

Appendix B: Uniform probability case

If one uses instead of (3.1) at each level l uniform probabilities $(u.p.)$, $p_i^{(l)} \;=\; q_j^{(l)} \;=\; 1/F_{l+2}$ independent of i and j, one finds from (4.1)

$$\Gamma^{(l)}_{u.p.}(q,\tau) \;=\; \frac{1}{(F_{l+2})^q}\ \Gamma^{(l)}(0,\tau) \quad . \tag{B.1}$$

With the result of (4.2) for $q = 0$ one has for large l

$$\Gamma^{(l)}_{u.p}(q,\tau) \;\sim \tilde{c}(q,\tau)\big(\lambda_+(0,\tau)/\varphi^q\big)^l \quad . \tag{B.2}$$

The requirement of finite, non-vanishing $\Gamma^{(l)}_{u.p.}$ in the $l \to \infty$ limit now becomes $\lambda_+(0,\tau) = \varphi^q$, or $q_{u.p.}(\tau) \;=\; ln\ \lambda_+(0,\tau)/ln\ \varphi$, or with (4.2c)

$$q_{u.p.}(\tau) \;=\; \big((\tau-1)\ ln\ 2 \;+\; ln(1+\sqrt{1+4(4/3)^\tau}\ )\ \big)/ln\ \varphi \quad . \tag{B.3}$$

The $f_{u.p.}(\alpha)$ spectrum obtained from this expression *via* (4.28) coincides with the result one finds using the free energy $F(\beta)$ approach of ref.[3] with the replacements $\beta \leftrightarrow -\tau$, $-F(\beta) \leftrightarrow q$. See *Fig.6* for this spectrum $f_{u.p.}(\alpha)$.

Finally a remark concerning the four scale Cantor set proposed in ref.[3]. This is also an example of a self-similar bicolored multifractal where white and black intervals are generated according to the substitution rule $1 \to 1,0$ and $0 \to 1,0$. This means that odd, resp. even numbered intervals are white, resp. black. One can compute the implicit q,τ relation for this example, also for a nonuniform probability distribution, using the method of recursion relations applied in this paper. For the uniform probability case one finds an explicit result in accordance with the free energy expression given in ref.[3].

References

[1] Brown, T.C. "Description of the characteristic sequence of an irrational." *Canad. Math. Bull.,* Vol.*36* (1993): pp. 15-21.

[2] Falconer, K.J. *Fractal Geometry, Mathematical Foundations and Applications*, Wiley, Chichester, 1990, ch.17.

[3] Feigenbaum, M.J. "Some characterizations of strange sets." *J. Stat. Phys.,*Vol.*46* (1987): pp. 919-924.

[4] Frisch, U. and Parisi, G. "On the singularity structure of fully developed turbulence", Appendix to the article by U. Frisch in *Turbulence and Predictability in Geophysical Fluid Dynamics and Climate Dynamics*, edts. M. Ghil, R. Benzi and G. Parisi, North Holland, 1985, Proceedings of the "Enrico Fermi" School, course LXXXVIII, 1983.

[5] Halsey, Th.C., Jensen, M.H., Kadanoff, L.P., Procaccia, I. and Shraiman, B.I. "Fractal measures and their singularities: The characterization of strange sets." *Phys. Rev. A,* Vol.*33* (1986): pp. 1141-1151.

[6] Hentschel, H.G.E. and Procaccia, I. "The infinite number of generalized dimensions of fractals and strange attractors" Physica, Vol. *8D* (1983): pp. 435-444.

[7] Lang, W. "On the characteristic polynomials of Fibonacci chains." *J. Phys. A,* Vol.*25* (1992): pp. 5395-5413, *ibid.* Vol. *26* (1993): p. 1261.

[8] Lang, W. "The Wythoff and the Zeckendorf Representations of numbers are equivalent". in *Applications of Fibonacci Numbers*, Vol. 6, edts. Bergum, G.E., Philippou, A.N. and Horadam, A.F., Kluwer, Dordrecht, 1996, pp.321-337.

[9] Mandelbrot, B.B. *The Fractal Geometry of Nature*, W.H. Freeman, New York, 1983.

[10] Mandelbrot, B.B. "Intermittent turbulence in self-similar cascades: divergence of high moments and dimension of the carrier." *J. Fluid Mechanics,* Vol. *62* (1974): pp. 331-358.

[11] Mandelbrot, B.B. "An Introduction to multifractal distribution functions." pp. 279-291 in ref.[15].

[12] McCauley, J.L. "Introduction to multifractals in dynamical systems theory and fully developed fluid turbulence." *Phys. Rep.,* Vol. *189* (1990): pp. 225-266.

[13] Paladin, G. and Vulpiani, A. "Anomalous scaling laws in multifractal objects." *Phys. Rep.,* Vol. *156* (1987): pp. 147-225.

[14] Schroeder, M. *Fractals, Chaos, Power Laws*, W.H. Freeman, New York, 1991

[15] Stanley, H.E. and Ostrowsky, N. edts. *Random Fluctuations and Pattern Growth : Experiments and Models*, Kluwer, Dordrecht, 1988. Cargèse proceedings 1988.

Additional Reference: Feder, Jens. *Fractals*, Plenum, New York, 1988.