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On Polynomials Related to Powers and Derivatives of the Generating Function of Catalan's Numbers

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Abstract

Arbitrary powers of the generating function $c(x)$ of *Catalan*'s numbers are written as $c^n(x) := -(\frac{1}{\sqrt{x}})^n\ S_{n-2}(\frac{1}{\sqrt{x}})\ +\ (\frac{1}{\sqrt{x}})^{n-1}\ S_{n-1}(\frac{1}{\sqrt{x}})\ c(x)$, with *Chebyshev*'s polynomials of the second kind $S_n(y) = U_n(y/2)$ which are also defined for real (or complex) n. This formula leads to four sets of identities involving *Catalan* numbers.

The nth derivative of this generating function $c(x)$ is expressed as $\frac{1}{n!}\frac{d^n c(x)}{dx^n}\ =\ \big(a_{n-1}(x)\ +\ b_n(x)\ c(x)\big)/(x(1-4x))^n$, with certain polynomial systems $\{a_n\}$ and $\{b_n\}$ which are given explicitly. The coefficients of the $\{a_n\}$ polynomials furnish a triangle of numbers $A(n,k)$ which generalizes *Catalan*'s numbers. It is related to a convolution of the *Catalan* sequence with $2k$-fold convolutions of the central binomial coefficient sequence. Also, an associated rectangular array $\hat{A}(n,k)$ of numbers is defined. The triangle of numbers of the $\{b_n\}$ coefficients is related to the $(2k+1)$-fold convolution of the central binomial number sequence. This formula for the derivatives of $c(x)$ implies identities involving *Catalan*'s numbers as well as central binomial coefficients.

1 Introduction and Summary

Catalan's sequence of numbers $\{C_n\}_0^\infty\ =\ \{1,1,2,5,14,42,...\}$ (nr.1459 and $A000108$ of [10]) emerges in the solution of many combinatorial problems (see [1],[2],[3],[11] (also for further references). It also shows up in the asymptotic moments of zeros of scaled *Laguerre* and *Hermite* polynomials [6]. The ordinary generating function $c(x) = \sum_{n=0}^{\infty} C_n\ x^n$ is the solution of the quadratic equation $x\ c^2(x) - c(x) + 1\ =\ 0$ with $c(0) = 1$. Therefore, every positive integer power of $c(x)$ can be written as

$$c^n(x)\ =\ p_{n-1}(x)1\ +\ q_{n-1}(x)\ c(x)\ \ , \tag{1}$$

with certain polynomials p_{n-1} and q_{n-1}, both of degree $(n-1)$, in $1/x$. In *section* 2 they are shown to be related to Chebyshev's polynomials of the second kind:

$$p_{n-1}(x)\ =\ -(\frac{1}{\sqrt{x}})^n\ S_{n-2}(\frac{1}{\sqrt{x}})\ \ ,\ \ q_{n-1}(x)\ =\ (\frac{1}{\sqrt{x}})^{n-1}\ S_{n-1}(\frac{1}{\sqrt{x}})\ =\ -x\ p_n(x)\ , \tag{2}$$

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with $S_n(y) = U_n(y/2)$. It is therefore possible to extend the range of the power n to integers (or to real or complex numbers). Because powers of a generating function correspond to convolutions of the generated number sequence the given decomposition of $c^n(x)$ will determine convolutions of the *Catalan* sequence. In passing, an explicit expression for general convolutions in the form of nested sums will also be given. Contact with the works of refs. [4],[9],[12], [3] will be made.

Together with the known (*e.g.* [2],[8]) result (valid for real n)

$$c^n(x) \; = \; \sum_{k=0}^{\infty} C_k(n) \; x^k \quad \text{,with } C_k(n) \; := \; \frac{n}{n+2k} \binom{n+2k}{k} \; = \; \frac{n}{k+n} \binom{n-1+2k}{k} \; , \tag{3}$$

one finds from the alternative expression (1) for positive n two sets of identities:

$$(P1) \qquad \sum_{l=0}^{p} (-1)^l \binom{n-1-p+l}{p-l} \; C_l \; = \; \binom{n-2-p}{p} \; , \tag{4}$$

for $n \in \{2,3,...\}$, $p \in \{0,1,2,...\lfloor \frac{n}{2} \rfloor - 1\}$, and

$$(P2) \qquad \sum_{l=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^l \binom{n-1-l}{l} \; C_{k+n-1-l} \; = \; C_k(n) \; , \tag{5}$$

for $n \in \mathbf{N}$, $k \in \mathbf{N}_0$.

For negative powers in (1) two other sets of identities result:

$$(P3) \qquad \sum_{l=0}^{min(\lfloor \frac{n-1}{2} \rfloor, k-1)} (-1)^l \binom{n-1-l}{l} \; C_{k-1-l} \; = \; (-1)^{k+1} \binom{n-k-1}{k-1} \; , \tag{6}$$

for $n \in \mathbf{N}$, $k \in \{0,1,2,...\lfloor \frac{n}{2} \rfloor\}$, and

$$(P4) \qquad \sum_{l=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^l \binom{n-1-l}{l} \; C_{k-1-l} \; = \; -C_k(-n) \; = \; \frac{n}{k} \binom{2k-n-1}{k-1}, \tag{7}$$

for $n \in \mathbf{N}$, $k \in \mathbf{N}$ with $k \geq \lfloor \frac{n}{2} \rfloor + 1$. [2]

Another expression for the coefficients of negative powers of $c(x)$ is

$$C_k(-n) \; = \; \sum_{l=1}^{min(n,k)} (-1)^l \binom{n}{l} \; C_{k-l}(n) \; , \tag{8}$$

for $n, k \in \mathbf{N}$, and $C_0(-n) = 1$, $C_n(0) = \delta_{n,0}$. Also, from (3) $C_k(-n) = -C_{k-n}(n)$ for $n, k \in \mathbf{N}$ with $k \geq n$.

Section 3 deals with the derivatives of $c(x)$ where the following basic equation is used.

$$\frac{d \; c(x)}{dx} \equiv c'(x) \; = \; \frac{1}{x(1-4x)} \Big(1 + (-1+2x) \; c(x) \Big) \; . \tag{9}$$

[2] These identities can be continued for appropriate values of real n.

This eq. is equivalent to the simple recurrence relation valid for C_n: [3]

$$(n+2)\ C_{n+1}\ -\ 2(2n+1)\ C_n\ =\ 0\ \ ,\ \ n=-1,0,1,...,\ \text{with}\ \ C_{-1}=-1/2\ \ . \tag{10}$$

The result for the n-th derivative is of the form

$$\frac{1}{n!}\frac{d^n c(x)}{dx^n}\ =\ \frac{1}{(x(1-4x))^n}\Big(a_{n-1}(x)\ +b_n(x)\ c(x)\Big), \tag{11}$$

with certain polynomials a_{n-1} of degree $n-1$ and b_n of degree n. These polynomials are found to be $b_n(x)\ =\ \sum_{m=0}^{n}(-1)^m B(n,m)\ x^{n-m}$ with

$$B(n,m)\ :=\ \binom{2n}{n}\binom{n}{m}/\binom{2m}{m}\ , \tag{12}$$

which defines a triangle of numbers for $n,m\in\mathbf{N}$, $n\geq m\geq 0$. Its head is depicted in *TAB. 1* with $B(n,m)=0$ for $n<m$. Another representation for these b_n polynomials is also found, *viz*

$$b_n(x)\ =\ -2\sum_{k=0}^{n} C_{k-1}\ x^k\ (4x-1)^{n-k}\ . \tag{13}$$

Equating both forms of $b_n(x)$ leads to a formula involving convolutions of Catalan numbers with powers of an arbitrary constant $\lambda:=(4x-1)/x$. This formula is given in *section 3* as eq.(71).

The other family of polynomials is $a_n(x)=\sum_{k=0}^{n}(-1)^k\ A(n+1,k+1)\ x^{n-k}$ with the triangular array $A(n,m)$ defined for $m=0$ by $A(n,0)=C_n$, and for $n\in\mathbf{N},m\in\mathbf{N}$ with $n\geq m>0$ by the numbers

$$A(n,m)\ =\ \frac{1}{2}\binom{n}{m-1}\Big[4^{n-m+1}\ -\ \binom{2n}{n}/\binom{2(m-1)}{m-1}\Big]\ . \tag{14}$$

The head of this triangular array of numbers is shown in *TAB.2* with $A(n,m)=0$ for $n<m$. These results are solutions to recurrence relations which hold for $b_n(x)$ and $a_n(x)$ and their respective coefficients $B(n,m)$ and $A(n,m)$.

The triangle of numbers $A(n,m)$ is related to a rectangular array of numbers $\hat{A}(n,m)$, with $\hat{A}(0,0)=1$, $\hat{A}(n,0)=-C_n$ for $n\in\mathbf{N}$, and for $m\in\mathbf{N},n\in\mathbf{N}_0$ by

$$A(n,m)\ =\ -\hat{A}(n-m,m)\ +\ 2^{2(n-m)+1}\binom{n-1}{m-1}\ , \tag{15}$$

or with (14), for $m\in\mathbf{N},n\in\mathbf{N}_0$, by

$$\hat{A}(n,m)\ =\ \frac{1}{2}\binom{n+m}{n+1}\Big[\binom{2(n+m)}{n+m}/\binom{2(m-1)}{m-1}\ -\ 4^{n+1}\ \frac{m-1}{n+m}\Big]. \tag{16}$$

Part of the array $\hat{A}(n,m)$ is shown in *TAB. 3*, where it is called $C4(n,m)$.

It turns out that the mth column of the number triangle $A(n,m)$ is for $m=0,1,...$ determined by the generating function $c(x)(\frac{x}{1-4x})^m$. The mth column of the number triangle $B(n,m)$ is, for $m=0,1,...,$ generated by $\frac{1}{\sqrt{1-4x}}\ (\frac{x}{1-4x})^m$.

[3] Eq.(9) can, of course, also be found from the explicit form $c(x)\ =\ (1-\sqrt{1-4x}\ )/(2x)$.

Because differentiation of $c(x) = \sum_{k=0}^{\infty} C_k \, x^k$ leads to

$$\frac{1}{n!} \frac{d^n c(x)}{dx^n} = \sum_{k=0}^{\infty} C(n,k) \, x^k \ , \text{ with } C(n,k) := \frac{1}{n!} \prod_{j=1}^{n} (k+j) \, C_{n+k} = \frac{(2(n+k))!}{n!k!(n+k+1)!} \ , \tag{17}$$

with $C(0,k) = C_k$, one finds, together with (11), the following identities, for $n \in \mathbf{N}$, $p \in \{0, 1, 2, ..., n-1\}$

$$(D1): \sum_{k=0}^{p} (-1)^k \, C_k \binom{n}{p-k} / \binom{2(n-p+k)}{n-p+k} = \frac{1}{2} \binom{n}{p+1} \{ 2^{2(p+1)} / \binom{2n}{n} - 1 / \binom{2(n-p-1)}{n-p-1} \}$$

$$= A(n, n-p) / \binom{2n}{n} \ , \tag{18}$$

and, for $n \in \mathbf{N}, k \in \mathbf{N}_0$,

$$(D2): \sum_{j=0}^{n} (-1)^j \, (\binom{n}{j} / \binom{2j}{j}) \sum_{l=0}^{k} 4^l \binom{n+l-1}{n-1} C_{k+j-l} = C(n,k) / \binom{2n}{n} \ . \tag{19}$$

The remainder of this paper provides proofs for the above given statements. *Section 2* deals with integer (and real) powers of the generating function $c(x)$. Convolutions of general sequences are expressed there in terms of nested sums. In *Section 3* derivatives of $c(x)$ are treated.

2 Powers

The equation $x \, c^2(x) - c(x) + 1 = 0$ whose solution defines the generating function of *Catalan*'s numbers if $c(0) = 1$ can be considered as characteristic equation for the recursion relation

$$x \, r_{n+1} - r_n + r_{n-1} = 0 \ , \ n = 0, 1, ... \ , \tag{20}$$

with arbitrary inputs $r_{-1}(x)$ and $r_0(x)$. A basis of two linearly independent solutions is given by the *Lucas*-type polynomials $\{\mathcal{U}_n\}$ and $\{\mathcal{V}_n\}$, with standard inputs $\mathcal{U}_{-1} = 0$, $\mathcal{U}_0 = 1$, ($\mathcal{U}_{-2} = -x$), and $\mathcal{V}_{-1} = 1$, $\mathcal{V}_0 = 2$, ($\mathcal{V}_1 = 1/x$), in the *Binet* form

$$\mathcal{U}_{n-1}(x) = \frac{c_+^n(x) - c_-^n(x)}{c_+(x) - c_-(x)} \ , \tag{21}$$

$$\mathcal{V}_n(x) = c_+^n(x) + c_-^n(x) = \frac{1}{x} (\mathcal{U}_{n-1}(x) - 2 \, \mathcal{U}_{n-2}(x)) \ , \tag{22}$$

with the two solutions of the characteristic equation, *viz* $c_{\pm}(x) := (1 \pm \sqrt{1-4x}\ )/(2x)$. $c(x) := c_-(x)$ satisfies $c(0) = 1$, and $c_+(x) = 1/(xc(x))$, as well as $c_+(x) + c(x) = 1/x$. From the recurrence (20) it is clear that for positive $n \neq 0$ $\mathcal{U}_n$ is a polynomial in $1/x$ of degree $n-1$. If $c_+(x) - c_-(x) = 0$, *i.e.* $x = 1/4$, eq.(21) is replaced by $\mathcal{U}_n(1/4) = 2^n(n+1)$. The second eq. in (22) holds because both sides of the eq. satisfy recurrence (20) and the inputs for $\mathcal{V}_0$ and $\mathcal{V}_1$ match. One may associate with the recurrence relation (20) a transfer matrix

$$\mathbf{C}(x) = \begin{pmatrix} 1/x & -1/x \\ 1 & 0 \end{pmatrix} \ , \ Det \, \mathbf{C}(x) = 1/x \ . \tag{23}$$

With this matrix one can rewrite (20) as

$$\begin{pmatrix} r_n \\ r_{n-1} \end{pmatrix} = \mathbf{C}(x) \begin{pmatrix} r_{n-1} \\ r_{n-2} \end{pmatrix} = \mathbf{C}^n(x) \begin{pmatrix} r_0(x) \\ r_{-1}(x) \end{pmatrix} \tag{24}$$

Because $\mathbf{C}^n = \mathbf{C}\,\mathbf{C}^{n-1}$ with input $\mathbf{C}^1 = \mathbf{C}(\mathbf{x})$ given by (23), one finds from the recurrence relation (20) with $r_n = \mathcal{U}_n$

$$\mathbf{C}^n(x) = \begin{pmatrix} \mathcal{U}_n(x) & -\frac{1}{x}\,\mathcal{U}_{n-1}(x) \\ \mathcal{U}_{n-1}(x) & -\frac{1}{x}\,\mathcal{U}_{n-2}(x) \end{pmatrix} . \tag{25}$$

Note that for $x = 1$ one has $c_{\pm}(1) = (1 \pm i\sqrt{3})/2$, which are 6th roots of unity, and the related period 6 sequences are $\{\mathcal{U}_n(1)\}_{-1}^{\infty} = \{\overline{0, 1, 1, 0, -1, -1,}\}$, as well as $\{\mathcal{V}_n(1)\}_0^{\infty} = \{\overline{2, 1, -1, -2, -1, 1,}\}$. This follows from eqs. (21) and (22). It is convenient to map the recursion relation (20) to the familiar one for *Chebyshev's* $S_n(x) = U_n(x/2)$ polynomials of the second kind, *viz*

$$S_n(x) = x\,S_{n-1}(x) - S_{n-2}(x) \ , \quad S_{-1} = 0,\ S_0 = 1 \ , \tag{26}$$

with characteristic equation $\lambda^2 - x\lambda + 1 = 0$ and solutions $\lambda_{\pm}(x) = \frac{x}{2}(1 \pm \sqrt{1 - (2/x)^2}\ )$, satisfying $\lambda_+(x)\ \lambda_-(x) = 1$ and $\lambda_+(x) + \lambda_-(x) = x$. The relation to $c_{\pm}(x)$ is

$$\sqrt{x}\ c_{\pm}(x) = \lambda_{\pm}(1/\sqrt{x}) \ . \tag{27}$$

The *Binet* form of the corresponding two independent polynomial systems is

$$S_{n-1}(x) = \frac{\lambda_+^n(x) - \lambda_-^n(x)}{\lambda_+(x) - \lambda_-(x)} \ , \tag{28}$$

$$2\,T_n(x/2) = \lambda_+^n(x) + \lambda_-^n(x) \ , \tag{29}$$

and $T_n(x/2) = (S_n(x) - S_{n-2}(x))/2$ are *Chebyshev*'s polynomials of the first kind.

The extension to negative integer indices runs as follows

$$\mathcal{U}_{-n}(x) = -x^{n-1}\,\mathcal{U}_{n-2}(x) \ , \tag{30}$$

$$S_{-(n+2)}(x) = -S_n(x) \ . \tag{31}$$

This follows from (21) and (28). Note that from (20) $\mathcal{U}_n$ is for positive n a monic polynomial in $1/x$ of degree n, and for negative n in general a non-monic polynomial in x of degree $\lfloor -\frac{n}{2} \rfloor$. It is possible to extend the range of n to complex numbers using the *Binet* forms.

Connection between both systems of polynomials is made, after using (21), (27) and (28),by

$$\mathcal{U}_n(x) = \left(\frac{1}{\sqrt{x}}\right)^n S_n(1/\sqrt{x}) \ . \tag{32}$$

This holds for $n \in \mathbf{Z}$, in accordance with (30) and (31).

After these preliminaries we are ready to state:

Proposition 1: The nth power of $c(x)$, the generating function of *Catalan*'s numbers, can, for $n \in \mathbf{Z}$, be written as

$$c^n(x) = -\frac{1}{x}\,\mathcal{U}_{n-2}(x) + \mathcal{U}_{n-1}(x)\ c(x) \ , \tag{33}$$

$$= -(\frac{1}{\sqrt{x}})^n\ S_{n-2}(1/\sqrt{x}) + (\frac{1}{\sqrt{x}})^{n-1}\ S_{n-1}(1/\sqrt{x})\ c(x). \tag{34}$$

Proof: Due to $c^2(x) = (c(x)-1)/x$ and $c^{-1}(x) = 1 - x\ c(x)$ one can, for $n \in \mathbf{Z}$, write $c^n(x) = p_{n-1}(x) + q_{n-1}(x)\ c(x)$. From $c^n(x) = c(x)\ c^{n-1}(x)$ one is led to $q_{n-1} = p_{n-2} + \frac{1}{x}\ q_{n-2}$ and $p_{n-1} = -\frac{1}{x}\ q_{n-2}$, or $q_{n-1} = (q_{n-2} - q_{n-3})/x$ with input $q_{-1} = 0$, $q_0 = 1$. Therefore, $q_{n-1}(x) = \mathcal{U}_{n-1}(x)$ and $p_{n-1}(x) = -\ \mathcal{U}_{n-2}(x)/x$. (34) then follows from (32). [4] $\square$

Note 1: An alternative proof of *proposition 1* can be given starting with eqs.(28) and (29) which show, together with $\lambda_+(x) - \lambda_-(x) = \sqrt{x^2-4}$, that

$$\lambda_\pm^n(x) \ = \ T_n(x/2) \ \pm \ \sqrt{(x/2)^2-1}\ S_{n-1}(x)\ , \tag{35}$$

or, from $\pm\sqrt{(x/2)^2-1} \ = \ \lambda_\pm(x) - x/2$ and the S_n recurrence relation (26)

$$\begin{aligned} \lambda_\pm^n(x) \ &= \ T_n(x/2) \ - \ \tfrac{1}{2}\,(S_n(x) + S_{n-2}(x)) \ + \ S_{n-1}(x)\ \lambda_\pm(x) && (36)\\ &= \ -S_{n-2}(x) \ + \ S_{n-1}(x)\ \lambda_\pm(x)\ . && (37)\end{aligned}$$

Now (34) follows from (27). This also proves that one may replace in *proposition 1* $c(x)$ by $c_+(x) = 1/(xc(x))$ from which one recovers the c^{-n} formula for $n \in \mathbf{N}$ in accordance with (30) and (31).

Note 2: For the transfer matrix $\mathbf{C(x)}$, defined in (25), one can prove for $n \in \mathbf{N}$ in an analogous manner

$$\mathbf{C}^n \ = \ -(\frac{1}{\sqrt{x}})^n\ S_{n-2}(1/\sqrt{x})\ \mathbf{1} \ + \ (\frac{1}{\sqrt{x}})^{n-1}\ S_{n-1}(1/\sqrt{x})\ \mathbf{C}(x), \tag{38}$$

by employing the *Cayley-Hamilton* theorem for the 2×2 matrix $\mathbf{C}$ with $tr\ \mathbf{C} \ = \ \frac{1}{x} \ = \ det\ \mathbf{C}$ which states that $\mathbf{C}$ satisfies the characteristic equation $\mathbf{C}^2 \ - \ \frac{1}{x}\ \mathbf{C} \ + \ \frac{1}{x}\ \mathbf{1} \ = \ 0$.

Powers of a function which generates a sequence generate convolutions of this sequence. Therefore, *proposition 1* implies that convolutions of the *Catalan* sequence can be expressed in terms of *Catalan* numbers and binomial coefficients. Before giving this result we shall present an explicit formula for the nth convolution of a general sequence $\{C_n\}$ generated by $c(x) \ = \ \sum_{l=0}^{\infty}\ C_l\ x^l$. Usually the convolution coefficients $C_l(n)$, defined by $c^n(x) = \sum_{l=0}^{\infty}\ C_l(n)\ x^l$, are written as

$$C_l(n) \ = \ \sum_{\sum_{j=1}^{n} i_j \ = \ l} \ C_{i_1}\ C_{i_2}\cdots C_{i_n} \quad , \quad \text{with } i_j \in \mathbf{N}_0\ . \tag{39}$$

An explicit formula with $(l-1)$ nested sums is the content of the next lemma.

Lemma 1: General convolutions

For $l = 2, 3, \ldots$

$$C_l(n) \ = \ C_0^{n-l}\ C_1^l \Big(\prod_{k=2}^{l}\ \sum_{i_k=a_k}^{\lfloor b_k\rfloor}\Big) \ < n, l, \{i_j\}_2^l > \ \prod_{j=2}^{l}\Big((\frac{C_j\ C_0}{C_1^j})^{i_j}\ \frac{1}{i_j!}\Big)\ , \tag{40}$$

with

$$b_2 = l/2 \quad , \quad b_k \ = \ (l - \sum_{j=2}^{k-1}\ j\ i_j)/k\ , \tag{41}$$

[4]Because $S_n(y) = \sum_{j=0}^{\lfloor n/2\rfloor}(-1)^j\binom{n-j}{j}y^{n-2j}$ the explicit form of these polynomials (2) is $p_{n-1}(x) = \sum_{j=0}^{\lfloor n/2\rfloor -1}(-1)^{j+1}\binom{n-2-j}{j}x^{-(n-1-j)}$, $p_{-1} = 1$, $p_0 = 0$, and $q_{n-1}(x) = \sum_{j=0}^{\lfloor (n-1)/2\rfloor}(-1)^j\binom{n-1-j}{j}x^{-(n-1-j)}$, $q_{-1} = 0.$ For negative index one has, due to (31), $p_{-(n+1)}(x) = (\sqrt{x})^n\ S_n(1/\sqrt{x}) = \sum_{j=0}^{\lfloor n/2\rfloor}(-1)^j\ \binom{n-j}{j}\ x^j$, and $q_{-(n+1)}(x) = -(\sqrt{x})^{n+1}\ S_{n-1}(1/\sqrt{x}) = -x\ \sum_{j=0}^{\lfloor (n-1)/2\rfloor}(-1)^j\ \binom{n-1-j}{j}\ x^j$.

$$a_k = 0 \quad , \quad \text{for } k = 2, 3, ..., l-1; \quad a_l \;=\; max\Big(0, \lceil \frac{l - n - \sum_{j=2}^{l-1}(j-1)\; i_j)}{l-1} \rceil\Big) \tag{42}$$

$$< n, l, \{i_j\}_2^l > \;=\; \frac{n!}{(n - l + \sum_{j=2}^{l}(j-1)\; i_j)!(l - \sum_{j=2}^{l} j\; i_j)!} . \tag{43}$$

The first product in (40) is understood to be ordered such that the sums have indices $i_2, i_3, ...i_l$ when written from the left to the right. In addition: $C_0(n) = C_0^n$ and $C_1(n) = n\; C_0^{n-1}\; C_1$.

Proof: $C_l(n)$ of (39) is rewritten first as

$$C_l(n) \;=\; \sum\; (n, l, \{i_j\}_0^l)\; C_0^{i_0}\; C_1^{i_1} \cdots C_l^{i_l} \quad , \quad i_j \in \mathbf{N}_0 \quad , \tag{44}$$

where the sum is restricted by

$$(i): \qquad \sum_{j=0}^{l} j\; i_j \;=\; l \qquad \text{and} \qquad (ii): \qquad \sum_{j=0}^{l} i_j \;=\; n\; . \tag{45}$$

$(n, l, \{i_j\}_0^l)$ is a combinatorial factor to be determined later on. (*E.g.* for $n = 3, l = 5$ one has 4 terms in the sum: $i_5 = 1, i_0 = 2$; $i_4 = 1, i_1 = 1, i_0 = 1$; $i_3 = 1, i_2 = 1, i_0 = 1$; $i_3 = 1, i_2 = 2$, with other indices vanishing, and the combinatorial factors are $3, 6, 6, 3$, respectively.) (ii) restricts the sum to terms with n factors, and (i) produces the correct weight l. These restrictions are solved by $(i') : \quad i_1 \;=\; l - \sum_{j=2}^{l} j\; i_j$ and $(ii') : \quad i_0 \;=\; n - i_1 - \sum_{j=2}^{l} i_j \;=\; n - l + \sum_{j=2}^{l}(j-1)\; i_j$. From $i_1 \geq 0$,*i.e.* $l - \sum_{j=2}^{l} j\; i_j \;\geq 0$, one infers $i_2 \leq \lfloor \frac{l}{2} \rfloor$, thus $i_2 \in [0, \lfloor \frac{l}{2} \rfloor\;]$. For given i_2 in this range $i_3 \;\leq \lfloor \frac{l - 2 i_2}{3} \rfloor$, *etc.*, in general $0 \leq i_k \leq \lfloor (l - \sum_{j=2}^{k-1}\; j\; i_j)/k \rfloor$ for $k = 2, 3, ..., l$ with the sum replaced by zero for $k = 2$. This accounts for the upper boundaries $\lfloor b_k \rfloor$ in (41). Now, because $i_0 \geq 0$ (ii') implies a lower bound for i_l, the index of the last sum, *viz* $i_l \geq \lceil (l - n - \sum_{j=2}^{l-1}\; (j-1)\; i_j)/(l-1) \rceil$ with the ceiling function $\lceil \cdot \rceil$. In any case $i_l \geq 0$, therefore, the lower boundary for the i_l-sum is a_l as given in (42). All restrictions have then be solved and the lower boundaries of the other sums are given by $a_k = 0$, for $k = i_2, ..., i_{l-1}$. As to the combinatorial factor, it now depends only on $n, l, \{i_j\}_2^l$ and is written as $< n, l, \{i_j\}_2^l >$. It counts the number of possibilities for the occurence of the considered term of the sum which is given by $\binom{n}{i_0}\binom{n-i_0}{i_1}\cdots\binom{n-\sum_{j=2}^{l-1} i_j}{i_l} \;=\; n!/(\prod_{j=0}^{l}\; i_j!\; (n - \sum_{j=0}^{l}\; i_j)!$. Inserting i_0 and i_1 from (ii') and (i'), respectively, remembering (ii), produces $< n, l, \{i_j\}_2^l >$ as given in (43). Finally, $\sum < n, l, \{i_j\}_2^l > \; C_0^{i_0}\; C_1^{i_1} \cdots C_l^{i_l}$ is transformed into $(l-1)$ nested sums with boundaries a_k and $\lfloor b_k \rfloor$ after replacement of i_1 and i_0. This completes the proof of (40) for the non-trivial $l \geq 2$ cases. $\square$

Corollary 1: *Catalan* convolutions

For *Catalan*'s sequence $\{C_n\}_0^\infty$ the n-th convolution sequence is for $n \in \mathbf{N}$ given by $C_0(n) = 1$, $C_1(n) = n$ and, for $l = 2, 3, ..$, by

$$C_l(n) \;=\; \Big(\prod_{k=2}^{l} \sum_{i_k = a_k}^{\lfloor b_k \rfloor}\Big) \;< n, l, \{i_j\}_2^l >\; \prod_{j=2}^{l}\Big(\frac{C_j^{i_j}}{i_j!}\Big) , \tag{46}$$

with (41), (42) and (43).

Proof: This is *lemma 1* with $C_0 = 1 = C_1$. $\square$

Example 1: $C_4(3) \;=\; 3C_4 \;+\; 6C_3 \;+\; 3C_2^2 \;+\; 3C_2 \;=\; 90$.

Corollary 2: With the *Catalan* generating function $c(x)$ and the definition $c^{-n}(x) =: \sum_{l=0}^{\infty}\; C_l(-n)\; x^l$, for $n \in \mathbf{N}$, one has for $l = 2, 3, \ldots$

$$C_l(-n) \;=\; (-1)^l \Big(\prod_{k=2}^{l} \sum_{i_k=a_k}^{\lfloor b_k \rfloor} \frac{(-1)^{(k-1)i_k}}{i_k!} \Big) \; < n, l, \{i_j\}_2^l > \; \prod_{j=2}^{l-1} C_j^{i_{j+1}} \;, \tag{47}$$

with (41), (42), (43) and *Catalan*'s numbers C_k. In addition: $C_0(-n) = 1 \;, C_1(-n) = -n$.

Proof: *Lemma 1* is used for powers of $c(x)$ replaced by those of $c^{-1}(x) = 1 - x\; c(x)$, with the *Catalan* generating function $c(x)$. Hence $c^{-1}(x) = \sum_{k=0}^{\infty} \; C_k(-1)\; x^k$ with

$C_k(-1) = \begin{cases} 1 & \text{for } k = 0 \\ -C_{k-1} & \text{for } k = 1, 2, \ldots \end{cases}$. Then in *lemma 1* C_k is replaced by $C_k(-1)$. □

Example 2: $C_4(-3) \;=\; -3C_3 \;+\; 6C_2 \;-\; 3 \;+\; 3 \;=\; -3$.

Convolutions of *Catalan*'s sequence have been encountered in various contexts. For example, in the enumeration of non-intersecting path pairs on a square lattice [9], [12], [3], and in the problem of inverting triangular matrices with *Pascal* triangle entries [4] (and earlier works cited there). [5]

Lemma 2: Explicit form of *Catalan* convolutions [9],[12], [4],[2],[8],[3]

For $n \in \mathbf{R}$, $l \in \mathbf{N}_0$:

$$C_l(n) \;=\; \frac{n}{l} \binom{2l+n-1}{l-1} \;=\; \frac{n}{n+2l} \binom{n+2l}{l} \;=\; \frac{n}{l+n} \binom{2l+n-1}{l} \;. \tag{48}$$

Proof: Three equivalent expressions have been given for convenience. See [2], p. 201, eq.(5.60), with $\mathcal{B}_2(z) \;=\; c(z)$, $t \to 2, k \to l, r \to n$. The proof of this eq.(5.60) appears as (7.69) on p.349, with $m = 2, n = l \in \mathbf{R}$.

The same formula occurs as exercise nr. 213 in Vol.1 of [8] for $\beta = 2$ as a special instant of exercises nrs. 211, 212. Put $\alpha = n$ and $n = l$ in the solution of exercise nr. 213 on p. 301.

In order to prove this lemma from [9] or [12] one can use $C_l(n) \;=\; \sum_{j=0}^{\min(l,n)} \binom{n}{j}\; \hat{C}_l(j)$ obtained from $c(x) =: 1 + \hat{c}(x)$ with $\hat{c}^n(x) =: \sum_{k=n}^{\infty} \hat{C}_k(n)\; x^{k-n}$. The result in [9] and [12] is, with this notation, $\hat{C}_l(j) \;=\; B_{l,j} \;=\; b(l,j) \;=\; \frac{j}{l}\binom{2l}{l-j}$. Inserting this in the given sum, making use of the identity $j\,\binom{n}{j} \;=\; n\binom{n-1}{j-1}$ and the *Vandermonde* convolution identity, leads to *lemma 2* at least for positive integer n but one can continue this formula to real (or complex) n.

In [4] one finds this result as eq.(3.1), p.402, for $i = 1$: $s_1(l,n) = C_l(n)$.

In [3] ${}_2d_{2-n,l+1} \;=\; C_l(n)$ with the result given in theorem 2.3, eq. (2.6), p.71. □

[5] *Shapiro*'s *Catalan* triangle has entries $B_{n,k} \;=\; \frac{k}{n}\binom{2n}{n-k}$ for $n \geq k \geq 1$, and $B_{n,k} \;=\; [x^n]\big(x^k\; \hat{c}^k(x)\big)$, with $[x^n]f(x)$ denoting the coefficient of x^n in the expansion of $f(x)$ around $x = 0$. Here $\hat{c}(x) = (c(x)-1)/x \;=\; c^2(x)$. (See [9], propositions (2.1) and (3.3) with $i_j \in \mathbf{N}$, *not* $\mathbf{N}_0$.) In [12] this triangle of numbers from [9] reappears as $b(n,k)$ and it is shown there that $B_{n,k} \equiv b(n,k) \;=\; [x^n](x\; c^2(x))^k$, in accordance with the identity $\hat{c}(x) \;=\; c^2(x)$. Therefore, only even powers of $c(x)$ appear in *Shapiro*'s *Catalan* triangle. In [3] $C_l(n)$ appears as special case ${}_2d_{2-n,l+1}$. In [4] all powers of $c(x)$ show up as convolutions for the special case of the S_1 sequence there. The entries of the S_1-array, p. 397, are $[x^n]c^{k+1}(x)$ for $n, k \in \mathbf{N}_0$.

We now compute the coefficients $C_l(n) = [x^l]c^n(x)$ (see footnote 5 for this notation) from our formula given in *proposition 1.* This can be done for $n \in \mathbf{Z}$.

First consider $n \in \mathbf{N}_0$. For $n = 0$ and $n = 1$ there is nothing new due to the inputs $S_{-2} = -1$, $S_{-1} = 0$ and $S_0 = 1$. $C_l(n) = 0$ for negative integer l. Therefore, terms proportional to $1/x^l$ with $l \in \mathbf{N}$ have to cancel in (34). For $n = 2, 3, ...$ terms of the type $1/x^{n-j}$ occur for $j \in \{1, 2, ..., \lfloor n/2 \rfloor\}$. The coefficient of $1/x^{n-j}$ in $p_{n-1}(x)$ is $(-1)^j \binom{n-1-j}{j-1}$ (see footnote 4 for the explicit form of p_{n-1}). For the $1/x^{n-j}$ coefficient in $q_{n-1}(x)\ c(x)$ one finds the convolution $\sum_{l=0}^{j-1} (-1)^{j-l-1} \binom{n-(j-l)}{j-l-1} C_l$. Compensation of both coefficients leads to identity $(P1)$ given in (4), after $(j-1)$ has been traded for p. Thus:

Proposition 2: Identity $(P1)$

For $n = 2, 3, ...$ and $p = 0, 1, ..., \lfloor \frac{n}{2} \rfloor - 1$ identity $(P1)$, given in eq.(4) holds.

Example 3: $n = 2k,\ p = k - 1$, and $n = 2k + 1,\ p = k - 1$ for $k \in \mathbf{N}$

$\sum_{l=0}^{k-1} (-1)^l \binom{k+l}{2l+1} C_l = 1$, $\sum_{l=0}^{k-1} (-1)^l \binom{k+l+1}{2(l+1)} C_l = k$.

For $n = 2, 3, ...$ terms in (1), or (31), proportional to x^k with $k \in \mathbf{N}_0$ arise only from $q_{n-1}(x)\ c(x)$, and they are given by the convolution (cf. footnote 4) $\sum_{l=0}^{\lfloor (n-1)/2 \rfloor} (-1)^l \binom{n-1-l}{l} C_{k+n-1-l}$. For $n = 1$ this is C_k. The l.h.s. of (1) contributes $C_k(n)$, and $C_k(1) = C_k$. Therefore:

Proposition 3: Identity $(P3)$

For $n \in \mathbf{N}$, $k \in \mathbf{N}_0$ identity $P(2)$, given in eq.(5) with (3) holds.

Example 4: $k = 0$, $(n-1) \to n$: $\sum_{l=1}^{\lfloor n/2 \rfloor} (-1)^{l+1} \binom{n-l}{l} C_{n-l} = C_n - 1$

Now consider negative powers in (1), *i.e.* $c^{-n}(x)$, $n \in \mathbf{N}$. No negative powers of x appear (*cf.* footnote 4 for the explicit form of $p_{-(n+1)}(x)$ and $q_{-(n+1)}(x)$). The coefficient of x^k, $k \in \mathbf{N}_0$, of the *rhs.* of (1) is $(-1)^k \binom{n-k}{k} - \sum_{l=0}^{\lfloor (n-1)/2 \rfloor} (-1)^l \binom{n-1-l}{l} C_{k-1-l}$, where the first term, arising from $p_{-(n+1)}(x)$, contributes only for $k \in \{0, 1, ..., \lfloor n/2 \rfloor\}$. The *lhs.* of (1) has $[x^k]c^{-n}(x) = C_k(-n)$. From the last eq. in (48) one finds $C_k(-n) = \frac{n}{n-k} \binom{2k-n-1}{k} = (-1)^k \frac{n}{n-k} \binom{n-k}{k}$. In the last eq. the upper index in the binomial has been negated (*cf.* [2], (5.14)). Two sets of identities follow, depending on the range of k:

Proposition 4: Identity $(P3)$

For $n \in \mathbf{N}$, $k \in \{0, 1, ..., \lfloor \frac{n}{2} \rfloor\}$ identity $(P3)$, given in eq.(6) holds.

Proposition 5: Identity $(P4)$

For $n \in \mathbf{N}$, $k \in \mathbf{N}$ with $k \geq \lfloor \frac{n}{2} \rfloor + 1$ identity $(P4)$, given in eq.(7) holds.

In $(P4)$ only the $q_{-(n+1)}(x)\ c(x)$ part of (1) contributed and we used the first expression for $C_k(-n)$ in (48). In $(P3)$, where also $p_{-(n+1)}(x)$ contributed, we used the negated binomial coefficient for $C_l(-n)$ and absorption in the resulting one.

Note that (48) implies $C_k(-n) = -C_{k-n}(n)$ for $k, n \in \mathbf{N}$, and $k \geq n$. $C_k(0) = \delta_{k,0}$.

If one uses the binomial formula for $c^{-n}(x) = (1 - x\ c(x))^n$ and $c^n(x) = \sum_{k=0}^{\infty} C_k(n)\ x^k$ one arrives at eq.(8).

We close this section by presenting some sequences of positive integers which are defined with the help of the $\mathcal{U}_n$ polynomials (21).

$$a_n(m) \ := \ \mathcal{U}_n(1/m) \ = \ (\sqrt{m}\ )^n \ S_n(\sqrt{m}\ ) \ . \tag{49}$$

The last eq. is due to (32). It will be shown that $a_n(m)$ is for each $m = 4, 5, ...$ and $n = -1, 0, ...$ a non-negative integer. Also negative integers $-m$, $m \in \mathbf{N}$ are of interest. In this case we add a sign factor.

$$b_n(m) \ := \ (-1)^n \ \mathcal{U}_n(-1/m) \ = \ (-i\sqrt{m}\ )^n \ S_n(i\sqrt{m}\ ) \ . \tag{50}$$

From the S_n recursion relation (26) one infers those for the $a_n(m)$ and $b_n(m)$ sequences.

$$a_n(m) \ = \ m \ (a_{n-1}(m) \ - \ a_{n-2}(m)) \ \ , \ \ a_{-1}(m) \equiv 0 \ \ , \ \ a_0(m) \equiv 1 \ , \tag{51}$$

$$b_n(m) \ = \ m \ (b_{n-1}(m) \ + \ b_{n-2}(m)) \ \ , \ \ b_{-1}(m) \equiv 0 \ \ , \ \ b_0(m) \equiv 1 \ . \tag{52}$$

This shows that $b_n(m)$ constitutes a non-negative integer sequences for positive integer m. It describes certain generalized *Fibonacci* sequences (see *e.g.* [5] with $b_n(m) \ = \ W_{n+1}(0,1;m,m)$). Of course, one can define in a similar manner generalized *Lucas* sequences using the polynomials $\{\mathcal{V}_n\}$ given in (22). Each $a_n(m)$ sequence (which is identified with $W_{n+1}(0,1;m,-m)$ of [5]) turns out to be composed of two simpler sequences, *viz* $a_{2k}(m) =: m^k \ \alpha_k(m)$ and $a_{2k-1} =: m^k \ \beta_k(m)$, $k \in \mathbf{N}_0$. These new sequences, which are, due to (49) and (50), given by $\alpha_k \ = \ S_{2k}(\sqrt{m}\ )$ and $\beta_k(m) \ = \ S_{2k-1}(\sqrt{m}\ )/\sqrt{m}$, satisfy therefore the following relations.

$$\beta_{k+1}(m) \ = \ (m-2) \ \beta_k(m) \ - \ \beta_{k-1}(m) \ \ , \ \ \beta_0(m) \equiv 0 \ \ , \ \ \beta_1(m) \equiv 1 \ , \tag{53}$$

and

$$\alpha_{k-1}(m) \ = \ \beta_k(m) \ + \beta_{k-1}(m) \ \ . \tag{54}$$

From (53) it is now clear that $\beta_n(m)$ is a non-negative integer sequence for $m = 4, 5, ...$ (In [5] $\beta_n(m) = W_n(0,1;m-2,-1)$.) This property is then inherited by the $\alpha_n(m)$ sequences due to (54), and then by the composed sequence $a_n(m)$. (Of course, one could also consider sequences built from negative and positive numbers, but we refrain from doing so here).

The ordinary generating functions are [6]

$$g_\beta(m;x) := \sum_{n=0}^{\infty} \ \beta_n(m) \ x^n = \frac{1}{x^2-(m-2)x+1} \ \ , \ \ g_\alpha(m;x) := \sum_{n=0}^{\infty} \ \alpha_n(m) \ x^n = \frac{1+x}{x^2-(m-2)x+1} \ , \tag{55}$$

$$g_a(m;x) \ := \sum_{n=0}^{\infty} \ a_n(m) \ x^n \ = \frac{1}{1-m \ x+m \ x^2} \ \ , \ \ g_b(m;x) \ := \sum_{n=0}^{\infty} \ b_n(m) \ x^n \ = \frac{1}{1-m \ x-m \ x^2} \ . \tag{56}$$

[6]The $\{\beta_n(m)\}$ sequences for $m = 4, 5, 6, 7, 8, 10$ appear in the book [10]. The case $m = 4$ produces the sequence of non-negative integers, $m = 5$ are the even indexed *Fibonacci* numbers. The $m = 9$ sequence appears only in *Sloane's* On-Line-Encyclopedia [10] as $A004187$. The $\{\alpha_n(m)\}$ sequences for $m = 4, 5, 6$ and 8 appear in the book [10]. $m = 4$ yields the positive odd integer sequence, $m = 5$ the odd indexed *Lucas* number sequence. The $m = 7$ sequence appears now as $A030221$ in the data bank [10]. The composed sequences $\{a_n(m)\}$ are not in the book but some of them are found in the data bank [10]. $m = 4$ is the sequence $(n+1)\ 2^n$, $A001787$, and $m = 5, 6, 7$ appear now as $A030191$, $A030192$, $A030240$, respectively. As mentioned above $\{b_{n+1}(1)\}$ is the *Fibonacci* sequence. The instances $m = 2$ and 3 appear as $A002605$ and $A030195$, respectively, in the data bank [10].

3 Derivatives

The starting point is eq.(9) which can either be verified from the explicit form of the generating function $c(x)$ (*cf.* footnote 3), or by converting the recursion relation (10) for *Catalan*'s numbers into an eq. for their generating function. A computation of $\frac{1}{(n+1)!}\,\frac{d^{n+1}c(x)}{dx^{n+1}} = \frac{1}{n+1}\,\frac{d}{dx}\big(\frac{1}{n!}\,\frac{d^n c(x)}{dx^n}\big)$ with Ansatz (11) and eq. (9) produces the following mixed relations between the quantities $a_n(x)$ and $b_n(x)$ and their first derivatives, valid for $n \in \mathbf{N}_0$,

$$(n+1)\ a_n(x) \ = \ x(1-4x)\ a'_{n-1}(x)\ +\ b_n(x)\ +\ n(8x-1)\ a_{n-1}(x)\ , \tag{57}$$

$$(n+1)\ b_{n+1}(x) \ = \ x(1-4x)\ b'_n(x)\ +\ (-(n+1)+2(1+4n)x)\ b_n(x)\ , \tag{58}$$

with inputs $a_{-1}(x) \equiv 0$ and $b_0(x) \equiv 1$.

From (58) and the input it is clear by induction that $b_n(x)$ is a polynomial in x of degree n. With this information (57) and the input show, again by induction, that the same statement holds for $a_n(x)$. Therefore we write, for $n \in \mathbf{N}_0$, [7]

$$a_n(x) \ = \ \sum_{k=0}^{n} (-1)^k\ a(n,k)\ x^{n-k}\ , \tag{59}$$

$$b_n(x) \ = \ \sum_{k=0}^{n} (-1)^k\ B(n,k)\ x^{n-k}\ , \tag{60}$$

with the triangular arrays of numbers $a(n,k)$ and $B(n,k)$ with row number n and column number $k \leq n$.

We first solve the $b_n(x)$ eq.(58) by inserting (60) and deriving the recursion relation for the coefficients $B(n,m)$ after comparing coefficients of x^{n+1}, x^0, and x^{n-k} for $k = 0, 1, ..., n-1$.

$$x^{n+1}\ :\quad (n+1)\ B(n+1,0)\ =\ 2(2n+1)\ B(n,0)\ , \tag{61}$$

$$x^{0}\ :\quad B(n+1,n+1)\ =\ B(n,n)\ , \tag{62}$$

$$x^{n-k}\ :\quad (n+1)\ B(n+1,k+1)\ =\ (k+1)\ B(n,k)\ +\ 2(2(n+k)+3)\ B(n,k+1)\ . \tag{63}$$

With the input $B(0,0) = 1$ one deduces from (61) for the leading coefficient of $b_n(x)$

$$B(n,0) \ = \ 2^n\,\frac{(2n-1)!!}{n!} \ = \ \frac{(2n)!}{n!\ n!} \ = \ \binom{2n}{n}\ , \tag{64}$$

and from (62)

$$B(n,n) \ \equiv 1 \quad , \quad \text{i.e. } \ b_n(0) = (-1)^n\ . \tag{65}$$

In order to solve (63) we inspect the $B(n,m)$ triangle of numbers *TAB.1*, and conjecture that for $n, m \in \mathbf{N}$

$$B(n,m) \ = \ 4\ B(n-1,m)\ +\ B(n-1,m-1)\ , \tag{66}$$

with input $B(n,0) = \binom{2n}{n}$ from (64).

If we use this conjecture in (63), written with $n \to n-1,\ k \to m-1$ we are led to consider the simple recursion

$$B(n,m) \ = \ \frac{n+1-m}{2(2m-1)}\ B(n,m-1)\ , \tag{67}$$

[7] The triangular array $a(n,k)$ will later be enlarged to another one which will then be called $A(n,k)$.

with input $B(n,0) = \binom{2n}{n}$ from (64).

The solution of this recursion is, for $n, m \in \mathbf{N}_0$, [8]

$$B(n,m) \;=\; \frac{1}{2^m(2m-1)!!}\frac{n!}{(n-m)!}\binom{2n}{n} \;=\; \frac{m!\;n!}{(2m)!\;(n-m)!}\binom{2n}{n} \;=\; \binom{2n}{n}\binom{n}{m}\Big/\binom{2m}{m}\;. \tag{68}$$

This result satisfies (61), *i.e.* (64), as well as (62), *i.e.* (65). It is also the solution to (63) provided we prove the conjecture (66) for $B(n,m)$ of (68). This can be done by using the form $B(n,m) = \frac{(2n)!\;m!}{(2m)!\;n!\;(n-m)!}$ and extracting this expression on the *rhs.* of (66). Then one is left to prove $1 \;=\; \frac{4}{2}\,\frac{n-m-1}{2n-1} \;+\; \frac{2m-1}{2n-1}$, which is trivial. Thus we have proved:

Proposition 6: Explicit form of $b_n(x)$

$B(n,m)$ given by eq. (68) is the solution to eqs.(61), (62), and (63). Hence $b_n(x)$, defined by (60) with $B(n,m)$ from (68), solves eq. (58) with $b_0(x) \equiv 1$.

One can derive another explicit representation for the $b_n(x)$ polynomials by converting the simple recurrence relation (67) into the following eq. for $b_n(x)$ defined by (60).

$$(1-4x)\;b_n'(x) \;+\; 2(2n-1)\;b_n(x) \;+\; 2\binom{2n}{n}\;x^n \;=\; 0\;. \tag{69}$$

Now this first order linear and inhomogeneous differential eq. for $b_n(x)$ can be solved.

Proposition 7: Alternative form for $b_n(x)$

The solution to eq.(69) with input $b_n(0) \equiv (-1)^n$ is given by eq.(13), with $C_{-1} = -1/2$ and the *Catalan* numbers C_k for $k \in \mathbf{N}_0$.

Proof: This eq. is of the standard type $y' + f(x)\;y = g(x)$ with $y \equiv b_n$, $f(x) = 2(2n-1)/(1-4x)$ and $g(x) = 2(n+1)C_n x^n/(1-4x)$. $F(x) := \int dx\;f(x) \;=\; -\frac{1}{2}(2n-1)\;ln(1-4x) + const(n)$. $y = exp(-F(x))\;\{Const(n) - 2\;(n+1)C_n I_n(x)\}$ with $I_n(x) := \int dx\;x^n/(1-4x)^{n+1/2}$ and $exp(-F(x)) = (1-4x)^{n-1/2}$. The integral $I_n(x)$ can be computed by repeated partial integration, and it is found to be

$$I_n(x) \;=\; \frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\;\frac{C_{n-k-1}}{C_n}\;x^{n-k}/(1-4x)^{n-k-1/2}\;, \tag{70}$$

where we used $C_{-1} := -1/2$, compatible with the recursion (10). This leads to the desired result for $y \equiv b_n(x)$ if the integration constant $Const(n)$ is put to zero in order to satisfy $b_n(0) = (-1)^n$ and a resummation $k \to k-n$ is performed. $\square$

Comparing this alternative form (13) for $b_n(x)$ with the one given by (60), together with (68), proves the following identity in n and $\lambda := (4x-1)/x$. The term $k = 0$ in the sum (13) has been written separately.

Corollary 3: Convolution of *Catalan* sequence and powers of λ

$$s_{n-1}(\lambda) \;:=\; \lambda^{n-1}\sum_{k=0}^{n-1}\frac{C_k}{\lambda^k} \;=\frac{1}{2}\Big(\lambda^n \;-\; \binom{2n}{n}\sum_{k=0}^{n}(-1)^k\;(4-\lambda)^k\;\binom{n}{k}\Big/\binom{2k}{k}\Big)\;, \tag{71}$$

[8] With the *Pochhammer* symbol $(a)_n := \Gamma(n+a)/\Gamma(a)$ this result can also be written as $B(n,m) = ((2m+1)/2)_{n-m}\;4^{m-n}/(n-m)!$.

for $n \in \mathbf{N}$ and $\lambda \neq \infty$. Observe that $s_n(\lambda)$ is the convolution of the *Catalan* sequence with the sequence of powers of λ. Therefore, the (ordinary) generating function for the sequence $s_n(\lambda)$ is $g(\lambda;x) := \sum_{n=0}^{\infty} s_n(\lambda)\ x^n \ = \ c(x)/(1-\lambda\ x)$. [9]

The case $\lambda = 0$ ($x = 1/4$) is also covered by this formula. It produces from $s_n(0) = C_n$ the following identity.

Example 5: Case $\lambda = 0 \quad (x = 1/4)$ [10]

$$\sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} 4^k / \binom{2k}{k} \ = \ \frac{1}{2n-1} \ . \tag{72}$$

We note that from (13) one has $-2b_{n+1}(1/4) = C_n/4^n$. [11]

If one puts in (13) $4x - 1 = x$, *i.e.* $x = 1/3$, one can identify the partial sum of *Catalan* numbers, $s_n(1)$ [12], as follows.

$$s_n(1) \ = \sum_{k=0}^{n} C_k = \ \frac{1}{2}(1 \ - \ 3^{n+1}\ b_{n+1}(1/3)\ ). \tag{73}$$

If one puts $\lambda = 1$ in *Corollary 3* one finds also

Example 6:

$$2\ s_{n-1}(1) \ = \ 1 \ + \ \binom{2n}{n} \sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} 3^k / \binom{2k}{k} \ . \tag{74}$$

Another interesting example is the case $\lambda = 4$ ($x = \infty$). Here one finds a simple result for the convolution of *Catalan*'s sequence with powers of 4, *viz* [13]

Example 7: $\lambda = 4 \quad (x = \infty)$

$$2\ s_{n-1}(4) \ = \ 4^n \ - \ \binom{2n}{n} \ . \tag{75}$$

The sequence for $\lambda = -1$ ($x = 1/5$) is also non-negative, as can be seen by writing $s_{2k}(-1) = C_2 + \sum_{l=2}^{k}(C_{2l} - C_{2l-1})$ for $k \in \mathbf{N}$ and $s_{2k+1}(-1) = \sum_{l=1}^{k} (C_{2l+1} - C_{2l})$, and using $\triangle C_n := C_n - C_{n-1} = 3\frac{n-1}{n+1}\ C_{n-1} \geq 0$. [14]

Recursion (66) for $B(n,m)$ can be transformed into an eq. for the (ordinary) generating function for the sequence appearing in the mth column of the $B(n,m)$ triangle

$$G_B(m;x) := \sum_{n\geq m} B(n,m)\ x^n \quad , \tag{76}$$

[9] From the generating function the recurrence relation is found to be $s_n(\lambda) = \lambda\ s_{n-1}(\lambda) + C_n$, $s_{-1}(\lambda) \equiv 0$. The connection to the $b_n(x)$ polynomial is $s_n(\lambda) \ = \ \frac{1}{2}\Big(\lambda^{n+1} \ - \ (4-\lambda)^{n+1}\ b_{n+1}(1/(4-\lambda))\Big)$.

[10] This identity occurs in one of the exercises 2.7, 2, p.32, in [7].

[11] The large n behaviour of this sequence is known to be $C_n/4^n \sim \frac{1}{\sqrt{\pi}}\ \frac{1}{n^{3/2}}$, *cf.* [2], Exercise 9.60.

[12] This sequence $\{1, 2, 4, 9, 23, 65, 197, 626, 2056, ...\}$, appears as A014137 in the on-line encyclopedia [10].

[13] This sequence $\{1, 5, 22, 93, 386, 1586, 6476, ...\}$ appears in the book [10] as Nr. 3920 and as A000346 in the on-line encyclopedia. It will show up again in this work as $A(n+1,1)$, the second column in the $A(n,m)$ triangle (*cf. TAB.2*).

[14] This is the sequence $\{1, 0, 2, 3, 11, 31, 101, 328, 1102, 3760, ...\}$ which appears now as *A*032357 in the on-line encyclopedia [10] .

with input $G_B(0;x) \ = \ \sum_{n=0}^{\infty} \binom{2n}{n} \ x^n \ = \ 1/\sqrt{1-4x}$, the generating function for the central binomial numbers. (66) implies for $m \in \mathbf{N}_0$ [15]

$$G_B(m;x) \ = \ \Big(\frac{x}{1-4x}\Big)^m \ \frac{1}{\sqrt{1-4x}}. \tag{77}$$

Therefore, we have proved:

Proposition 8: Column sequences of the $B(n,m)$ triangle

The sequence $\{B(n,m)\}_{n=m}^{\infty}$, defined, for fixed $m \in \mathbf{N}_0$, by (68) for $n \in \mathbf{N}_0$ is the convolution of the central binomial sequence $\{\binom{2k}{k}\}_0^{\infty}$ and the mth convolution of the (shifted) power sequence $\{0,1,4^1,4^2,...\}$.

In a similar vein we solve the $a_n(x)$ eq.(57) with $b_n(x)$ given by (60) and (68). The coefficients $a(n,k)$, defined by (59), have to satisfy, after comparing coefficients of x^n, x^0, and x^{n-k} for $k = 1,2,...,n-1$ and $n \in \mathbf{N}_0$:

$$x^n \ : \qquad a(n,0) \ = \ 4\ a(n-1,0) \ + \ C_n, \tag{78}$$

$$x^0 \ : \qquad (n+1)\ a(n,n) \ = \ 1 \ + \ n\ a(n-1,n-1)\ , \tag{79}$$

$$x^{n-k} \ : \qquad (n+1)\ a(n,k) \ = \ k\ a(n-1,k-1) \ + \ 4(n+1+k)\ a(n-1,k) \ + \ B(n,k)\ . \tag{80}$$

We have used (64), *i.e.* $B(n,0) = (n+1)\ C_n$ in (78), as well as (65), *i.e.* $B(n,n) \equiv 1$, in (79). From (78) one finds with input $a(0,0) = 1$ [16]

$$a(n,0) \ = \ \sum_{k=0}^{n} C_k\ 4^{n-k}\ , \tag{81}$$

and from (79)

$$a(n,n) \equiv 1, \ \text{or} \ \ a_n(0) \ = \ (-1)^n\ . \tag{82}$$

It is convenient to define $a(n-1,-1) := C_n$, $n \in \mathbf{N}_0$. Then the sequence $\{a(n,0)\}_{-1}^{\infty}$ is, with $a(-1,0) := 0$, the convolution of the sequence $\{a(k,-1)\}_{-1}^{\infty}$ and the shifted power sequence $\{0,1,4^1,4^2,...\}$. Before solving (80) with inserted $B(n,k)$ from (68) we therefore add to the trianglular array of numbers $a(n,m)$ the $m = -1$ column and an extra row for $n = -1$, and define a new enlarged triangular array for $n,m \in \mathbf{N}_0$ as

$$A(n,m) \ := \ a(n-1,m-1) \tag{83}$$

with $A(n,0) = a(n-1,-1) = C_n$ and $A(0,m) = a(-1,m-1) = \delta_{0,m}$. An inspection of the $A(n,m)$ triangular array, partly depicted in *TAB. 2*, leads to the conjecture

$$A(n,m) \ = \ 4\ A(n-1,m) \ + \ A(n-1,m-1)\ , \tag{84}$$

with $A(n,0) = C_n$ and $A(n,m) \equiv 0$ for $n < m$. [17] This conjecture is correct for $A(n+1,1) = a(n,0)$ found in (81), as well as for $A(n+1,n+1) = a(n,n) \equiv 1$ known from (82). The (ordinary) generating function for the sequence appearing in the mth column,

$$G_A(m;x) \ = \ \sum_{n=m}^{\infty} A(n,m)\ x^n\ , \tag{85}$$

[15] For $x\frac{d}{dx}G_B(m;x)$ see (92).
[16] $a(n,0) = s_n(4)$ of (71) with solution (75).
[17] This recursion relation can be employed to extend the array $A(n,m)$ to negative integer m values.

satisfies due to (84) $G_A(m;x) = \frac{x}{1-4x}\, G_A(m-1;x)$, remembering that $A(m-1,m) \equiv 0$, or because of $G_A(0;x) \;=\; c(x)$

$$G_A(m;x) \;=\; \Big(\frac{x}{1-4x}\Big)^m\, c(x)\ . \tag{86}$$

Because of (77) and $\sqrt{1-4x}\; c(x) \;=\; 2-c(x)$ these generating functions of the conjectured $A(n,m)$ column sequences obey

$$G_A(m;x) \;=\; (2-c(x))\; G_B(m;x)\ . \tag{87}$$

If we use the conjecture (84) in (80) which is written with (83) in the form $(n+1)\; A(n+1,m+1) \;=\; m\; A(n,m) \;+\; 4(n+m+1)\; A(n,m+1) \;+\; B(n,m)$, for $n \in \mathbf{N}_0,\; m \in \{1,2,..,n-1\}$, we have

$$m\; A(n+1,m+1) \;-\; (n+1)\; A(n,m) \;+\; B(n,m) \;=\; 0\ . \tag{88}$$

This recursion relation can be written with the help of the generating functions (76) and (85) as

$$(x\frac{d}{dx}+1)\; G_A(m;x) \;-\; \frac{m}{x}\; G_A(m+1;x) \;=\; G_B(m;x)\ , \tag{89}$$

or with (86) (*i.e.* the conjecture) as

$$\Big(x\frac{d}{dx}+1 \;-\; \frac{m}{1-4x}\Big)\; G_A(m;x) \;=\; G_B(m;x)\ . \tag{90}$$

Together with (87) this means

$$x\frac{d}{dx}\Big((2-c(x))\; G_B(m;x)\Big) \;=\; \Big[(\frac{m}{1-4x}-1)(2-c(x)) \;+1\Big]\; G_B(m;x)\ . \tag{91}$$

If we can prove this eq. with $G_B(x)$ given by (77) we have shown that (80) is equivalent to the conjecture (84). In order to prove (91) we first compute from (77), for $m \in \mathbf{N}_0$,

$$x\frac{d}{dx}G_B(m;x) \;= (2+\frac{m}{x})\; G_B(m+1;x) \;=\; \frac{2x+m}{1-4x}\; G_B(m;x)\ . \tag{92}$$

With this result (91) reduces to

$$\Big(-x\; c'(x) \;+\; (2-c(x))\,\frac{1-2x}{1-4x} \;-\; 1\Big)\; G_B(m;x) \;=\; 0\ , \tag{93}$$

and with (9) the factor in front of $G_B(m;x)$ finally vanishes identically for $x \neq 1/4$. Therefore, we have proved the following two propositions.

Proposition 9: Column sequences of the $A(n,m)$ triangular array

The triangular array of numbers $A(n,m)$, defined for $n,m \in \mathbf{N}_0$ by eq.(84), $A(n,0) = C_n$, $A(n,m) \equiv 0$ for $n < m$ has as mth column sequence $\{A(n,m)\}_{n=m}^{\infty}$ the convolution of *Catalan*'s sequence and the mth convolution of the shifted power sequence $\{0,1,4^1,4^2,...\}$.

Proof: (86) with (85). □

Proposition 10: Triangular $A(n,m)$ array

The triangular array $A(n,m)$ of *proposition 9* coincides with the one defined by (83) and (78), (79) and (80) with $B(n,m)$ given by (68).

Proof: $a(n,0) = A(n+1,1)$ and $a(n,n) = A(n+1,n+1) \equiv 1$ of (78) and (79), *i.e.* (81) and (82), respectively, coincide with (84). (80) is rewritten with the aid of (83) as (88), and (88) has been proved by (89) to (93). $\square$

It remains to find the explicit expression for the $a_n(x)$ coefficients $a(n,k)$ defined by (59). Because of (83) we try to find a formula for $A(n,m)$. By *propositions 9* and *10* we may consider the recursion (84) with inputs $A(n,0) = C_n$, $A(n,m) \equiv 0$ for $n < m$, and $A(n,n) \equiv 1$ from (83) and (82).

Proposition 11: Explicit form of $a_n(x)$

$A(n,m)$ given by $A(n,0) = C_n$, $A(n,m) \equiv 0$ for $n < m$, and (14) is the solution to (84) with $A(n,n) \equiv 1$.

Therefore, $a_n(x)$ is given by (59) with $a(n,k) = A(n+1,k+1)$ from (14).

Proof: The first term of $A(n,m)$, $\frac{1}{2}\, 4^{n-m+1}\, \binom{n}{m-1}$, satisfies the recursion (84) because of the binomial identity $\binom{n}{m-1} = \binom{n-1}{m-1} + \binom{n-1}{m-2}$ (*Pascal*'s triangle). For the second term of $A(n,m)$ in (14) one has to prove

$$\binom{n}{m-1}\binom{2n}{n} \;=\; 4\binom{n-1}{m-1}\binom{2(n-1)}{n-1} \;+\; \binom{n-1}{m-2}\binom{2(n-1)}{n-1}\frac{2(2m-3)}{m-1}\;, \tag{94}$$

or after division by $\binom{2(n-1)}{n-1}$

$$\frac{2n-1}{n}\binom{n}{m-1} \;=\; 2\binom{n-1}{m-1} \;+\; \binom{n-1}{m-2}\frac{2m-3}{m-1}\;, \tag{95}$$

which reduces to the trivial identity $2n-1 = 2\,(n-m+1) + 2m-3$.

Both terms together, *i.e.* (14), satisfy the input $A(n,n) \equiv 1$. $\square$

Note 3: $A(n,m)$ was found originally after iteration in the form (with $n \geq m > 0$ and $(-1)!! := 1$)

$$A(n,m) \;=\; 2\cdot 4^{n-m}\binom{n}{m-1} \;-\; \frac{\prod_{k=1}^{m}(2(n-m)+2k-1)}{(2m-3)!!}\, C_{n-m}\;. \tag{96}$$

$A(n,0) = C_n$. It is easy to establish equivalence with (14).

In the original derivation of the $A(n,m)$ formula (14) it turned out to be convenient to introduce a rectangular array of integers $\hat{A}(n,m)$ for $n,m \in \mathbf{N}_0$ as follows. $\hat{A}(0,0) := 1$, $\hat{A}(n,0) := -C_n$ for $n \in \mathbf{N}$, and for $m \in \mathbf{N}$ and $n \in \mathbf{N}_0$ $\hat{A}(n,m)$ is defined by (15), or equivalently, by (16). The $A(n,m)$ recursion (84) translates (with the help of the above mentioned *Pascal*-triangle identity) to

$$\hat{A}(n,m) \;=\; 4\,\hat{A}(n-1,m) \;+\; \hat{A}(n,m-1)\;. \tag{97}$$

This leads, after iteration and use of $\hat{A}(0,m) \equiv 1$ from (15) with $A(n,n) \equiv 1$, to

$$\hat{A}(n,m) \;=\; 4^n \sum_{k=0}^{n} \hat{A}(k,m-1)/4^k\;. \tag{98}$$

Thus, the following proposition holds.

Proposition 12: Column sequences of the $\hat{A}(n,m) \equiv C4(n,m)$ array

The mth column sequence of the $\hat{A}(n,m)$ array, $\{\hat{A}(n,m)\}_{n=0}^{\infty}$, is the convolution of the sequence $\{\hat{A}(n,0)\}_0^{\infty} = \{1,-1,-2,-5,...\}$, generated by $2-c(x)$, and the mth convolution of the power sequence $\{4^k\}_0^{\infty}$.

Proof: Iteration of (98) with the $\hat{A}(n,0)$ input. $\square$

Corollary 4: Generating functions for columns of the $\hat{A}(n,m) \equiv C4(n,m)$ array

The ordinary generating function of the mth column sequence of the $\hat{A}(n,m)$ array (16) is for $m \in \mathbf{N}_0$ given by

$$G_{\hat{A}}(m;x) \ := \ \sum_{n=0}^{\infty} \hat{A}(n,m)\ x^n \ = \ (2-c(x)) \left(\frac{1}{1-4x}\right)^m . \tag{99}$$

Proof: Proposition 12 written for generating functions. $\square$

Because of the convolution of the (negative) *Catalan* sequence with powers of 4 we shall call this array $\hat{A}(n,m)$ also $C4(n,m)$. A part of it is shown in *TAB.3.* [18]

Finally, we derive identities by using, for $n \in \mathbf{N}_0$, eq.(17) for the *lhs.* of (11) and the results for a_{n-1} and b_n for the *rhs.*

Because there are no negative powers of x on the *lhs.* of (11), such powers have to vanish on the *rhs.* This leads to the first family of identities. Because $(1-4x)^{-n} = \sum_{k=0}^{\infty} \frac{(n)_k}{k!}\ 4^k\ x^k$, with *Pochhammer*'s symbol defined in footnote 8, this means that $[x^p]\ (a_{n-1}(x) + b_n(x)\ c(x))$, the coefficient proportional to x^p, has to vanish for $p = 0,1,...,n-1$, $n \in \mathbf{N}$. This requirement reads

$$(-1)^{n-1-p}\ a(n-1,n-1-p) \ + \ \sum_{k=0}^{p}(-1)^{n-k}\ B(n,n-k)\ C_{p-k} \ \equiv 0\ . \tag{100}$$

The sum is restricted to $k \leq p (< n)$ because no C_l number with negative index is found in $c(x)$. Inserting the known coefficients this produces identity $(D1)$ of (18).

Proposition 13: Identity $(D1)$ of (18)

For $n \in \mathbf{N}$ and $p \in \{0,1,...,n-1\}$ identity $(D1)$, given by (18), holds.

Proof: With (83) (100) becomes

$$\sum_{k=0}^{p}\ (-1)^{p-k}\ C_{p-k}\ B(n,n-k) \ = \ A(n,n-p)\ , \tag{101}$$

which is $(D1)$ of (18) if the summation index k is changed into $p-k$, and symmetry of the binomial coefficients is used. $\square$.

[18] The second column sequence is given by $\hat{A}(n,1) \equiv C4(n,1) = \binom{2n+1}{n}$ and appears as nr.2848 in the book [10], or as $A001700$ in the on-line encyclopedia [10]. The sequence of the third column $\{\hat{A}(n,2) \equiv C4(n,2)\}_0^{\infty} = \{1,7,38,187,...\}$ is from (98) and (96) with (15) determined by $4^n\ \sum_{k=0}^{n} \binom{2k+1}{k}/4^k = (2n+3)\ (2n+1)\ C_n - 2^{2n+1}$, and is listed as $A000531$ in the mentioned on-line encyclopedia. There the fourth column sequence is now listed as $A029887$.

Example 8: $(D1)$ identity for $p = n-1 \in \mathbf{N}_0$ [19]

$$\sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} \frac{1}{2k+1} = 4^n / \binom{2n}{n} - 1 = 2A(n,1)/\binom{2n}{n} \quad . \tag{102}$$

The second family of identities, $(D2)$ of (19), results from comparing powers x^k with $k \in \mathbf{N}_0$ on both sides of eq.(11) after expansion of $(1-4x)^{-n}$ as given above in the text before eq. (100). Only the second term $b_n(x)\ c(x)$ contributes because $a_{n-1}(x)/x^n$ has only negative powers of x. Thus, with definition (17) one finds for $k \in \mathbf{N}_0$ and $n \in \mathbf{N}$,

$$C(n,k) = \sum_{l=0}^{k} \frac{(n)_l\ 4^l}{l!} \sum_{j=0}^{n} (-1)^{n-j}\ B(n, n-j)\ C_{n-j+k-l} \tag{103}$$

which is, after interchange of the summations and insertion of $B(n, n-j)$ from (12) the desired identity $(D2)$ if also the summation index j is changed to $n-q$.

Thus we have shown:

Proposition 14: Identity $(D2)$ of (19)

For $k \in \mathbf{N}_0$ and $n \in \mathbf{N}$ identity $(D2)$ of (19) with $C(n,k)$ defined by (17) holds.

Example 9: Identity $(D2)$ for $k = 0$, $n \in \mathbf{N}$

$$\sum_{j=0}^{n} (-1)^j \binom{n+1}{j+1} \equiv 1 \ , \tag{104}$$

which is elementary.

Acknowledgements

The author likes to thank Dr. Stephen Bedding for a collaboration on power of matrices. In *section 2* a result for 2×2 matrices (here $\mathbf{C}$) was recovered.

[19]With this identity we have found a sum representation for the convolution of the *Catalan* sequence and powers of 4: $s_{n-1}(4) := 4^{n-1} \sum_{k=0}^{n-1} C_k/4^k = \frac{1}{2} \binom{2n}{n} \sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} \frac{1}{2k+1}$ (*cf.* (75) with (71)).

TAB. 1 : $B(n,m)$ Central Binomial Triangle

n\m	0	1	2	3	4	5	6	7	8	9	10
0	1	0	0	0	0	0	0	0	0	0	0
1	2	1	0	0	0	0	0	0	0	0	0
2	6	6	1	0	0	0	0	0	0	0	0
3	20	30	10	1	0	0	0	0	0	0	0
4	70	140	70	14	1	0	0	0	0	0	0
5	252	630	420	126	18	1	0	0	0	0	0
6	924	2772	2310	924	198	22	1	0	0	0	0
7	3432	12012	12012	6006	1716	286	26	1	0	0	0
8	12870	51480	60060	36036	12870	2860	390	30	1	0	0
9	48620	218790	291720	204204	87516	24310	4420	510	34	1	0
10	184756	923780	1385670	1108536	554268	184756	41990	6460	646	38	1

TAB. 2 : $A(n,m)$ Catalan triangle

n\m	0	1	2	3	4	5	6	7	8	9	10
0	1	0	0	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0	0	0
2	2	5	1	0	0	0	0	0	0	0	0
3	5	22	9	1	0	0	0	0	0	0	0
4	14	93	58	13	1	0	0	0	0	0	0
5	42	386	325	110	17	1	0	0	0	0	0
6	132	1586	1686	765	178	21	1	0	0	0	0
7	429	6476	8330	4746	1477	262	25	1	0	0	0
8	1430	26333	39796	27314	10654	2525	362	29	1	0	0
9	4862	106762	185517	149052	69930	20754	3973	478	33	1	0
10	16796	431910	848830	781725	428772	152946	36646	5885	610	37	1

TAB. 3 : $C4(n,m)$ Catalan array

n\m	0	1	2	3	4	5	6
0	1	1	1	1	1	1	1
1	-1	3	7	11	15	19	23
2	-2	10	38	82	142	218	310
3	-5	35	187	515	1083	1955	3195
4	-14	126	874	2934	7266	15086	27866
5	-42	462	3958	15694	44758	105102	216566
6	-132	1716	17548	80324	259356	679764	1546028
7	-429	6435	76627	397923	1435347	4154403	10338515
8	-1430	24310	330818	1922510	7663898	24281510	65635570
9	-4862	92378	1415650	9105690	39761282	136887322	399429602
10	-16796	352716	6015316	42438076	201483204	749032492	2346750900

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