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On Polynomials Related to Powers and Derivatives of the Generating Function of Catalan's Numbers

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Abstract

Arbitrary powers of the generating function $c(x)$ of *Catalan*'s numbers are written as $c^n(x) := -(\frac{1}{\sqrt{x}})^n\ S_{n-2}(\frac{1}{\sqrt{x}})\ +\ (\frac{1}{\sqrt{x}})^{n-1}\ S_{n-1}(\frac{1}{\sqrt{x}})\ c(x)$, with *Chebyshev*'s polynomials of the second kind $S_n(y) = U_n(y/2)$ which are also defined for real (or complex) n. This formula leads to four sets of identities involving *Catalan* numbers.

The nth derivative of this generating function $c(x)$ is expressed as $\frac{1}{n!}\frac{d^n c(x)}{dx^n}\ =\ \big(a_{n-1}(x)\ +\ b_n(x)\ c(x)\big)/(x(1-4x))^n$, with certain polynomial systems $\{a_n\}$ and $\{b_n\}$ which are given explicitly. The coefficients of the $\{a_n\}$ polynomials furnish a triangle of numbers $A(n,k)$ which generalizes *Catalan*'s numbers. It is related to a convolution of the *Catalan* sequence with $2k$-fold convolutions of the central binomial coefficient sequence. Also, an associated rectangular array $\hat{A}(n,k)$ of numbers is defined. The triangle of numbers of the $\{b_n\}$ coefficients is related to the $(2k+1)$-fold convolution of the central binomial number sequence. This formula for the derivatives of $c(x)$ implies identities involving *Catalan*'s numbers as well as central binomial coefficients.

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