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On Generalizations of the Stirling Number Triangles [1]

Wolfdieter L a n g [2]

Institut für Theoretische Physik
Universität Karlsruhe
Kaiserstraße 12, D-76128 Karlsruhe, Germany

Abstract

Sequences of generalized *Stirling* numbers of both kinds are presented. These sequences of triangles of numbers (or infinite dimensional lower triangular matrices) will be denoted by $S2(k;n,m)$ and $S1(k;n,m)$, with $k \in \mathbf{Z}$. The original *Stirling* number triangles of the second and first kind show up for $k=1$. $S2(2;n,m)$ is identical with the unsigned S1(2;n,m) triangle, called $S1p(2;n,m)$, which represents the triangle of signless *Lah* numbers. Associated number triangles, named $s2(k;n,m)$ and $s1(k;n,m)$, are also defined. $s2(2;n,m)$ as well as $s1(2;n+1,m+1)$ is *Pascal*'s triangle, and $s2(-1,n,m)$ turns out to be *Catalan*'s triangle [8].

Generating functions for the columns of these triangles are given. Each $\mathbf{S2}(k)$ and each $\mathbf{S1}(k)$ matrix is an example of a *Jabotinsky*-matrix. Therefore, the generating functions of the rows of these triangular arrays constitute exponential convolution polynomials. The sequences of the row-sums of these triangles are also considered.

The connection to the problem of finding finite transformations from infinitesimal ones generated by $x^k \frac{d}{dx}$, for $k \in \mathbf{Z}$, is made.

[1] In memory of my mother Else Gertrud Lang.

[2] E-mail: wolfdieter.lang@physik.uni-karlsruhe.de, http://www-itp.physik.uni-karlsruhe.de/~wl

1 Conspectus

Stirling's numbers of the second kind (also called subset numbers) $S2(n,m)$ (or $\left\{ \begin{array}{c} n \\ m \end{array} \right\}$ in the notation of [3], or $\mathcal{S}_n^{(m)}$ in [1], or A008277 in the data-base [10]) can be defined by

$$E_x{}^n \equiv (x\,d_x)^n \;=\; \sum_{m=1}^{n} S2(n,m)\, x^m\, d_x{}^m \quad , \quad n \in \mathbf{N}, \tag{1}$$

with the derivative operator $d_x \equiv \frac{d}{dx}$, and E_x is the *Euler*- operator satisfying $E_x\, x^k \;=\; k\, x^k$. A recursion relation can be derived from eq. 1 by considering $x\, d_x (x\, d_x)^{n-1}$, using the convention $S2(n,m) = 0$ if $n < m$, in order to interpret $S2(n,m)$ as lower triangular, infinite dimensional matrix **S2**. Thus,

$$S2(n,m) \;=\; m\, S2(n-1,m) \;+\; S2(n-1,m-1), \tag{2}$$

with inputs $S2(n,0) \equiv 0$ and $S2(1,1) = 1$. Because of eq. 1 these numbers emerge when one asks how finite scale transformations (dilations) look, given the infinitesimal ones. This is a special application of the exponentiation procedure available for *Lie*-groups. The generator of the abelian *Lie*-group of scale transformations $x' = \lambda\, x$, $\lambda \in \mathbf{R}_+$, is E_x. In order to exhibit these numbers within this framework consider first

$$\begin{aligned} e^{c\,x\,d_x} \;&=\; \sum_{n=0}^{\infty} \frac{c^n}{n!}\, E_x{}^n \;=\; 1 + \sum_{n=1}^{\infty} \frac{c^n}{n!} \sum_{m=1}^{n} S2(n,m)\, x^m\, d_x{}^m \\ &=\; 1 + \sum_{m=1}^{\infty} \left(\sum_{n=m}^{\infty} \frac{c^n}{n!}\, S2(n,m) \right) x^m\, d_x{}^m \;=\; 1 + \sum_{m=1}^{\infty} G2_m(c)\, x^m\, d_x{}^m \; . \end{aligned} \tag{3}$$

In the third step a resummation has been performed (here we are not concerned about convergence problems), and in the last step the exponential generating function (*e.g.f.*) for the $m-$th column of the number triangle, or lower triangular matrix, **S2** was introduced. The recursion relation will be seen to imply $G2_m(c) = \frac{1}{m!}\,(G2(c))^m$ with $G2(c) = exp(c) - 1$; therefore, one finds

$$e^{c\,x\,d_x} \;=\; \sum_{m=0}^{\infty} \frac{1}{m!}\, (G2(c)\, x)^m\, d_x{}^m \;=: e^{(exp(c)-1)x\,d_x} : \, , \tag{4}$$

where we have used the linear normal order symbol $: A :$ familiar from quantum physics. It instructs one to write in a product built from x and d_x all operators d_x to the righthand side (*rhs*) without worrying about the commutation rule $[d_x, x] \equiv d_x\, x - x\, d_x = 1$. *E.g.* $: (x\, d_x)^m : \;=\; x^m\, d_x^m$. This normal order prescription is applied to each term of the expanded exponential in eq. 4. For suitable functions f we finally have from *Taylor*'s theorem

$$e^{c\,x\,d_x}\, f(x) \;=: e^{(exp(c)-1)x\,d_x} :\, f(x) \;=\; f(x + (e^c - 1)x) \;=\; f(x') \quad , \tag{5}$$

with $x' = e^c\, x$. Therefore, the parameter λ for finite scale transformations is $\lambda = e^c$ if c is the one for infinitesimal transformations. Within the setting of *Lie*-groups this fact is found from integrating the ordinary differential eq. (*cf. e.g.*[2] and ref. given there.)

$$\frac{dx(\alpha)}{d\alpha} \;=\; c\, x(\alpha) \;\; \text{with } x' = x(\alpha = 1) \text{ and } x = x(\alpha = 0). \tag{6}$$

Here x' is the transformed variable x (not a derivative).
In this work we shall generalize to the case $E_{k;x} \equiv x^k\, d_x$ with $k \in \mathbf{Z}$. It is clear from the solution of the differential eq.

$$\frac{dx(\alpha)}{d\alpha} \;=\; c\, x^k(\alpha) \quad \text{with } x' = x(\alpha = 1) \text{ and } x = x(\alpha = 0) \tag{7}$$

that the scale transformation case $k = 1$ which has been treated above is special. After separation of the variables one finds for $k \neq 1$ the eq. $(x')^{1-k} \;-\; x^{1-k} \;=\; (1-k)\, c$. With the ansatz

$$x' \;=\; (1 + g(k; c; x))\, x \tag{8}$$

this means that

$$1 + g(k; c; x) \;=\; \left(1 - (k-1)\, c\, x^{k-1}\right)^{-\frac{1}{k-1}} \;. \tag{9}$$

Therefore,

$$e^{c\, x^k\, d_x}\, f(x) \;=\; f(x') \;=\; f\left(\left(1 - (k-1)\, c\, x^{k-1}\right)^{-\frac{1}{k-1}}\, x\right) \tag{10}$$

for $k \in \mathbf{Z} \setminus \{1\}$. The case $k = 1$ has been dealt with in eq. 5. It can be recovered from the $k \neq 1$ formula eq. 10 in the limit $k - 1 \to 0$.

$k-$*Stirling* numbers of the second kind which we shall call $S2(k; n, m)$, with $S2(1; n, m) = S2(n, m)$ the ordinary *Stirling* subset numbers, emerge in a proof, independent of the one implied by eq. 10, of the following operator identity, valid for $k \in \mathbf{Z}$,

$$e^{c\, x^k\, d_x} \;=: e^{g(k;c;x)\, x\, d_x} :\;, \tag{11}$$

where $g(k; c; x)$ is defined by eq. 9 for $k \neq 1$ and $g(1; c; x) = G2(c)$ defined in eq. 4. In analogy to eq. 1 the $S2(k; n, m)$ number triangle is defined by

$$E_{k;x}{}^n \equiv (x^k\, d_x)^n \;=\; \sum_{m=1}^{n} S2(k; n, m)\, x^{m+(k-1)\, n}\, d_x{}^m \quad,\quad n \in \mathbf{N},\quad k \in \mathbf{Z}\;, \tag{12}$$

with the additional convention $S2(k; n, m) = 0$ for $n < m$, and $S2(k; n, 0) = 0$. These numbers will be shown to satisfy the recursion relation

$$S2(k; n, m) \;=\; ((k-1)(n-1) + m)\, S2(k; n-1, m) \;+\; S2(k; n-1, m-1), \tag{13}$$

with $S2(k; 1, 1) = 1$ from eq. 12. The *e.g.f.* for the $m-$th column of the $\mathbf{S2}(k)$ triangle,

$$G2(k; m; x) \;:=\; \sum_{n=m}^{\infty} S2(k; n, m)\, \frac{x^n}{n!}\;, \tag{14}$$

satisfies for $k \neq 1$

$$G2(k; m; x) \;=\; \frac{1}{m!}\, (G2(k; x))^m \tag{15}$$

with

$$G2(k; x) \;=\; (k-1)\, g2(k; \frac{x}{k-1})\;, \tag{16}$$

where

$$g2(k; y) \;:=\; \sum_{n=1}^{\infty} s2(k; n, 1)\, y^n \tag{17}$$

is the ordinary generating function (*o.g.f.*) of the first column of the triangle of numbers $s2(k;n,m)$ which is associated to triangle $S2(k;n,m)$ according to [3]

$$s2(k;n,m) \;:=\; (k-1)^{n-m}\,\frac{m!}{n!}\,S2(k;n,m)\;. \tag{18}$$

These number triangles, or lower triangular infinite dimensional matrices, $\mathbf{s2}(k)$, which are here only defined for $k \in \mathbf{Z}\setminus\{1\}$, obey the recursion relation

$$s2(k;n,m) \;=\; \frac{k-1}{n}\,[(k-1)(n-1)+m]\;s2(k;n-1,m) \;+\; \frac{m}{n}\,s2(k;n-1,m-1)\;\;, \tag{19}$$

with

$$s2(k;n,m) \;=\; 0,\; n<m \;\;,\;\; s2(k;n,0) \;=\; 0,\;\; s2(k;1,1)=1\;. \tag{20}$$

It follows that these numbers are non-negative. At this stage it is not obvious that they are integers for every $k \in \mathbf{Z}\setminus\{1\}$.
The *o.g.f.* of the m-th column of the $\mathbf{s2}(k)$ matrix is

$$g2(k;m;y) \;:=\; \sum_{n=m}^{\infty} s2(k;n,m)\,y^n \;=\; (g2(k;y))^m\;, \tag{21}$$

with

$$g2(k;y) \;=\; y\,c2(1-k;y), \tag{22}$$

where, finally, for $l \in \mathbf{Z}\setminus\{0\}$,

$$c2(l;y) \;=\; \frac{1-(1-l^2\,y)^{\frac{1}{l}}}{l\,y}\;. \tag{23}$$

It is clear from eq. 21 that $\mathbf{s2}(k)$ is a convolution triangle generated from its first column (*cf. e.g.* [4],[6],[9]). Such ordinary convolution triangles will be called *Bell* matrices [9] (see *note* 7). Because for $k \in \mathbf{Z}\setminus\{1\}$ eq. 16 now yields

$$G2(k;x) \;=\; -1 \;+\; (1+(1-k)\,x)^{\frac{1}{1-k}} \tag{24}$$

one obtains the $k=1$ result $G2(1;x) \;=\; e^x-1 \;=\; G2(x)$ in the limit $k-1\to 0$. The infinite dimensional lower triangular matrices $\mathbf{S2}(k)$ with integer entries are examples of *Jabotinsky* matrices (*cf.* [4], also for original references). Therefore, the *o.g.f.* of the rows of triangle $\mathbf{S2}(k)$ are exponential (*alias* binomial) convolution polynomials. *I.e.* the polynomials

$$S2_n(k;x) \;:=\; \sum_{m=1}^{n} S2(k;n,m)\,x^m \;\;,\;\; S2_0(k;x):=1\;, \tag{25}$$

obey for $k \in \mathbf{Z}$

$$S2_n(k;x+y) \;=\; \sum_{p=0}^{n}\binom{n}{p}\,S2_p(k;x)\,S2_{n-p}(k;y) \;=\; \sum_{p=0}^{n}\binom{n}{p}\,S2_p(k;y)\,S2_{n-p}(k;x)\;. \tag{26}$$

In the umbral calculus notation (*cf.* [7]) the polynomials $S2_n(k;x)$ are a special type of *Sheffer* polynomials called *associated* polynomial sequences, *alias Sheffer* for $(1,\overline{G2(k;t)})$, where $\overline{G2(k;t)} \;=$

[3] These associated *Stirling* numbers of the second kind are not the ones of [5], p.76, Table 2.

$(-1+(1+t)^{1-k})/(1-k)$ for $k \neq 1$ is the compositional inverse of $G2(k;t)$ from eq. 24. $\overline{G2(1;t)} = ln(1+t)$ can also be obtained in the limit $(1-k) \to 0$.
For negative k the $\mathbf{S2}(k)$ matrices have also negative entries. The recursion eq. 13 shows that it is possible to define non-negative number matrices by

$$S2p(-k;n,m) \;:=\; (-1)^{n-m}\, S2(-k;n,m) \quad , \quad k \in \mathbf{N}_0. \tag{27}$$

The *e.g.f.* for column m of the triangle $\mathbf{S2}p(-|k|)$ is

$$G2p(-|k|;m;x) \;=\; \frac{1}{m!}\,(G2p(-|k|;x))^m \tag{28}$$

with

$$G2p(-|k|;x) \;=\; -\,G2(-|k|;-x) \;=\; 1 \;-\; (1-(|k|+1)\,x)^{\frac{1}{|k|+1}}. \tag{29}$$

Eqs. 19 and 20 will be seen to imply that the $\mathbf{s2}(-|k|)$ matrices have always non-negative entries. In the tables TAB. 1, resp. TAB. 2, we have listed for some of these $\mathbf{s2}(k)$, resp. $\mathbf{S2}(k)$, triangles the A-numbers under which they can be viewed in the on-line data-base [10]. This data-base will henceforth be quoted as *EIS* (Encyclopedia of Integer Sequences). In the second columns of these tables we have listed the $A-$numbers of the sequences of the first columns of the lower triangular matrices, and in the third column one finds the $A-$numbers of the sequences of the row-sums of these matrices.
The function $c2(l;y)$ defined in eq. 23 generates for $l = 2$ *Catalan*'s numbers. For $l \in \mathbf{Z} \setminus \{0\}$ it defines what we call $l-$*Catalan* numbers. For positive l these sequences have been introduced by *O. Gerard* in *EIS*, who called them Patalan numbers. It will be proven later (see *note 11*) that $c2(l;y)$ generates indeed integers. Their explicit form can be found in eq. 77.

Stirling numbers of the first kind, $S1(n,m)$ ($S_n^{(m)}$ in [1], *EIS*: A008275), can be defined from the inversion of eq. 1 by

$$x^n\, d_x{}^n \;=\; \sum_{m=1}^{n} S1(n,m)\,(x\,d_x)^m \quad . \tag{30}$$

In addition one puts $S1(n,0) \equiv 0$ and $S1(n,m) := 0$ for $n < m$. In (infinite dimensional) matrix notation one can write [4]

$$\mathbf{S1} \cdot \mathbf{S2} \;:=\; \mathbf{1} \;=\; \mathbf{S2} \cdot \mathbf{S1} \quad . \tag{31}$$

The signless *Stirling* numbers of the first kind, also known as cycle numbers, $S1p(n,m)$ (or $\left[\begin{array}{c} n \\ m \end{array} \right]$ in the notation of [3]), are

$$S1p(n,m) \;:=\; (-1)^{n-m}\, S1(n,m) \quad . \tag{32}$$

Their recurrence formula is

$$S1p(n+1,m) \;=\; n\,S1p(n,m) \;+\; S1p(n,m-1) \quad , \tag{33}$$

with $S1p(1,1) = 1$, $S1p(n,0) = 0$ and $S1p(n,m) = 0$ for $n < m$.

The generalized $k-$*Stirling* numbers of the first kind $S1(k;n,m)$ are defined analogously by inverting eq. 12, *i.e.*

$$x^{kn}\, d_x{}^n \;=\; \sum_{m=1}^{n} S1(k;n,m)\, x^{(k-1)(n-m)}\,(x^k\, d_x)^m \quad \text{for} \quad k \in \mathbf{Z}\ ,\ n \in \mathbf{N}\ . \tag{34}$$

In (infinite dimensional) matrix notation:

$$\mathbf{S1}(k) \cdot \mathbf{S2}(k) \;=\; \mathbf{1} \;=\; \mathbf{S2}(k) \cdot \mathbf{S1}(k) \quad \text{for} \quad k \in \mathbf{Z}. \tag{35}$$

For $k \in \mathbf{N}$ one defines non-negative $k-$*Stirling* numbers of the first kind by

$$S1p(k;n,m) \;:=\; (-1)^{n-m}\, S1(k;n,m)\ . \tag{36}$$

For $-k \in \mathbf{N}_0$ the numbers $S1(k;n,m)$ are non-negative.
The recurrence relation for $k-$*Stirling* numbers of the first kind is

$$S1(k;n,m) \;=\; -[(k-1)\,m+n-1]\,S1(k;n-1,m) \;+\; S1(k;n-1,m-1)\ \ , \tag{37}$$

with $S1(k;1,1)=1$, $S1(k;n,0)=0$ and $S1(k;n,m)=0$ for $n<m$. For $k \neq 1$ one introduces associated $k-$Stirling numbers of the first kind [4]

$$s1(k;n,m) \;:=\; (1-k)^{n-m}\,\frac{m!}{n!}\,S1(k;n,m)\ , \tag{38}$$

which turn out to be always non-negative. They satisfy the recursion

$$s1(k;n,m) \;=\; \frac{k-1}{n}\big((k-1)\,m+n-1\big)\,s1(k;n-1,m)+\frac{m}{n}\,s1(k;n-1,m-1) \tag{39}$$

with $s1(k;n,m)=0$ for $n<m$, $s1(k;1,1)=1$ and $s1(k;n,0):=0$. At this stage it is not obvious that the $s1(k;n,m)$ are in fact integers.
The *o.g.f.* for the $m-$th column sequence of the number triangle $s1(k;n,m)$ is for $k \in \mathbf{Z}\setminus\{1\}$ found to be

$$g1(k;m;y) \;=\; \sum_{n=m}^{\infty} s1(k;n,m)\,y^n \;=\; (g1(k;y))^m\ , \tag{40}$$

with

$$g1(k;y) \;=\; y\,c1(k-1;y), \tag{41}$$

where, for $l \in \mathbf{Z}\setminus\{0\}$,

$$c1(l;y) \;=\; \frac{-1\;+\;(1-l\,y)^{-l}}{l^2\,y}\ . \tag{42}$$

Hence, $\mathbf{s1}(k)$ is, like $\mathbf{s2}(k)$, a convolution triangle generated from its $m=1$ column, *i.e.* they are *Bell* matrices.
The function $c1(l;y)$ generates for $l \in \mathbf{N}$ the numbers

$$c1(l;y) \;=\; \sum_{n=0}^{\infty} c1_n^{(l)}\,y^n\ ,\quad c1_n^{(l)} \;=\; \binom{n+l}{l-1}\,l^{n-1}\ . \tag{43}$$

For $l=2$ this is the *EIS* sequence A001792 $\{1,3,8,20,48,...\}$. $c1(l;y)$ becomes polynomial for negative l; *e.g.* $c1(-2;y)=1+y$, or $g1(-1;y)=y+y^2$. The coefficients of these polynomials define a triangle of numbers found under the *EIS* nr. A049323. Their explicit form is, for $l \in \mathbf{N}$,

$$c1_n^{(-l)} \;=\; \binom{l}{n+1}\,l^{n-1} \quad \text{for } n=0,1,...,l-1 \text{ and 0 otherwise.} \tag{44}$$

Eq. 43 now implies *via* eqs. 41 and 40 that $s1(k;n,m)$ is indeed integer for every $k \in \mathbf{Z}\setminus\{1\}$. The explicit form for the first column entries is

$$s1(k;n,1) \;=\; \begin{cases} \binom{k-2+n}{k-2}\,(k-1)^{n-2} & \text{for } k=2,3,...,\text{a}nd\ n \in \mathbf{N} \\ \\ \binom{|k|+1}{n}\,(|k|+1)^{n-2} & \text{for } -k \in \mathbf{N}_0\,,\ n=1,2,...|k|+1\ . \end{cases} \tag{45}$$

[4] These associated *Stirling* numbers of the first kind are not the ones appearing in [5], p.75, table 2.

The *e.g.f.*s for the $m-$th column of the signless $k-$*Stirling* numbers of the first kind are then for $k = 2, 3, ...$

$$G1p(k;m;x) \quad := \quad \sum_{n=m}^{\infty} S1p(k;n,m)\,\frac{x^n}{n!}\ , \tag{46}$$

$$= \quad \frac{1}{m!}\left[(k-1)\,g1(k;\frac{x}{k-1})\right]^m\ . \tag{47}$$

The $k = 1$ instance corresponds to the ordinary unsigned *Stirling* numbers $S1p(n,m)$ with *e.g.f.* for column m given by $G1p(1;m;x) = \frac{1}{m!}\left(-ln(1-x)\right)^m$. Because from eqs. 47, 41 and 42

$$G1p(k;1;x) \ = \ (k-1)\,g1(k;\frac{x}{k-1}) \ = \ \frac{1}{k-1}\,(-1 \ + \ \frac{1}{(1-x)^{k-1}})\ , \tag{48}$$

one covers the $k = 1$ result with *l'Hospital*'s rule in the limit $(k-1) \to 0$.
Note that $G1(k;1;x) = -G1p(k;1;-x) = \overline{G2(k;1;x)}$ for $k \in \mathbf{Z}$.
For $-k \in \mathbf{N}_0$ the *e.g.f.* for the $m-$th column of the non-negative triangular matrix $\mathbf{S1}(-|k|)$ is, due to eqs. 36 and 46, $G1(-|k|;m;x) = (-1)^m\,G1p(-|k|;m;-x)$, hence

$$G1(k;1;x) \ = \ \frac{1}{1+|k|}\,(-1 \ + \ (1+x)^{1+|k|}) \quad \text{for} \quad -k \in \mathbf{N}_0\ . \tag{49}$$

For the signed matrix $\mathbf{S1}(-|k|)$ with the elements defined by $S1s(-|k|;n,m) := (-1)^{n-m}\,S1(-|k|,n,m)$ the *e.g.f.* of the $m-$th column is $G1s(-|k|;m;x) = (-1)^m\,G1(-|k|;m;-x)$, *i.e.* $G1s(-|k|;x) \equiv G1s(-|k|;1;x) = (1-(1-x)^{1+|k|})/(1+|k|)$ for $k \in \mathbf{N}_0$.

The tables TAB. 3, resp. TAB. 4, show the *EIS* A-numbers of some of the number triangles $\mathbf{s1}(k)$, resp. $\mathbf{S1}(k)$ and $\mathbf{S1p}(k)$. The A-number of the $m = 1$ column and of the sequence of row-sums for each triangle is also given there.

The *o.g.f.* of the row sequences of triangle $\mathbf{S1}(k)$ are also exponential (*alias* binomial) convolution polynomials. *I.e.* the polynomials

$$S1_n(k;x) \ := \ \sum_{m=1}^{n} S1(k;n,m)\,x^m \quad , \quad S1_0(k;x) := 1\ , \tag{50}$$

obey for $k \in \mathbf{Z}$ eq. 26 with $S2$ replaced by $S1$. In the umbral calculus notation (*cf.* [7]) the polynomials $S1_n(k;x)$ are a special type of *Sheffer* polynomials called *associated* polynomial sequences (*alias Sheffer*) for $(1, \overline{G1(k;t)})$, where $\overline{G1(k;t)} = G2(k;t)$, given, for $k \neq 1$, in eq. 24. $\overline{G1(1;t)} = G2(1;t) = exp(t) - 1$, as obtained in the limit $(1-k) \to 0$.

Each sequence of row-sums of a given triangle of the type considered in this work is generated by a function which depends on the generating function of the triangle's first $(m = 1)$ column. For the $\mathbf{s2}(k)$, *resp.* $\mathbf{s1}(k)$, triangles which can be considered as *Bell* matrices these *o.g.f.*s are for $k \in \mathbf{Z} \setminus \{1\}$

$$r2(k;x) \quad = \quad \frac{g2(k;x)}{1-g2(k;x)} \ = \ \frac{-1 \ + \ [1-(1-k)^2\,x]^{\frac{1}{1-k}}}{k \ - \ [1-(1-k)^2\,x]^{\frac{1}{1-k}}}, \tag{51}$$

$$r1(k;x) \quad = \quad \frac{g1(k;x)}{1-g1(k;x)} \ = \ \frac{1 \ - \ [1-(k-1)\,x]^{k-1}}{(1+(1-k)^2)\,[1-(k-1)\,x]^{k-1} \ - \ 1} \tag{52}$$

For the $\mathbf{S2}(k)$, $k \in \mathbf{N}_0$, and $\mathbf{S2p}(k)$, $-k \in \mathbf{N}_0$, triangles which can be interpreted as *Jabotinsky* matrices the *e.g.f.*s for the sequences of row-sums are

$$R2(k;x) \quad = \quad e^{G2(k;x)} \ - \ 1 \ = \ exp\left[-1+(1-(k-1)\,x)^{\frac{1}{1-k}}\right] \ - \ 1\ , \tag{53}$$

$$R2p(-|k|;x) \quad = \quad e^{G2p(-|k|;x)} \ - \ 1 \ = \ exp\left[1-(1-(1+|k|)\,x)^{\frac{1}{1+|k|}}\right] \ - \ 1\ . \tag{54}$$

For the $\mathbf{S1p}(k)$, $k \in \mathbf{N}_0$, and $\mathbf{S1}(k)$, $-k \in \mathbf{N}_0$, triangles which can also be interpreted as *Jabotinsky* matrices the *e.g.f.*s for the sequences of row-sums are

$$R1p(k;x) \;=\; e^{G1p(k;x)} \;-\; 1 \;=\; exp\Big(\frac{1}{k-1}\,[-1+(1-x)^{\frac{1}{k-1}}]\Big) \;-\; 1 \;\;, \tag{55}$$

$$R1(-|k|;x) \;=\; e^{G1(-|k|;x)} \;-\; 1 \;=\; exp\Big(\frac{1}{1+|k|}[-1+(1+x)^{1+|k|}]\Big) \;-\; 1 \;\;. \tag{56}$$

The special case $k = 1$ can be reached for $R2(k;x)$ as well as $R1p(k;x)$ in the limit $(k-1) \to 0$.

In the remainder of this work one will find in *sections 2* and *3* proofs for the above stated results related to *k-Stirling* numbers of the second and first kind.

2 k−Stirling numbers of the second kind

Definition 1: $S2(k;n,m)$
The *k-Stirling* numbers of the second kind $S2(k;n,m)$ are defined for $k \in \mathbf{Z}$ by eq. 12.
Lemma 1: The recursion relation eq. 13 follows from the definition of the *k−Stirling* numbers of the second kind, $S2(k;n,m)$, given in eq. 12.
Proof: Consider $(x^k\,d_x)^n \;=\; x^k\,d_x\,(x^k\,d_x)^{n-1}$ and use eq. 12 with $n \to n-1$ together with the lower triangular matrix conditions given after this eq. Then coefficients of $\{x^m\,{d_x}^m\}_1^n$ are compared. □

Note 1: It is obvious from the recursion 13 together with the input that the $S2(k;n,m)$ are always integers.

Definition 2: $s2(k;n,m)$
The *associated k-Stirling* numbers of the second kind $s2(k;n,m)$ are defined for $k \in \mathbf{Z} \setminus \{1\}$ by eq. 18.

Lemma 2: The recursion relation for the associated *k−Stirling* numbers of the second kind $s2(k;n,m)$ is given by eqs. 19 and 20.
Proof: Rewrite eq. 13 for $s2(k;n,m)$. □

Note 2: That the $s2(k;n,m)$ are indeed integers will be proved much later in *lemma 19*.

Note 3: For $k = 1$ one finds from the recurrence eq. 19 with eq. 20 the (infinite dimensional) unit matrix: $\mathbf{s2}(1) = \mathbf{1}$. This will be used to define $\mathbf{s2}(k)$ also for $k = 1$.

Lemma 3: Non-negativity of $\mathbf{s2}(k)$
The lower triangular matrix $\mathbf{s2}(k)$ has only non-negative elements for each $k \in \mathbf{Z}$.

Proof: If $k-1 \geq 0$ this is manifest from the recurrence relation eq. 19. For $1-k \in \mathbf{N}$ the first term in this recurrence becomes negative if and only if $(1-k)\,(n-1) < m$ and $n-1 \geq m$ (otherwise $s2(k;n-1,m)$ vanishes). But the first condition contradicts the second one. □

Lemma 4: $g2(k;m,y)$, the *o.g.f.* of the $m-$th column sequence of the lower triangular matrix $\mathbf{s2}(k)$ with $k \in \mathbf{Z} \setminus \{1\}$, defined in the first of eqs. 21, satisfies the first order linear differential-difference equation

$$[1-(k-1)^2\,y]\,g2'(k;m,y) \;-\; m\,(k-1)\,g2(k;m,y) \;-\; m\,g2(k;m-1,y) \;=\; 0 \;\;, \tag{57}$$

$$g2(k;m,0) = 0 \;,\; m \in \mathbf{N};\quad g2'(k;m,y)|_{y=0} = 0 \;,\; m \in \{2,3,...\} \;;\; g2'(k;1,y)|_{y=0} = s2(k;1,1) = 1. \tag{58}$$

Here the prime denotes differentiation with respect to the variable y.
Proof: Compute $y\,\frac{d}{dy}\,\sum_{n=m}^{\infty} n\,s2(k;n,m)\,y^n$ with the help of the recurrence relation eqs. 19 and 20 for $y \neq 0$. For $y = 0$ one finds the conditions given in eq. 58 from the definition of $g2(k;m,y)$. □

Lemma 5: With the convolution ansatz eq. 40, *i.e.* $g2(k;m,y) = (g2(k;1,y))^m$, one derives from *lemma 4* the following first order differential eq. for $g2(k;y) := g2(k;1,y)$, $k \in \mathbf{Z} \setminus \{1\}$,

$$[1-(k-1)^2\,y]\,g2'(k;y) \;-\; (k-1)\,g2(k;y) \;-\; 1 \;=\; 0\;. \tag{59}$$

Proof: Elementary. □

Lemma 6: The solution of the differential eq. 59 with the condition $g2(k;0) = 0$ is, for $k \in \mathbf{Z} \setminus \{1\}$,

$$g2(k;y) \;=\; \frac{1}{[1-(k-1)^2\,y]^{\frac{1}{k-1}}}\,\frac{1-[1-(k-1)^2\,y]^{\frac{1}{k-1}}}{k-1} \;=:\; \frac{y}{[1-(k-1)^2\,y]^{\frac{1}{k-1}}}\,c2(k-1;y)\;. \tag{60}$$

Proof: Standard integration of a first order inhomogeneous differential eq. of the type $g'(y)+f(y)\,g(y) = k(y)$. □

Note 4: Generalized *Catalan*-numbers

$$c2(l;x) \;:=\; \frac{1-[1-l^2\,x]^{\frac{1}{l}}}{l\,x} \tag{61}$$

is, for $l \in \mathbf{Z} \setminus \{0\}$, the *o.g.f.* for generalized *Catalan*-numbers, called $l-$*Catalan*-numbers. $l = 2$ corresponds to the ordinary *Catalan*-numbers *EIS* A000108. For positive l these numbers have been called Patalan-numbers by *Gerard* in *EIS* (*cf.* A025748-A025755 for $l = 3..10$).

That $c2(l;y)$ generates integers follows later on from the fact that $s2(k;n,m)$ is always integer (see *notes 2* and *11*).

Because of $c2(-l;x) \;=\; c2(l;x)/(1-l^2\,x)^{\frac{1}{l}}$ one can write $g2(k;y) \;=\; y\,c2(1-k;y)$, as stated in eq. 22.

Consider the expansion $1/(1-l^2\,x)^{1/l} \;=\; \sum_{n=0}^{\infty} b_n^{(l)}\,x^n$, where $b_n^{(l)} \;=\; l^n\,((l\,n\;+\;1\;-\;l)(!^l))/n! \;:=\; l^n\,(\prod_{j=1}^{n}(j\,l\;+\;1\;-\;l))/n!$ and $b_0^{(l)} \;=\; 1$, $n \geq 1$, where we have introduced the l-factorial notation $(!^l)$. Therefore, the sequence $\{c2_n^{(-l)}\}_{n=0}^{\infty}$ generated by $c2(-l;x)$ for $l \in \mathbf{N}$ is the (ordinary) convolution of the sequence $\{b_n^{(l)}\}_{n=0}^{\infty}$ with the sequence $\{c2_n^{(l)}\}_{n=0}^{\infty}$. See *e.g. EIS* A035323 for $l = -10$.

Since we have put $\mathbf{s2}(1) \;=\; \mathbf{1}$ we take $g2(1;y) \;=\; y$.

Lemma 7: The *e.g.f.* for the $m-th$ column sequence of the $k-$*Stirling* triangle of the second kind $\mathbf{S2}(k)$, defined in eq. 14, is $G2(k;m;x) \;=\; \frac{1}{m!}\,(G2(k;1;x))^m$, $m \in \mathbf{N}$, with $G2(k;1;x) \;\equiv\; G2(k;x) \;=\; (k-1)\,g2(k;\frac{x}{k-1})$ for $k \neq 1$ and $G2(1;1;x) \;\equiv\; G2(1;x) \;=\; exp(x)-1$.

Proof: For $k \neq 1$ insert $S2(k;n,m)$ from eq. 18 into the definition of $G2(k;m;x)$. Then employ the two eqs. 21. The last one has been proved above in the *lemmata* 3 and 4. For the ordinary *Stirling* numbers, *i.e.* for $k = 1$, the stated result is well-known [1]. □

Lemma 8: For $k \in \mathbf{Z} \setminus \{1\}$,

$$e^{c\,x^k\,d_x} \;=\; \sum_{0}^{\infty}\frac{1}{m!}\left[g2(k;\frac{c}{k-1}\,x^{k-1})\,(k-1)\,x\right]^m\,d_x{}^m \;=:\; e^{g(k;c;x)\,x\,d_x}\;:\;, \tag{62}$$

with the definition $g(k;c;x) := g2(k;\frac{c}{k-1}\,x^{k-1})\,(k-1)$, and the normal order $:A:$ notation has been explained in the *conspectus* after eq. 4.

Proof: This runs along the same line as for the ordinary *Stirling* numbers of the second kind explained in the *conspectus*, eqs. 3 and 4. Expand the exponential and insert the definition of the $S2(k;n,m)$ numbers from eq. 12 together with the triangle convention stated there. Then exchange the row-summation with the column summation (without bothering about convergence). After replacing the $S2(k;n,m)$ numbers by their associated ones, $s2(k;n,m)$, due to eq. 18 (remembering that $k \neq 1$) one finds the *o.g.f.* $g2(k;m;\frac{c}{k-1}\,x^{k-1})$ under the column summation symbol. The proven convolution property eq. 21

(*lemmata* 4,5 and 6) then yields the first eq. of the *lemma*. The second one is due to the definition of the normal order which should be applied to each term in the expanded exponential. □

Note 5: If one inserts for $k \neq 1$ the *o.g.f.* $g2(k;y)$ given in *lemma 6*, or eqs. 22 and 23, one will find for $g(k;c;x)$ the result given in eq. 9. The $k=1$ result from eq. 5 is $g(1;c;x) = exp(c) - 1$.

Corollary 1: The operator identity eq. 11, just proven, implies the shift property shown in eq. 10.

Proof: Applicaton of *Taylor*'s theorem. □

Note 6: A third proof for the shift property eq. 10 can be given by using the well-known multiple commutator formula for $exp(\mathbf{B})\, x^l\, exp(-\mathbf{B})$ for given $l \in \mathbf{N}_0$ with the operator $\mathbf{B} = c\, E_{k;x} = c\, x^k\, d_x$ for $k \in \mathbf{Z}$ and the commutator $[E_{k;x}, x^l] = l\, x^{l+k-1}$. For $k = 1$ one thus obtains $exp(c\, x\, d_x)\, x^l\, 1 = (exp(c)\, x)^l\, exp(c\, x\, d_x)\, 1 = (exp(c)\, x)^l$. For $k \neq 1$ one first derives $exp(c\, E_{k;x})\, x^l\, exp(-c\, E_{k;x}) = \sum_{n=0}^{\infty} \frac{1}{n!} (l/(k-1))_n\, (c\,(k-1)\, x^{k-1})^n$ with the rising factorial (*alias Pochhammer*) symbol $(\nu)_n := \nu\,(\nu+1)\cdots(\nu+n-1)$. This implies $exp(c\, x^k\, d_x)\, x^l\, 1 = [(1 + g(k;c;x))\, x]^l\, 1$ with $1 + g(k;c;x)$ given in eq. 9. The 1 standing on the *rhs* indicates application of the operator identity on an $x-$independent function. With these results the shift property eq. 10 holds for polynomials and formally for every power series $f(x)$.

Lemma 9: For $k \in \mathbf{Z}$ the *e.g.f.* of the row polynomials $S2_n(k;x)$, defined in eq. 25, is called $\mathcal{G}2(k;z,x) := \sum_{n=0}^{\infty} S2_n(k;x)\, z^n/n!$ and it is given by

$$\mathcal{G}2(k;z,x) \;=\; e^{x\, G2(k;z)}\;, \tag{63}$$

where $G2(k;z)$ is the *e.g.f.* for the first ($m=1$) column sequence of the triangular matrix $\mathbf{S2}(k)$ given in eq. 24.

Proof: Separate the $n=0$ term in the definition of $\mathcal{G}2(k;z,x)$ and insert in the remainder the definition of the row polynomials eq. 25. Then interchange the row with the column summation index (without bothering about convergence problems) and use the definition of the *e.g.f.* $G2(k;m;z)$ given in eq. 14. The convolution property *lemma 7*, or eq. 15, then leads to the desired result. □

Note 7: Another way to state *lemma* 9 is to write

$$S2(k;n,m) \;=\; \left[\frac{z^n}{n!}\right]\, [x^m]\, e^{x\, G2(k;z)}\;, \tag{64}$$

where $[y^k]\, f(y)$ denotes the coefficient of y^k in the expansion of $f(y)$. For each $k \in \mathbf{Z}$ a matrix constructed in this way from the entries of its first ($m=1$) column (collected in the *e.g.f.* $G2(k;z)$) is called a *Jabotinsky* matrix. (*cf.* [4] with references to the original works. Note that we use *Knuth*'s $n!\, F_n(x)$ as row polynomials (*alias Jabotinsky* polynomials). His $f(z)$ corresponds to our *e.g.f.* of the $m=1$ column sequence.)

Another notation is used in the umbral calculus (*cf.* [7]). The row polynomials $E_n(x) = \sum_{n=1}^{m} J(n,m)\, x^n$ built from a lower triangular *Jabotinsky* matrix $J(n,m)$ are there called *associated* polynomial sequences to $f_{Roman}(t) = \bar{f}(t)$, the compositional inverse of the *e.g.f.* $f(t)$ used by *Knuth* and in this work (*cf.* [7], p.53). $\{E_n(x)\}$ are special *Sheffer*-polynomials for $(1, \bar{f}(t))$ in the umbral notation (*cf.* [7], p.107).

Yet another description of such convolution polynomials can be found in [9] where *Jabotinsky* matrices appear as a special case of so-called *Riordan* matrices (if one uses exponential generating functions). The corresponding matrix product furnishes a so-called *Bell* subgroup of the *Riordan* group (*cf.* [9], p.238). In the sequel we shall reserve the name *Riordan*, resp. *Bell*, matrix for the case of ordinary convolutions.

Lemma 10: The exponential (*alias* binomial) convolution property shown in eq. 26 for polynomials $S2_n(k;x), n \in \mathbf{N}_0$ and fixed k, is equivalent to the functional eq.

$$\mathcal{G}2(k;z,x+y) \;=\; \mathcal{G}2(k;z,x)\, \mathcal{G}2(k;z,y)\;, \tag{65}$$

which follows from eq. 63 for the *e.g.f.* $\mathcal{G}2(k;z,x)$ defined in *lemma 9*.

Proof: Compare, for fixed k, the $z^n/n!$ coefficient functions on both sides of this eq. □

Proposition 1: Exponential convolution property of the $S2_n(k;x)$ polynomials
The row polynomials $S2_n(k;x)$, defined in eq. 25 for $n \in \mathbf{N}_0$, satisfy for each $k \in \mathbf{Z}$ the exponential (*alias* binomial) convolution property shown in eq. 26.

Proof: *Lemma 10* with *lemma 9*. □

Lemma 11: Row-sums of ordinary convolution matrices [6]
The *o.g.f.* $r(x) := \sum_{n=1}^{\infty} r_n\, x^n$ of the row-sums $r_n := \sum_{m=1}^{n} s(n,m)$ of a lower triangular ordinary convolution matrix $\{s(n,m)\}_{n\geq m\geq 1}$ is given by

$$r(x) \;:=\; \frac{g(x)}{1-g(x)}\;, \tag{66}$$

where $g(x)$ is the *o.g.f.* of the first $(m=1)$ column of the matrix $s(n,m)$.

Proof: A lower triangular ordinary convolution matrix has by definition the *o.g.f.* $g(m;x)$ for its $m-$th column sequence given by $g(m;x) = (g(1;x))^m$ for $m \in \mathbf{N}$. Inserting into $r(x)$ the definition of the row-sums r_n, interchanging the row with column summation index and using the definition and convolution property of $g(m;x)$ proves the statement with $g(x) := g(1;x)$. □

Lemma 12: Row-sums of exponential convolution matrices
The *e.g.f.* $R(x) := \sum_{n=1}^{\infty} R_n\, x^n/n!$ of the row-sums $R_n := \sum_{m=1}^{n} S(n,m)$ of a lower triangular exponential convolution matrix $\{S(n,m)\}_{n\geq m\geq 1}$ is given by

$$R(x) \;:=\; e^{G(x)} \;-\; 1\;, \tag{67}$$

where $G(x)$ is the *e.g.f.* of the first $(m=1)$ column sequence of the matrix $\{S(n,m)\}_{n\geq m\geq 1}$.

Proof: Analogous to the proof of *lemma 11*. □

Proposition 2: *O.g.f.* for row-sums of $\mathbf{s2}(k)$ triangles

For $k \in \mathbf{Z} \setminus \{1\}$ the *o.g.f.* of the sequence of row-sums of the lower triangular matrix $\mathbf{s2}(k)$ is given by eq. 51.

Proof: *Lemma 11* and the $g2(k;x)$ result from *lemma 6*, eq. 60. □

Proposition 3: *E.g.f.* of the sequence of row-sums of $\mathbf{S2}(k)$ and $\mathbf{S2p}(k)$ triangles

For $k \in \mathbf{N_0}$ the *e.g.f.* of the sequence of row-sums of the non-negative lower triangular matrix $\mathbf{S2}(k)$, resp. $\mathbf{S2p}(k)$, defined from eq. 13, resp. eq. 27, is given by eq. 53, resp. eq. 54.

Proof: *Lemma 12* and $G2(k;x)$, resp. $G2p(-k;x)$, from eq. 24, resp. eq. 29. □

3 k−Stirling numbers of the first kind

k−Stirling numbers of the first kind can be defined as the elements of the inverse (infinite dimensional, lower triangular) matrix $\mathbf{S1}(k)$ of $\mathbf{S2}(k)$ composed of the *k−Stirling* numbers of the second kind.

Definition 3: *k−Stirling* numbers of the first kind $S1(k;n,m)$: eq. 34

$$x^n\, d_x{}^n \;=\; \sum_{m=1}^{n} S1(k;n,m)\, x^{-m(k-1)}\, (x^k\, d_x)^m \quad, \text{ for } k \in \mathbf{Z},\; n \in \mathbf{N}\;. \tag{68}$$

First note that this eq. is obtained from eq. 34 after multiplication of $x^{-n(k-1)}$ from the left. Therefore, both eqs. are equivalent for every k provided $x \neq 0$. One puts $S1(k;n,m) = 0$ if $n < m$, *i.e.* $\mathbf{S1}(k)$ is a lower triangular matrix.

Lemma 13: $\mathbf{S2}(k) \cdot \mathbf{S1}(k) = \mathbf{1}$, or

$$\sum_{m=p}^{n} S2(k;n,m)\, S1(k;m,p) = \delta_{n,p} \tag{69}$$

for fixed $k \in \mathbf{Z}$, $n \in \mathbf{N}$ and $p \in \mathbf{N}$. $\delta_{n,p}$ is the *Kronecker*-symbol.

Proof: Insert eq. 68 with $n \to m$ and $m \to p$ into the defining eq. 12 of the $S2(k;n,m)$ numbers, and then extend the $p-$sum from m to n, using lower triangularity of each matrix $\mathbf{S1}(k)$. After interchanging the summations over m and p we find for all $k \in \mathbf{Z}$, $n \in \mathbf{N}$ and $x \neq 0$

$$\mathcal{O}_x(k;n) := x^{-(k-1)n}\,(x^k\, d_x)^n = \sum_{p=1}^{n} \delta(k;n,p)\, \mathcal{O}_x(k;p)\ , \tag{70}$$

with $\delta(k;n,p) := \sum_{m=p}^{n} S2(k;n,m)\, S1(k;m,p)$. Because the operators $\{\mathcal{O}_x(k;p)\}_{p=1}^{n}$ acting on functions $f \in C^n$ constitute a linearly independent basis [5], eq. 70 implies $\delta(k;n,p) = \delta_{n,p}$ for each k. □

Similarly, one finds

$$\mathbf{S1}(k) \cdot \mathbf{S2}(k) = \mathbf{1} \tag{71}$$

after inserting eq. 12 with $n \to m$, $m \to p$ into eq. 68. Here one has to compare coefficients of the operators $\{x^p\, {d_x}^p\}_1^n$.

Lemma 14: The recurrence formula for the $k-$*Stirling* numbers of the first kind is given by eq. 37.

Proof: Use ${x^n\, d_x}^n = (x\, d_x - (n-1))\, x^{n-1}\, {d_x}^{n-1}$ and insert eq. 68 on both sides of this identity. After differentiation, remembering the triangularity of $\mathbf{S1}(k)$, one compares coefficients of the linear independent operators $\{\mathcal{O}_x(k;m)\}_{m=1}^{n}$ defined in eq. 70. □

Note 8: It is obvious from the recurrence 37 together with the inputs that all $S1(k;n,m)$ are integer for $k \in \mathbf{Z}$.

Definition 4: $s1(k;n,m)$

Associated $k-$*Stirling* numbers of the first kind, $s1(k;n,m)$, are defined, for $k \in \mathbf{Z} \setminus \{1\}$, by eq. 38.

Lemma 15: The recurrence relation for the numbers $s1(k;n,m)$ is given by eq. 39.

Proof: Rewrite the recurrence relation eq. 37 for $S1(k;n,m)$ with $k \neq 1$. The lower triangularity of the matrix $\mathbf{s1}(k)$ is inherited from $\mathbf{S1}(k)$. □

Note 9: For $k = 1$ one finds from the recurrence eq. 39 the unit matrix: $\mathbf{s1}(1) = \mathbf{1}$. This will be used to define $\mathbf{s1}(k)$ also for $k = 1$.

Lemma 16: Non-negativity of $\mathbf{s1}(k)$.

The lower triangular matrix $\mathbf{s1}(k)$ has only non-negative elements for each k.

Proof: If $k - 1 \geq 0$ this is manifest from the recurrence relation eq. 39. For $1 - k \in \mathbf{N}$ this follows from the fact that $s1(k;n-1,m) = 0$ if $n - 1 > (1-k)\, m$, *i.e.* if the coefficient of the first term in the recurrence eq. 39 is negative. This is proved by induction over the column number m. For $m = 1$ the assertion is true because only the first term in the recurrence is present, and since $s1(k, 2-k, 1) = 0$, due to the vanishing coefficient of the first term in its recursion, the recurrence shows that $s1(k;n-1,1)$ vanishes for $n - 1 = 2 - k, 3 - k, ...$ (if $n - 1 = 2 - k$ the prefactor in the first recursion term vanishes).

[5] This linear independence can be proved by applying the differentiation operators $\frac{1}{p!}\,\mathcal{O}_x(k;p)$ for fixed $k \in \mathbf{Z}$ and $p = 1, ..., n$ to the monomials x^q, for $q = 1, .., n$. The linear independence is then inferred from the non-singularity of the $n \times n$ matrix $A_{q,p}(k) = \frac{1}{p!} \prod_{j=0}^{p-1}(q + j(k-1))$. In fact, $Det\, \mathbf{A}(k) = +1$, for each $k \in \mathbf{Z}$ and $n \in \mathbf{N}$.

Assuming that the assertion holds for given $m > 1$, *i.e.* $s1(k; n-1, m) = 0$ for $n-1 > (1-k)\,m$, leads to a vanishing second term in the $s1(k; n-1, m+1)$ recurrence for all $n-1 > (1-k)\,m + 1$. Therefore, $s1(k; (1-k)\,(m+1) + 1, m+1)$ will be zero because the coefficient of the first term of this recurrence vanishes and the second term is not present because $(1-k)\,(m+1) > (1-k)\,m$. $s1(k; n-1, m+1)$ is then seen to vanish recursively for all $n-1 \geq (1-k)\,(m+1) + 1$. □

Lemma 17: *o.g.f.* for the $m-$th column of $\mathbf{s1}(k)$: eqs. 40, 41 and 42.

$\mathbf{s1}(k)$ is a *Bell* matrix (see *note 7* for this name), *i.e.* the *o.g.f.* for the sequence $\{s1(k; n, m)\}_{n=1}^{\infty}$ is given by $g1(k; m; y) = (g1(k; 1; y))^m$ and

$$g1(k; y) := g1(k; 1; y) = \frac{-1 + (1-(k-1)\,y)^{-(k-1)}}{(k-1)^2} \quad \text{for} \quad k \in \mathbf{Z} \setminus \{1\}\ . \tag{72}$$

Because we have put $\mathbf{s1}(1) = \mathbf{1}$ we take $g1(1; y) = y$.

Proof: From the recurrence relation eq. 39 one finds, for $k \in \mathbf{Z}$, the first order linear differential-difference eq.

$$[1-(k-1)\,y]\,g1'(k; m; y) - m\,(k-1)^2\,g1(k; m; y) - m\,g1(k; m-1; y) = 0\ , \tag{73}$$

$$g1(k; m; 0) = 0\ ,\ m \in \mathbf{N};\quad g1'(k; m; y)|_{y=0} = s1(k; 1, 1)\,\delta_{m,1} = \delta_{m,1}. \tag{74}$$

The prime denotes differentiation *w.r.t.* y. The $y = 0$ conditions follow from the definition of $g1(k; m; y)$ in eq. 40. Eq. 73 is solved with the convolution ansatz $g1(k; m; y) = (g1(k; 1; y))^m$ which results in a standard linear inhomogeneous differential eq. for $g1(k; y) := g1(k; 1; y)$, *viz*

$$[1-(k-1)\,y]\,g1'(k; y) - (k-1)^2\,g1(k; y) - 1 = 0\ . \tag{75}$$

with the initial condition $g1(k; 0) = 0$. The solution is then found to be eq. 72 which is eq. 41 with eq. 42. □

Note 10: Generalized *EIS* A001792 sequences.

Similar to generalized *Catalan* numbers generated by $c2(l; y)$ of eq. 23 (see *note 4*) one uses $c1(l; y)$ defined in eq. 42 as *o.g.f.* for sequences $\{c1_n^{(l)}\}_{n=0}^{\infty}$. $c1(1; y) = 1/(1-y)$ generates *EIS* A000012 (powers of 1). $c1(2; y)$ is the *o.g.f.* for the sequence A001792(n). The *EIS* A-numbers of the sequences for $l = k-1$ are found in the second column of *TAB. 3* for $l = 1, ..., 5$ and $l = -1, ..., -6$. See also *EIS* A053113 where a list of all $l = 1..10$ sequences is given. In order to have $g1(1; y) = y$ one should put $c1(0; y) \equiv 1$ (see eq. 41). The explicit expression for $c1_n^{(l)}$ with $l \in \mathbf{N}$ is given in eq. 43. $c1_n^{(0)} = \delta_{n,0}$. $c1(-l; x)$ is polynomial for $l \in \mathbf{N}$. For example, $c1(-3; x) = 1 + 3\,x + 3\,x^2$. The triangle of coefficients of these polynomials can be found as *EIS* A049323 (increasing powers of x), or A033842 (decreasing powers of x). The explicit form of these coefficients is given in eq. 44.

Lemma 18: The matrix $\mathbf{s1}(k)$ has for all $k \in \mathbf{Z}$ only integer entries.

Proof: The first column of $\mathbf{s1}(k)$ consists of integers because $c1(k-1; y)$ generates the integers $c1_n^{(k-1)}$ given explicitly in eqs. 43 and 44, and $g1(k; y)$ is given according to *lemma 17* by eq. 41. The $k = 1$ case is trivial by definition. Because $\mathbf{s1}(k)$ is an ordinary convolution triangle (*Bell* matrix) the proof of the *lemma* for the first column is sufficient. □

Lemma 19: The matrix $\mathbf{s2}(k)$ has for all $k \in \mathbf{Z}$ only integer entries.

Proof: Once this has been proved, all $s2(k; n, m)$ entries will be guaranteed to be non-negative integers due to *lemma 3*. For the proof we first rewrite eq. 69 for $\mathbf{s2}(k)$ and $\mathbf{s1}(k)$ defined by eqs. 18 and 38,

respectively. If one defines, for $k \in \mathbf{Z}$, the signed matrix $\mathbf{s2s}(k)$ by $s2s(k;n,m) := (-1)^{n-m}\, s2(k;n,m)$ then eq. 69 implies

$$\mathbf{s2s}(k) \cdot \mathbf{s1}(\mathbf{k}) \;=\; \mathbf{1}\;. \tag{76}$$

Based on the fact that the $s1(k;n,m)$ are integers from the previous *lemma* (from *lemma 16* they are even known to be non-negative) this eq. allows one to perform the proof recursively. For this write out the the entries $(\mathbf{s2s}(k) \cdot \mathbf{s1}(\mathbf{k}))_{n,m} \;=\; \delta_{n,m}$ in turn for $m = n, n-1, ..., 1$. Remember that both, $\mathbf{s2}(k)$ as well as $\mathbf{s1}(k)$, are triangular matrices. For $m = n$ one finds $s2s(k;n,n) \;=\; 1$ from $s1(k;n,n) \;=\; 1$ for all n, hence $s2(k;n,n) \;=\; 1$, which is also clear from the $\mathbf{s2}(k)$ recurrence. Descending with m one finds at level $m = n-l$ that $s2s(k;n,n-l)$ is integer if the information from the previous steps $m = n, n-1, ..., n-(l-1)$ is used which showed that all these $s2s(k;n,m)$ are integer because all $s1(k;n,m)$ are already known to be integers. Therefore all $s2(k;n,m)$ are integers. □

Note 11: Due to *lemmata 16* and *19* eqs. 21 and 22 show that $c2(l;y) \;=\; \sum_{n=0}^{\infty} c2_n^{(l)}\, y^n$ defined in eq. 23 generates positive integers for all $l \in \mathbf{Z} \setminus \{0\}$. Their explicit form is given by

$$c2_n^{(l)} \;=\; l^n \prod_{j=1}^{n} (j\,l-1)/(n+1)! \;=:\; l^n\,(l\,n-1)(!^l)/(n+1)!\;, \tag{77}$$

where the $l-$factorial notation $(!^l)$ has been defined like in *note 4*. By definition $c2(0;y) := 1$.

Lemma 20: The *e.g.f.* for the $m-th$ column sequence of the unsigned $k-$*Stirling*-triangle of the first kind, $\mathbf{S1p}(k)$, defined in eq. 36 for $k \in \mathbf{N}$, is $G1p(k;m;x) \;=\; \frac{1}{m!}\,(G1p(k;1;x))^m$, $m \in \mathbf{N}$, with $G1p(k;1;x) \;\equiv\; G1p(k;x) \;=\; (k-1)\,g1(k;\frac{x}{k-1})$ for $k = 2, 3, ...$ and $G1p(1;1;x) \;\equiv\; G1p(1;x) \;=\; -\,ln(1-x)$.

Proof: For $k \geq 2$ insert $S1p(k;n,m)$ from eqs. 36 and 38 into the definition of $G1p(k;m;x)$ given in eq. 46. In this way the *o.g.f.* $g1(k;m;y)$ enters in the claimed form. The result for the ordinary unsigned *Stirling* numbers $(k = 1)$ is well-known [1]. □

Note 12: Explicit form of $G1p(k;m;x)$, $k > 1$: eq. 48 and *lemma 20*.

With the result for the *o.g.f.* $g1(k;m;y)$ in eqs. 40 and 72 from *lemma 17* one finds eq. 48. This shows that $G1p(k;1;x) \;=\; -\overline{G2(k;-x)}$, the negative compositional inverse of $G2(k;-x)$ of eq. 24. Inverse *Jabotinsky* matrices like **S2** and **S1** (*cf.* eqs. 69 and 71) have first column *e.g.f.*s which are inverse to each other in the compositional sense [4].

Lemma 21: Row polynomials of $\mathbf{S1}(k)$

For $k \in \mathbf{Z}$ the *e.g.f.* of the row polynomials $S1_n(k;x) \;:=\; \sum_{m=1}^{n} S1(k;n,m)\, x^m$, $n \in \mathbf{N}$, and $S1_0(k;x) := 1$ are

$$\mathcal{G}1(k;z,x) := \sum_{n=0}^{\infty} S1_n(k;x)\, z^n/n! \;=\; e^{x\,G1(k;z)}\;, \tag{78}$$

where $G1(k;z) \;=\; (-1 \,+\, (1+z)^{1-k})/(1-k)$ for $k \neq 1$, and $G1(1;z) \;=\; ln(1+z)$, are the *e.g.f.*s for the first $(m = 1)$ column sequences of the triangular matrices $\mathbf{S1}(k)$.

Proof: Analogous to the one of *lemma 9*.

Note 13: $S1(k;n,m) \;=\; \left[\frac{z^n}{n!}\right]\,[x^m]\,e^{x\,G1(k;z)}$ (*cf. note 7*).

Proposition 5: Exponential convolution property of the $S1_n(k;x)$ polynomials

The row polynomials $S1_n(k;x)$, defined in *lemma 21* for $n \in \mathbf{N}_0$, satisfy for each $k \in \mathbf{Z}$ the exponential (*alias* binomial) convolution property shown in eq. 26 with $S2$ replaced everywhere by $S1$.

Proof: Compare, for fixed k, the $z^n/n!$ coefficient functions on both sides of the eq. $\mathcal{G}1(k;z,x+y) \;=\; \mathcal{G}1(k;z,x)\,\mathcal{G}1(k;z,y)$ which follows from eq. 78. □

Note 14: In the umbral calculus notation (*cf.* [7]) the polynomials $S1_n(k;x)$ are called *associated* polynomial sequences (*alias Sheffer*) for $(1, \overline{G1(k;t)} = G2(k;t))$. $G2(k;t)$ is given, for $k \neq 1$, in eq. 24. $\overline{G1(1;t)} = G2(1;t) = exp(t) - 1$.

Proposition 6: *O.g.f.* for row-sums of $\mathbf{s1}(k)$ triangles

For $k \in \mathbf{Z} \setminus \{1\}$ the *o.g.f.* of the sequence of row-sums of the lower triangular matrix $\mathbf{s1}(k)$ is given by eq. 52.

Proof: *Lemma 11* and the $g1(k;x)$ result from *lemma 17*. □

Proposition 7: *E.g.f.* of the sequence of row-sums of $\mathbf{S1p}(k)$ and $\mathbf{S1}(-|k|)$ triangles

For $k \in \mathbf{N_0}$ the *e.g.f.* of the sequence of row-sums of the non-negative lower triangular matrix $\mathbf{S1p}(k)$, resp. $\mathbf{S1}(-|k|)$, defined from eq. 36, resp. eq. 37, is given by eq. 55, resp. eq. 56.

Proof: *Lemma 12* and $G1p(k;x)$, resp. $G1(-|k|;x)$, from *lemma 20*, *i.e.* eq. 48, resp. eq. 49. □

Note 15: Row-sums of signed $\mathbf{S1}(k)$, $k \in \mathbf{N}$, resp. $\mathbf{S1s}(-|k|)$ triangles

Here *lemma 12* applies with the *e.g.f.*s $G1(k;x)$, resp. $G1s(-|k|;x)$, given in the first line after eq. 78, resp. in the paragraph after eq. 49.

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TAB. 1: Associated k-Stirling number triangles of the second kind

$$\mathbf{s2(k)},\ \mathbf{k \neq 1} \quad \mathbf{s2(1) := 1}$$

k	A-number of triangle	A-number of sequence of first column	A-number of sequence of row-sums
⋮			
-5	A049224	A025751 (Gerard)	A025759 (Gerard)
-4	A049223	A025750 (Gerard)	A025758 (Gerard)
-3	A049213	A025749 (Gerard)	A025757 (Gerard)
-2	A048966	A025748 (Gerard)	A025756 (Gerard)
-1	A033184 (Catalan)	A000108(n-1)	A000108 (Catalan)
0	A023531 (**1** matrix)	A000007(n-1)	A000012 (powers of 1)
2	A007318(n-1,m-1) (Pascal)	A000012	A000079 (powers of 2)
3	A035324	A001700(n-1)	A049027
4	A035529	A034171(n-1)	A049028
5	A048882	A034255(n-1)	A048965
6	A049375	A034687	A039746
⋮			

TAB. 2: k-Stirling number triangles of the second kind

$\mathbf{S2(k), k = 0, 1, 2, ...,}$ $\mathbf{S2p(k), k = 0, -1, -2, ...}$

k	A-number of triangle	A-number of sequence of first column	A-number of sequence of row-sums
⋮			
-5	A013988	A008543(n-1) (Keane)	A028844
-4	A011801	A008546(n-1) (Keane)	A028575
-3	A000369	A008545(n-1) (Keane)	A016036
-2	A004747	A008544(n-1) (Keane)	A015735
-1	A001497(n-1,m-1) (Bessel)	A001147(n-1) (double factorials)	A001515 (Riordan)
0	A023531 (**1** matrix)	A000007(n-1)	A000012 (powers of 1)
1	A008277 (Stirling 2nd kind)	A000012 (powers of 1	A000110 (Bell)
2	A008297 (unsigned Lah)	A000142 (factorials)	A000262 (Riordan)
3	A035342	A001147 (2-factorials)	A049118
4	A035469	A007559 (3-factorials)	A049119
5	A049029	A007696 (4-factorials)	A049120
6	A049385	A008548 (5-factorials)	A049412
⋮			

TAB. 3: Associated k-Stirling number triangles of the first kind

$\mathbf{s1(k)}$, $\mathbf{k \neq 1}$ $\quad$ $\mathbf{s1(1) := 1}$

k	A-number of triangle	A-number of sequence of first column	A-number of sequence of row-sums
⋮			
-5	A049327	A049323(5,m)	A049351
-4	A049326	A049323(4,m)	A049350
-3	A049325	A049323(3,m)	A049349
-2	A049324	A049323(2,m)	A049348
-1	A030528	A019590=A049323(1,m)	A000045(n+1) (Fibonacci)
0	A023531 (**1** matrix)	A000007(n-1)=A049323(0,m)	A000012 (powers of 1)
2	A007318(n-1,m-1) (Pascal)	A000012 (powers of 1)	A000079 (powers of 2)
3	A030523	A001792	A039717
4	A030524	A036068	A043553
5	A030526	A036070	A045624
6	A030527	A036083	A046088
⋮			

TAB. 4: k-Stirling number triangles of the first kind

$\mathbf{S1p(k), k = 0, 1, 2, ...,}$ $\mathbf{S1(k), k = 0, -1, -2, ...}$

k	A-number of triangle	A-number of sequence of first column	A-number of sequence of row-sums
⋮			
-5	A049411	A008279(5,n-1) (numbperm)	A049431
-4	A049424	A008279(4,n-1) (numbperm)	A049427
-3	A049410	A008279(3,n-1) (numbperm)	A049426
-2	A049404	A008279(2,n-1) (numbperm)	A049425
-1	A049403	A008279(1,n-1) (numbperm)	A000085
0	A023531 (**1** matrix)	A000007(n-1)	A000012 (powers of 1)
1	A008275 (unsigned Stirling 1st kind)	A000142(n-1)	A000142 (factorials)
2	A008297 (unsigned Lah)	A000142 (factorials)	A000262 (Riordan)
3	A046089	A001710(n+1) (Mitrinovic^2)	A049376
4	A049352	A001715(n+2) (Mitrinovic^2)	A049377
5	A049353	A001720(n+3) (Mitrinovic^2)	A049378
6	A049374	A001725(n+4) (Mitrinovic^2)	A049402
⋮			