KA-TP-13-2005
November 2005

Two Normal Ordering Problems and Certain Sheffer Polynomials

Wolfdieter L a n g [1]

Institut für Theoretische Physik
Universität Karlsruhe
D-76128 Karlsruhe, Germany

Abstract

The first normal ordering problem involves bosonic harmonic oscillator creation and annihilation operators (*Heisenberg* algebra). It is related to the problem of finding the finite transformation generated by $L_{k-1} := -z^k \partial_z, k \in \mathbb{Z}, z \in \mathbb{C}$ (conformal algebra generators). It can be formulated in terms of a subclass of *Sheffer*-polynomials called *Jabotinsky*-polynomials. The coefficients of these polynomials furnish generalized *Stirling*-number triangles of the second kind, called $S2(k; n, m)$ for $k \in \mathbb{Z}$. Generalized *Stirling*-numbers of the first kind, $S1(k; n, m)$ are also defined.

The second normal ordering problem appears in thermo-field dynamics for the harmonic *Bose* oscillator. Again *Sheffer*-polynomials appear. They relate to *Euler* numbers and iterated sums of squares. In a different approach to this problem one solves the differential-difference equation

$$f_{n+1} = f'_n + n^2 f_{n-1}, n >= 1, \text{ with certain inputs } f_0 \text{ and } f_1 = f'_0.$$

In this case the integer coefficients of the special *Sheffer*-polynomials which emerge have an interpretation as sum over multinomials for some subset of partitions.

1 Introduction

Two exercises from physics lectures on quantum field theory will be discussed.

Problem 1: Normal ordering of harmonic *Bose* oscillator operators related to the exponential $exp(c\, z^k\, \partial_z)$, $z \in \mathbb{C}$, $k \in \mathbb{Z}$. This will introduce a family of generalized *Stirling* numbers of both kinds, called $S2(k; n, m)$ and $S1(k; n, m)$. The problem is related to the *Witt*-algebra (conformal *Lie*-algebra for $\mathbb{C}$).

Problem 2: Rightsided normal ordering in thermal quantum field theory of the harmonic Bose oscillator. In both problems *Jabotinsky* and *Sheffer*-number triangles, *resp.* polynomials will show up.

2 Problem 1

The *Heisenberg* algebra $[\mathbf{a}, \mathbf{a}^+] = \mathbf{1}$ is considered in the (infinite dimensional) holomorphic representation [5], [2], [14], [15]:

$$\mathbf{a} \doteq \partial_z = \frac{\partial}{\partial z}, \quad \mathbf{a}^+ \doteq z \in \mathbb{C}. \tag{1}$$

These operators act on the space of holomorphic functions $\mathbb{C} \to \mathbb{C}$ (entire functions) endowed with a scalar product.

[1] E-mail: wolfdieter.lang@physik.uni-karlsruhe.de, http://www-itp.physik.uni-karlsruhe.de/~wl

The problem is to solve the following equation for g:

$$\boxed{:\, exp(g(k;c;z)\, z\, \partial_z) : \,=\, exp(c\, z^k\, \partial_z)}\;, \text{ with } c \in \mathbb{C}\,, \tag{2}$$

and the linear normal ordering symbol $:\mathbf{1}: \,=\, \mathbf{1}$ and $:(z\,\partial_z)^p: \,=\, z^p\,\partial_z^p$ for $p \in \mathbb{N}$.
This is useful because if g is known, one has, *via Taylor*'s expansion for holomorphic functions,

$$exp(c\, z^k\, \partial_z)\, f(z) \;=\; f(z') \;=\; f(z\,(1 \,+\, g(k;c;z)))\;. \tag{3}$$

It rephrases the problem of finding the finite conformal transformations on $\mathbb{C}$ generated by $L_{k-1} \;:=\; -z^k\,\partial_z$, with $k \in \mathbb{Z}$. These generators obey the conformal *Lie* algebra

$$[L_m, L_n] \;=\; (m-n)\, L_{m+n}\,,\quad m,n \;\in\; \mathbb{Z}\,.$$

Together with the complex conjugated generators, this is the *Witt Lie* algebra.
Some special cases:

$k \,=\, 0 \,:\, g(0;c;z) \,=\, c/z\,;\; z' \,=\, z \,+\, c$; L_{-1} generates translations.

$k \,=\, 1 \,:\, g(1;c;z) \,=\, exp(c) \,-\, 1\,;\; z' \,=\, exp(c)\, z$; L_0 generates scalings (dilations) and rotations.

$k \,=\, 2 \,:\, g(2;c;z) \,=\, c\,z/(1 \,-\, c\,z)\,;\; z' \,=\, z/(1 \,-\, c\,z)$; L_{+1} generates special conformal transformations.

$\{L_{-1}\,,\, L_0\,,\, L_{+1}\}$ generate the globally defined $SL(2,\mathbb{C})$ *Möbius* transformations.

There are at least 4 different ways to solve for $g(k;c;z)$:

1) To derive from the *Lie* algebra the *Lie* group one solves the differential eq.

$$\frac{dz(\alpha)}{d\alpha} \;=\; c\, z^k(\alpha) \;\text{ with } z' = z(\alpha = 1) \text{ and } z = z(\alpha = 0), \tag{4}$$

and finds the solution $z' \;=\; (1 + g(k;c;z))\, z$ with

$$\boxed{1 + g(k;c;z) \;=\; \left(1 - (k-1)\, c\, z^{k-1}\right)^{-\frac{1}{k-1}}\;.} \tag{5}$$

For all $k \;\neq\; 0, 1, 2$ there appear $|k-1|$-th roots defined on *Riemann* sheets.

2) One may also use a transformation of variables, *viz.* $y \;:=\; -\frac{1}{k-1}\,\frac{1}{z^{k-1}}\;,\; k \;\neq\; 1$. This reduces the problem to a translation in the y-variable *e.g.* [4] .

3) The physicist's solution:
Use the multiple commutator formula for $exp(\mathbf{B})\, z^l\, exp(-\mathbf{B})$ with $\mathbf{B} \;=\; -cL_{k-1}$ and $[L_{k-1}, z^l] \;=\; -l\, z^{l+k-1}$.
A resummation of the ensuing series in powers of $c\,(k-1)\, z^{k-1}$ with coefficients $(l/(k-1))^{\overline{n}}/n!$ leads to the above given result for $1 \,+\, g(k;c;z)$.

4) Direct solution generalizing *Stirling* numbers of the second kind (the case $= 1$ is reached as a limit). This approach has been used in [9].
Case $k \;=\; 1$:

$$e^{c\,z\,\partial_z} \;=\; \sum_{n=0}^{\infty} \frac{c^n}{n!}\, E_z{}^n \;=\; 1 + \sum_{n=1}^{\infty} \frac{c^n}{n!} \sum_{m=1}^{n} S2(n,m)\, z^m\, \partial_z{}^m \tag{6}$$

$$\;=\; 1 + \sum_{m=1}^{\infty} \left(\sum_{n=m}^{\infty} \frac{c^n}{n!}\, S2(n,m) \right) z^m\, \partial_z{}^m \;=\; 1 + \sum_{m=1}^{\infty} G2_m(c)\, z^m\, \partial_z{}^m\,. \tag{7}$$

with $E_z{}^n \;:=\; (z\,\partial_z)^n \;=\; \sum_{m=1}^{n} S2(n,m)\, z^m\, \partial_z{}^m\;,\; n \in \mathbf{N}$.

$G2_m(c) = \frac{1}{m!}\,(G2(c))^m$ with $G2(c) = exp(c) - 1$ the *e.g.f.* (exponential generating function) of the first $(m = 1)$ column

of the $S2(n,m)$ number triangle. For this triangle see [18] nr. A008277.

$e^{c\,z\,\partial_z} \;=: exp(G2(c)\,z\,\partial_z) :=: exp[(exp(c)-1)\,z\,\partial_z] :,\;$ *i.e.* $1\,+\,g(1;c;z)\;=\;exp(c)\,.$

This signals that $S2(n,m)$ is a special instance of a *Sheffer* triangle, called by *D. E. Knuth* [8] *Jabotinsky* matrix [7].

The row polynomials $S2_n(x) \;:=\; \sum_{m=1}^{n} S2(n,m)\,x^m$ are therefore exponential (also called binomial) convolution polynomials, satisfying, with $S2_0(x) := 1$,

$$S2_n(x+y) \;=\; \sum_{p=0}^{n}\binom{n}{p}\,S2_p(x)\,S2_{n-p}(y) \;=\; \sum_{p=0}^{n}\binom{n}{p}\,S2_p(y)\,S2_{n-p}(x)\;. \tag{8}$$

General $k-$case: One can write everywhere $S2(k;n,m)$ with $k \in \mathbb{Z}$. With

$$E_{k;z}{}^{n} \equiv (z^k\,\partial_z)^n \;=\; \sum_{m=1}^{n} S2(k;n,m)\,z^{m+(k-1)\,n}\,\partial_z{}^{m} \quad,\quad n\in\mathbb{N}\,, \tag{9}$$

and the triangle convention: $S2(k;n,m) = 0$ for $n < m$, and $S2(k;n,0) \;=\; \delta_{n,0}$. The recurrence relation for each k is:

$$\boxed{\text{S2(k;n,m)} \;=\; \text{((k-1)(n-1)+m)}\,\text{S2(k;n-1,m)} \;+\; \text{S2(k;n-1,m-1)}}\,. \tag{10}$$

Number triangles of this type have been investigated by Carlitz [3].

Special cases:

The $k = 0$ triangle is the lower part of the unit matrix.

The $k = 2$ triangle was known as (unsigned) *Lah* number triangle. [18] nr. A008297.

The $k = -1$ triangle is related to a *Bessel* triangle. [18] nr. A001497.

The *e.g.f.*s for the first columns $(k \neq 1)$: (for *Jabotinsky* triangles this is all what is needed):

$$G2(k;x) \;=\; (k-1)\,g2(k;\frac{x}{k-1})\;,\text{with } g2(k;y) \;=\; \left(1\,-\,(1\,-\,(1-k)^2\,y\,)^{\frac{1}{1-k}}\right)/(1-k)\,. \tag{11}$$

$g2(k;y)$ is the *o.g.f.* (ordinary generating function) of the first column of the associated triangles

$$s2(k;n,m) \;:=\; (k-1)^{n-m}\,\frac{m!}{n!}\,S2(k;n,m)\;. \tag{12}$$

with recurrence

$$s2(k;n,m) \;=\; \frac{k-1}{n}\,[(k-1)(n-1)+m]\;s2(k;n-1,m)\;+\;\frac{m}{n}\,s2(k;n-1,m-1)\;, \tag{13}$$

where $s2(k;n,m) \;=\; 0,\; n<m \;\;,\;\; s2(k;n,0) \;=\; \delta_{n,0},\;\; s2(k;1,1) = 1\;.$

$$\boxed{c2(l;y) \;:=\; \frac{1-(1-l^2\,y)^{\frac{1}{l}}}{l\,y}}\,, \tag{14}$$

which appears here as $g2(k;y) \;=\; y\,c2(1-k;y)$ is, for $l \neq 0$ the *o.g.f.* for generalized *Catalan*-numbers $(l \;=\; 2$ corresponds to the usual case).

3 Sheffer group and Jabotinsky subgroup

Before commenting on generalized *Stirling* numbers of the first kind, $S1(k;n,m)$, an interlude on the *Sheffer* group and its *Jabotinsky* subgroup. This is similar to the case of ordinary convolution polynomials where the corresponding group has been called in [16] *Riordan* group with its associated subgroup.

Elements of the *Sheffer* group are (g,f), with e.g.f.s $g(y) := 1 + \sum_{k=1}^{\infty} g_k\, y^k/k!$ and $f(y) = y + \sum_{n=2}^{\infty} f_n\, y^n/n!$, standing for the infinite dimensional, lower triangular matrix $\mathbf{S}$

$$\boxed{S(n,m) := \left[\frac{y^n}{n!}\right] g_m(y), \text{with } g_m(y) := g(y)\,\frac{(f(y))^m}{m!} \text{ for } n \geq m \geq 0, \text{ and } 0 \text{ if } n < m\,.} \tag{15}$$

Multiplication is defined as matrix multiplication $S^{(1)} \cdot S^{(2)} = S^{(3)}$ which produces the law

$$(g^{(1)}, f^{(1)}) \cdot (g^{(2)}, f^{(2)}) = (g^{(3)}, f^{(3)})\ \text{, with} \tag{16}$$
$$g^{(3)} = g^{(1)}\,(\,g^{(2)} \circ f^{(1)}\,) \text{ and } f^{(3)} = f^{(2)} \circ f^{(1)} \tag{17}$$

- The unit element is matrix $\mathbf{1}_{\infty}$, corresponding to $(1,y)$.
- The inverse element to (g,f) is $(g,f)^{-1} := (1/(g \circ \overline{f})\,,\,\overline{f})$ with the compositional inverse $\overline{f} \equiv f^{-1}$ of f.

The *Sheffer* polynomials [13] $s_n(x) = \sum_{m=0}^{n} S(n,m)\, x^m$ have *e.g.f.* $\boxed{\mathrm{g(y)\,exp(x\,f(y))}}$.

- $(1,f)$ are the elements of the *Jabotinsky* subgroup $\mathcal{I}$: $(1,f^{(1)}) \cdot (1,f^{(2)}) = (1, f^{(2)} \circ f^{(1)})$. $(1,f)^{-1} = (1,\overline{f})$.
- The $\{s_n(x)\}$, together with the associated *Jabotinsky* polynomials $\{p_n(x)\}$ with $(1,f)$ are exponential convolution polynomials:

$$s_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} s_k(x)\, p_{n-k}(y) = \sum_{k=0}^{n} \binom{n}{k} s_k(y)\, p_{n-k}(x)\,. \tag{18}$$

- A multinomial $M3$ expression for *Jabotinsky* matrix elements is [8]

$$J(n,m) = n! \sum_{\vec{\alpha} \in Pa(n,m)} \prod_{k=1}^{n} f_k^{\alpha_k} / (\alpha_k!\,(k!)^{\alpha_k})\,. \tag{19}$$

where $Pa(n,m)$ denotes the partitions of n with m parts written in the exponential language $(1^{\alpha_1} \cdots n^{\alpha_n})$ with $\alpha_j \in \mathbb{N}_0$ and $f_k := [y^k]\, f(y)$.

Example: $(1, exp(y)-1)$ for $S2(n,m)$ and its inverse $(1, ln(1+y))$ for $S1(n,m)$.

$$\mathbf{S2} \cdot \mathbf{S1} = \mathbf{1}_{\infty} = \mathbf{S1} \cdot \mathbf{S2}\,.$$

Also for the general $k \in \mathbb{Z}$ case: $\mathbf{S2}(k) \cdot \mathbf{S1}(k) = \mathbf{1}_{\infty} = \mathbf{S1}(k) \cdot \mathbf{S2}(k)$.

Neuwirth [11] (also private communication) observed that for $k \neq 1$ one has $\mathbf{S2}(k) = \mathbf{kS1} \cdot \mathbf{S2}$, as well as $\mathbf{S1}(k) = \mathbf{S1} \cdot \mathbf{kS2}$, with $kS1(n,m) := (1-k)^{n-m}\, S1(n,m)$, and $kS2(n,m) := (1-k)^{n-m}\, S2(n,m)$.

- The e.g.f. for $\mathbf{S}$ row sums is $r_n := \sum_{m=0}^{n} S(n,m)$ is $g(x)\, exp(f(x))$.
- A recurrence relation is given by $s_n(x) = [x + (ln(g(\bar{f}(t))))'/\bar{f}'(t)]\big|_{t^k = d_x^k}\, s_{n-1}(x)\,,\ n \geq 1,\ s_0(x) = 1$. See [12] p.50.
- Orthogonal polynomial systems of the *Sheffer* type have been classified by *Meixner* [10].

4 Problem 2

The second problem involves rightsided normal ordering in thermal quantum field theory for harmonic *Bose* oscillator operators [19]. One wants to prove the following identity.

$$\boxed{exp(\theta\,(\mathbf{A}^+ \,-\, \mathbf{A}))\,|\,0) \;=\; \frac{1}{cosh\,\theta}\,exp(tanh(\theta)\,\mathbf{A}^+)\,|\,0)\;.} \tag{20}$$

This is the thermo-vacuum $|0;\beta)$ with the inv. temperature $\beta \,=\, 1/(k\,T), and\; tanh(\theta) \,=\, exp(-\beta\,\hbar\,\omega/2)$ where ω the frequency of the oscillator. The operators $\mathbf{A}^+$ and $\mathbf{A}$ act on a direct product space according to

$$\mathbf{A}^+ \;:=\; \mathbf{a}^+ \otimes \tilde{\mathbf{a}}^+\,,\quad \mathbf{A} \;:=\; \mathbf{a} \otimes \tilde{\mathbf{a}}\;,\;|\,0) \;:=\; |\,0 > \otimes |\,\tilde{0} >\;\;. \tag{21}$$

The tilde-system $\tilde{\mathbf{a}}$, $\tilde{\mathbf{a}}^+$ with $|\,\tilde{0} >$ is a twin version of the harmonic *Bose* oscillator. One has a *Lie* algebra $su(1,1)$ with $\mathbf{J}_- \;:=\; \mathbf{A}\,,\;\; \mathbf{J}_+ \;:=\; \mathbf{A}^+ \;=\; (\mathbf{J}_-)^+,\;\; \mathbf{J}_3 \;:=\; (\mathbb{1} \,+\, \mathcal{N})/2 \;=\; (\mathbf{J_3})^+$.

$$[\mathbf{A}\,,\,\mathbf{A}^+] \;=\; \mathbb{1} \,+\, \mathcal{N}\,,\quad \text{with}\;\; \mathcal{N} \;:=\; \mathbf{N} \otimes \tilde{\mathbf{1}} \;+\; \mathbf{1} \otimes \tilde{\mathbf{N}},\; \text{where}\; \mathbf{N} \;:=\; \mathbf{a}^+\,\mathbf{a}\,,\;\; \tilde{\mathbf{N}} \;:=\; \tilde{\mathbf{a}}^+\,\tilde{\mathbf{a}} \tag{22}$$

$$[\mathcal{N}\,,\,\mathbf{A}\,] \;=\; -2\,\mathbf{A}\,,\;\; [\mathcal{N}\,,\,\mathbf{A}^+\,] \;=\; +2\,\mathbf{A}^+\;,\;\; \mathbb{1} \;:=\; \mathbf{1} \otimes \tilde{\mathbf{1}}\;. \tag{23}$$

A holomorphic representation is, *cf.* [4] eq. (I.3.43)

$$\mathbf{A}^+ \;\doteq\; (1/2)\,\partial_z^2\;,\;\; \mathbf{A} \;\doteq\; (1/2)\,z^2\;,\;\; (\mathbb{1} \,+\, \mathcal{N})/2 \;\doteq\; -(1/2)\,(z\,\partial_z \,+\, 1/2)\;. \tag{24}$$

Compute the above given l.h.s. of the thermo-vacuum with $\mathcal{N}\,|\,0) \;=\; 0$ and $\mathbf{A}\,|\,0) \;=\; 0$, keep $\mathbb{1}\,|\,\mathbf{0})$ and $\mathbf{A}^+\,|\,0)$. Rightsided normal ordering means to write for every monomial all $\mathbf{A}$ and $\mathcal{N}$ to the righthand side.

Polynomial functions of $\mathbf{A}$ and $\mathbf{A}^+$ are first rewritten, using the commutation relation, in a such a form that all $\mathbf{A}^+$'s are moved to the left of the $\mathbf{A}$: $\mathcal{O}(\mathbf{A},\mathbf{A}^+) \;=\; \mathcal{U}(\mathcal{O})$. This expression $\mathcal{U}(\mathcal{O})$ is then decomposed according to $\mathcal{U}(\mathcal{O}) \;=\; Nr(\,\mathcal{O}\,) \,+\, R(\mathcal{O})$ with $Nr(\,\mathcal{O}\,)|\,0) \;=\; 0$. Here Nr is the rightsided normal ordering symbol and R stands for the remainder.

Example: $\mathcal{U}\left((\mathbf{A}^+ \,-\, \mathbf{A})^2\right) \;=\; \mathbf{A}^{+\,2} \,-\, \mathbf{A}^+\,\mathbf{A} \,-\, (\mathbf{A}^+\,\mathbf{A} \,+\, \mathbb{1} \,+\, \mathcal{N}) \,+\, \mathbf{A^2}$, *i.e.* $Nr(\,(\mathbf{A}^+ \,-\, \mathbf{A})^2\,) \;=\; -2\,\mathbf{A}^+\,\mathbf{A} \,+\, \mathbf{A}^2 \,-\, \mathcal{N}$ and $R(\,(\mathbf{A}^+ \,-\, \mathbf{A})^2\,) \;=\; \mathbf{A}^{+\,\mathbf{2}} \,-\, \mathbb{1}\;$.
The interest is in $R(\mathcal{O})$. If x is used instead of $\mathbf{A}^+$, and 1 instead of $\mathbb{1}$ then $R((\mathbf{A}^+ \,-\, \mathbf{A})^n)$ becomes a polynomials $R_n(x)$. *E.g.* $R_2(x) \;=\; x^2 \,-\, 1$. One finds an integer coefficient triangle for $R(n,m)$ $:=\; [x^m]\,R_n(x)$. See [18] nr. A060081.

$R_n(x) \;=\; \sum_{k=0}^{\lfloor n/2\rfloor}\,(-1)^k\,a(n-(2\,k-1),k)\,x^{n-2\,k}$, where $a(n,k) \;=\; \sum_{j=1}^{n}\,a(j+1,k-1)\,j^2$, with input $a(n,0) \;=\; 1$. is a rectangular array satisfying the following recurrence.

$$\boxed{a(n,k) \;=\; a(n-1,k) \,+\, n^2\,a(n+1,k-1)\;,} \tag{25}$$

with input $a(n,-1) \;=\; 0\,,\, a(0,k) \;=\; \delta_{0,k}$.
Example: $R_6(x) \;=\; x^6 \,-\, a(5,1)\,x^4 \,+\, a(3,2)\,x^2 \,-\, a(1,3)\,1 \;\;=\; x^6 - 55\,x^4 + 331\,x^2 - 61\;$.

The $R_n(x)$ polynomials are *Sheffer* for $(\mathbf{1/cosh\,y}, \mathbf{tanh\,y})$, *i.e.*

$$\sum_{n=0}^{\infty} R_n(x)\,y^n/n! \;=\; (1/cosh\,y)\,exp(x\,tanh\,y)\;\;. \tag{26}$$

For $x \to \mathbf{A}^+\,, y \to \theta\,$ this then proves the thermo-vacuum identity.

Euler numbers $\bar{E}_n$ (signed, aerated) appear in the first (and second) column of the $a(n,m)$ array. In symbolic notation (exponents instead of indices) they are defined by

$$(\bar{E} \,+\, 1)^k \,+\, (\bar{E} \,-\, 1)^k \;=\; 0,\;\; k \in \mathbb{N}\,,\, \bar{E}_0 \;=\; 1. \tag{27}$$

To the author's knowledge one obtains here a new representation for *Euler* numbers $E_n = (-1)^n \bar{E}_{2n}$, $n \in \mathbb{N}_0$ and their generalizations, *via* iterated sums of squares:

$$\boxed{a(n,m) = \sum_{j_m=1}^{n} j_m^2 \sum_{j_{m-1}=1}^{j_m+1} j_{m-1}^2 \cdots \sum_{j_1=1}^{j_2+1} j_1^2\,,\; a(n,0) := 1,\; a(0,m) = \delta_{m,0}\,.} \tag{28}$$

The usual *Euler* numbers are $E_{m+1} = a(2,m)$ and the last sum extends only up to $n = 2$.

Note: The trigonometric version *Sheffer* $(\mathbf{1/\cos y, \tan y})$ is used for the *Moyal* star product for the harmonic *Bose* oscillator [17]. $f = f(\bar{a},a)$, $g = g(\bar{a},a)$; $f * g := f\, exp(i\,\hbar/2\, \overleftrightarrow{P}_{a,\bar{a}})\, g$ with $\overleftrightarrow{P}_{a,\bar{a}} := -i\,(\overleftarrow{\partial}_a \overrightarrow{\partial}_{\bar{a}} - \overleftarrow{\partial}_{\bar{a}} \overrightarrow{\partial}_a)$, the *Poisson* bidifferential. $[a,\bar{a}]_* := a * \bar{a} - \bar{a} * a = 1$. $U(t) := exp_*(-i\,H\,t/\hbar)$ with $H = \omega\,\bar{a}\,a$, with $H^{*n} := \underbrace{H * H * ... * H}_{n\,\text{times}}$ leads to

$$U(t) = \frac{1}{cosh\,y}\, exp(x\, tanh\, y) = \frac{1}{cos(\omega\, t/2)}\, exp(-i\,(2\,\bar{a}\,a/\hbar)\, tan(\omega\, t/2)\,)\,, \tag{29}$$

where $x \equiv 2\,\bar{a}\,a/\hbar$ and $y \equiv -i\,\omega\, t/2$. This results from

$$\boxed{x^{*n} = R_n(x) = \sum_{m=0}^{n} R(n,m)\, x^m}\,. \tag{30}$$

5 Alternative Approach to Problem 2

Define with *Umezawa* et al. [19] $f_n(\theta) := (0\,|\,\mathbf{A}^n\, exp(-\theta\,(\mathbf{A}^+ - \mathbf{A}))\,|\,0) \equiv (0\,|\,\mathbf{A}^n\,\mathbf{U}(-\theta)\,|\,0)$. Consider

$$f_0'(\theta) = -(0\,|\,(\mathbf{A}^+ - \mathbf{A})\,\mathbf{U}(-\theta)|\,0) = -(0\,|\,\mathbf{U}(-\theta)(\mathbf{A}^+ - \mathbf{A})|\,0)$$

and derive, using *Bogoliubov* transformations like

$$\mathbf{U}(\theta)\,\mathbf{a}\,\mathbf{U}(-\theta) = cosh\,\theta\,\mathbf{a} - sinh\,\theta\,\tilde{\mathbf{a}}^+, \text{ etc.}$$

the differential-difference eq.

$$\boxed{f_{n+1}(\theta) = f_n'(\theta) + n^2\, f_{n-1}(\theta)\,,} \tag{31}$$

with inputs $f_0(\theta) = 1/cosh\,\theta$ and $f_1(\theta) = f_0'(\theta)$.
For general input $f_0(\theta)$ one uses
$f_n(\theta) = \sum_{m=0}^{n} f(n,m)\, \frac{d^m}{d\theta^m}\, f_0(\theta) = s_n(\frac{d}{d\theta})\, f_0(\theta)$, with

$$\boxed{s_n(\theta) = \theta\, s_{n-1}(\theta) + (n-1)^2\, s_{n-2}(\theta)\,, \quad s_0(\theta) = 1\,,\; s_{-1}(\theta) = 0\,.} \tag{32}$$

Thus $\{s_n(\theta \to x)\}$ become *Sheffer* polynomials for $(\mathbf{\frac{1}{\sqrt{1-y^2}}, Artanh\, y})$.
The $f(n,m)$ triangle is the inverse of the $R(n,m)$ triangle. See [18], nr. A060524 (V. Jovovic)
There is the following combinatorial interpretation: $f(n,m) = \sum_{\vec{\alpha} \in Pao(n,m)} M2(\vec{\alpha})$.
$Pao(n,m)$ stands for partitions of n with m odd parts (and possibly even ones). Again, partitions are written in the exponential form with exponents $\vec{\alpha} := (\alpha_1, ..., \alpha_n)$.
$M2$ are the multinomial numbers [1], pp. 823, 831: $n!/\prod_{j=1}^{n} j^{a_j}\, a_j!$.
Example: $5 = f(3,1) = M2([3]) + M2([1,2]) = 2 + 3$.
This is a reformulation of exercise 3.3.13. on p.189 of [6]
For the considered physical problem the input is $f_0(\theta) = 1/cosh\,\theta$. This leads to

$$f_n(\theta) = n!(1/cosh\,\theta)\,(-tanh\,\theta)^n\,, \tag{33}$$

which coincides with the matrix elements $(0\,|\,\mathbf{A}^n\,\mathbf{U}(-\theta)\,|\,0)$ with $\mathbf{U}(-\theta)|\,0) = (1/cosh\,\theta)\, exp(-tanh(\theta)\,\mathbf{A}^+)\,|\,0)$ due to $(0\,|\mathbf{A}^n\,(\mathbf{A}^+)^m\,|\,0) = (n!)^2\,\delta_{n,m}$.

6 Conclusion

- ★ Two simple harmonic quantum oscillator problems feature some nice elements of the *Sheffer* group.
- ★ **Problem 1**: Sometimes it is rewarding not to take the *diretissima*.
- ★ **Problem 2**: Sometimes it is rewarding to take different routes to the same summit.

Acknowledgements

The author would like to thanks the DESFA (ISDE-SIDE-OPSFA) organizing committee for the invitation, the interesting program and the stimulating atmosphere.
Thanks go also to Dr. Stefan Gieseke (ITP Karlsruhe) for sharing his foil program.

TAB. 1: R(n,m) Sheffer triangle [18], nr. A060081

n/m	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	0	1									
2	-1	0	1								
3	0	-5	0	1							
4	5	0	-14	0	1						
5	0	61	0	-30	0	1					
6	-61	0	331	0	-55	0	1				
7	0	-1385	0	1211	0	-91	0	1			
8	1385	0	-12284	0	3486	0	-140	0	1		
9	0	50521	0	-68060	0	8526	0	-204	0	1	
10	-50521	0	663061	0	-281210	0	18522	0	-285	0	1
⋮											

TAB. 2: a(n,m) array (as triangle [18], nr. A060074)

n/m	0	1	2	3	4	5	6	7
0	1	0	0	0	0	0	0	0
1	1	1	5	61	1385	50521	2702765	199360981
2	**1**	**5**	**61**	**1385**	**50521**	**2702765**	**199360981**	**19391512145**
3	1	14	331	12284	663061	49164554	4798037791	596372040824
4	1	30	1211	68060	5162421	510964090	64108947631	9954077496120
5	1	55	3486	281210	28862471	3706931865	584856590956	111432850130020
6	1	91	8526	948002	127838711	20829905733	4059150905356	935210483855284
7	1	140	18522	2749340	475638163	96508175400	22882712047924	6296554692590120
8	1	204	36762	7097948	1544454483	384154309032	109415187933364	35575114290521256
⋮								

TAB. 3: associated Sheffer triangle (1,tanh y), [18], nr. A111593

n/m	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	0	1									
2	0	0	1								
3	0	-2	0	1							
4	0	0	-8	0	1						
5	0	16	0	-20	0	1					
6	0	0	136	0	-40	0	1				
7	0	-272	0	616	0	-70	0	1			
8	0	0	-3968	0	2016	0	-112	0	1		
9	0	7936	0	-28160	0	5376	0	-168	0	1	
10	0	0	176896	0	-135680	0	12432	0	-240	0	1
⋮											

TAB. 4: f(n,m) Sheffer triangle [18], nr. A060524

n/m	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	0	1									
2	1	0	1								
3	0	5	0	1							
4	9	0	14	0	1						
5	0	89	0	30	0	1					
6	225	0	439	0	55	0	1				
7	0	3429	0	1519	0	91	0	1			
8	11025	0	24940	0	4214	0	140	0	1		
9	0	230481	0	122156	0	10038	0	204	0	1	
10	893025	0	2250621	0	463490	0	21378	0	285	0	1
⋮											

TAB. 5: associated Sheffer triangle (1,Artanh y) [18], nr. A111594

n/m	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	0	1									
2	0	0	1								
3	0	2	0	1							
4	0	0	8	0	1						
5	0	24	0	20	0	1					
6	0	0	184	0	40	0	1				
7	0	720	0	784	0	70	0	1			
8	0	0	8448	0	2464	0	112	0	1		
9	0	40320	0	52352	0	6384	0	168	0	1	
10	0	0	648576	0	229760	0	14448	0	240	0	1
⋮											

References

[1] Abramowitz and I. A. Stegun, eds., *Handbook of Mathematical Functions*, National Bureau of Standards Applied Math. Series 55, Tenth Printing, reprinted as Dover publication, New York, 1972

[2] V. Bargmann, “On a Hilbert Space of Analytic Functions and an Associated Integral Transform, Part I”, Commun. Pure & Applied Math. 14, 3 (1961) 187-214 (K. O. Friedrichs anniversary issue)

[3] L. Carlitz, “On Arrays of Numbers”, Am. J. of Mathematics 54,4 (1932) 739-752

[4] G. Dattoli, P.L. Ottaviani, A. Torre and L. Vázquez, “Evolution operator equations: integration with algebraic and finite difference methods [...]”,*Riv. del Nuovo Cimento*, 20,2 (1997) 1-133

[5] V. A. Fock, ,, Verallgemeinerung und Loesung der Diracschen statistischen Gleichung‘‘, Z. Phys. 49 (1928) 339-357

[6] I. P. Goulden and D. M. Jackson, “*Combinatorial Enumeration*”, Wiley, 1983

[7] E. Jabotinsky, “Analytic Iteration”,Trans. Amer. Math. Soc.108 (1963) 457-477

[8] D. E. Knuth, “Convolution polynomials”, *The Mathematica J.*, 2.1 (1992) 67-78

[9] W. Lang, “On Generalizations of the Stirling Number Triangles”, on-line *Journal of Integer Sequences*, 3 (2000) Article 00.2.4

[10] J. Meixner, ,, Orthogonale Polynomsysteme mit einer besonderen Gestalt der erzeugenden Funktion‘‘, *J. London Math. Soc.*, 9 (1934) 6-13

[11] E. Neuwirth, “Recursively defined combinatorial functions: extending Galton’s board”, Discrete Maths.239 (2001) 33-51

[12] S. Roman, “*The Umbral Calculus*”, Academic Press, New York, 1984

[13] I. M. Sheffer, “Some Properties of Polynomial Sets of Type Zero”, Duke J. of Maths. 5 (1939) 590-622

[14] I. E. Segal, “Mathematical characterization of the physical vacuum for a linear Bose-Einstein field”, (Foundations of the dynamics of infinite systems. III), Illinois J. Math. 6 (1962) 500-523

[15] I. E. Segal, “The complex wave representation of the free Boson field”, in “*Topics in functional analysis*: essays dedicated to M. G. Krein on the occasion of his 70th birthday”, Advances in Mathematics: Supplementary studies, Vol. 3 (I. Gohberg and M. Kac, Eds.), Academic Press, New York, 1978, pp. 321-344.

[16] L. W. Shapiro, Seyoum Getu, Wen-Jin Woan and L. C. Woodson, “The Riordan Group”, *Discrete Appl. Maths.* 34 (1991) 229-239.

[17] Th. Spernat, „Clifford-Algebren in der Quantenmechanik‘‘, Diplomarbeit 2004, Dortmund,

[18] N.J.A. Sloane’s On-Line Encyclopedia of Integer Sequences, http//:www.research.att.com/˜njas/sequences/index.html

[19] H. Umezawa, H. Matsumoto and M. Tachiki, “*Thermo Field Dynamics and Condensed States* ”, North-Holland, 1982

AMS MSC numbers: 05A17, 26C99, 30C35.