Two Normal Ordering Problems and Certain Sheffer Polynomials

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Two exercises from lectures on quantum field theory.

- ★ **Problem 1**: Normal ordering of harmonic Bose oscillator operators.
  $exp(c\, z^k\, \partial_z)$, $z \in \mathbb{C}$, $k \in \mathbb{Z}$, and $k-$*Stirling* numbers of both kinds:
  $S2(k;n,m)$ and $S1(k;n,m)$. *Witt*-algebra (conformal *Lie*-algebra for $\mathbb{C}$).
- ★ **Problem 2**: Rightsided normal ordering in thermal quantum field theory of harmonic Bose oscillator.
- ★ *Jabotinsky* and *Sheffer*-number triangles (polynomials).

http://www-itp.physik.uni-karlsruhe.de/~ wl

Problem 1

- *Heisenberg* algebra $[\mathbf{a}, \mathbf{a}^+] = \mathbf{1}$ in (infinite dimensional) holomorphic representation:
$\mathbf{a} \doteq \partial_z = \frac{\partial}{\partial z}$, $\mathbf{a}^+ \doteq z \in \mathbb{C}$. **V. Fock (1928), V. Bargman (1961), I.E. Segal (1963)**
acting on the space of holomorphic functions $\mathbb{C} \longrightarrow \mathbb{C}$ (entire functions) endowed with a scalar product.
Solve for g:
$$\boxed{: exp(g(k;c;z)\, z\, \partial_z) : = \ exp(c\, z^k\, \partial_z)}\ , \text{ with } c \in \mathbb{C}\,,$$
and the linear normal ordering symbol $:\mathbf{1}: = \mathbf{1}$ and $:(z\,\partial_z)^p: = z^p\,\partial_z^p$ for $p \in \mathbb{N}$.
Useful because if g is known, one has, *via* *Taylor*'s expansion for holomorphic functions,
$$exp(c\, z^k\, \partial_z)\, f(z) = f(z') = f(z\,(1 + g(k;c;z)))\ .$$

- This is a rephrasing of the problem to find the finite conformal transformations in $\mathbb{C}$ generated by $L_{k-1} := -z^k\,\partial_z$, with $k \in \mathbb{Z}$. These generators obey the conformal *Lie* algebra
$$[L_m, L_n] = (m-n)\, L_{m+n}\,,\ \ m, n \in \mathbb{Z}\,.$$
Together with the c.c. generators, this is the *Witt Lie* algebra.

Problem 1

- Special cases:
  $k = 0 : g(0;c;z) = c/z\,;\ z' = z + c$; L_{-1} generates translations.
  $k = 1 : g(1;c;z) = exp(c) - 1\,;\ z' = exp(c)\,z$; L_0 generates scalings (dilations) and rotations.
  $k = 2 : g(2;c;z) = c\,z/(1 - c\,z)\,;\ z' = z/(1 - c\,z)$; L_{+1} generates special conformal transformations.
  $\{L_{-1}\,,\ L_0\,,\ L_{+1}\}$ generate the globally defined $SL(2,\mathbb{C})$ *Möbius* transformations.

- There are at least 4 different ways to solve for $g(k;c;z)$:

1) *Lie* algebra, *Lie* group: solution to differential eq.

$$\frac{dz(\alpha)}{d\alpha} = c\,z^k(\alpha) \ \text{ with } z' = z(\alpha = 1) \text{ and } z = z(\alpha = 0),$$

with the solution $z' = (1 + g(k;c;z))\,z$

$$\boxed{1 + g(k;c;z) = \left(1 - (k-1)\,c\,z^{k-1}\right)^{-\frac{1}{k-1}}\ .}$$

For all $k \neq 0,1,2$ there appear $|k-1|$-th roots defined on *Riemann* sheets.

2) Transformation of variables: $y := -\frac{1}{k-1}\,\frac{1}{z^{k-1}}\,,\ k \neq 1$. Reduce to translation in y-variable.

G. Dattoli, P.L. Ottaviani, A. Torre and L. Vázquez, Riv. NC 20 (1997) 1

Problem 1

3) The physicist's solution:
Using the multiple commutator formula for
$exp(\mathbf{B})\, z^l\, exp(-\mathbf{B})$ with $\mathbf{B} = -cL_{k-1}$ and $[L_{k-1}, z^l] = -l\, z^{l+k-1}$.
A resummation of the ensuing series in powers of $c\,(k-1)\, z^{k-1}$ with coefficients $(l/(k-1))^{\overline{n}}/n!$
leads to the above given result for $1 + g(k;c;z)$.

4) Direct solution generalizing *Stirling* numbers of the second kind (case $= 1$, reached as limit)
Case $k = 1$:

$$e^{c\,z\,\partial_z} = \sum_{n=0}^{\infty} \frac{c^n}{n!} E_z^{\ n} = 1 + \sum_{n=1}^{\infty} \frac{c^n}{n!} \sum_{m=1}^{n} S2(n,m)\, z^m\, \partial_z^{\ m}$$

$$= 1 + \sum_{m=1}^{\infty} \left(\sum_{n=m}^{\infty} \frac{c^n}{n!} S2(n,m) \right) z^m\, \partial_z^{\ m} = 1 + \sum_{m=1}^{\infty} G2_m(c)\, z^m\, \partial_z^{\ m}\,.$$

with $E_z^{\ n} := (z\,\partial_z)^n = \sum_{m=1}^{n} S2(n,m)\, z^m\, \partial_z^{\ m}\,,\ n \in \mathbf{N}$.
$G2_m(c) = \frac{1}{m!}\,(G2(c))^m$ with $G2(c) = exp(c) - 1$ the *e.g.f.* of the first ($m = 1$) column
of the $S2(n,m)$ number triangle. OEIS A008277
$e^{c\,z\,\partial_z} =: exp(G2(c)\, z\,\partial_z) :=: exp[(exp(c)-1)\, z\,\partial_z] :\,,$ *i.e.* $1 + g(1;c;z) = exp(c)\,.$

Problem 1

This signals that $S2(n,m)$ is a special instance of a *Sheffer* triangle, called (by *D. E. Knuth*) *Jabotinsky* matrix. **E. Jabotinsky, Trans. AMS 108 (1963) 457-477, D. E. Knuth, The Mathematica J. 2.1 (1992) 67-78.**

The row polynomials $S2_n(x) \ := \ \sum_{m=1}^{n} \ S2(n,m)\, x^m$ are therefore exponential (also called binomial) convolution polynomials, satisfying, with $S2_0(x) := 1$,

$$S2_n(x+y) \ = \ \sum_{p=0}^{n} \binom{n}{p} S2_p(x)\, S2_{n-p}(y) \ = \ \sum_{p=0}^{n} \binom{n}{p} S2_p(y)\, S2_{n-p}(x) \ .$$

General $k-$case: One can write everywhere $S2(k;n,m)$ with $k \in \mathbb{Z}$. With

$$E_{k;z}{}^{n} \equiv (z^k\, \partial_z)^n \ = \ \sum_{m=1}^{n} S2(k;n,m)\, z^{m+(k-1)\,n}\, \partial_z{}^{m} \quad , \quad n \in \mathbb{N}\, ,$$

and the triangle convention: $S2(k;n,m) = 0$ for $n < m$, and $S2(k;n,0) = \delta_{n,0}$. Recurrence relation for each k:

$$\boxed{\text{S2(k;n,m)} = \text{((k-1)(n-1)+m)}\,\text{S2(k;n-1,m)} \ + \text{S2(k;n-1,m-1)}}\, .$$

Problem 1

- Special cases:
  The $k = 0$ triangle is the lower part of the unit matrix.
  The $k = 2$ triangle was known as (unsigned) *Lah* number triangle. OEIS A008297
  The $k = -1$ triangle is related to a *Bessel* triangle. OEIS A001497

The $e.g.f.$s for the first columns ($k \neq 1$): (sufficient, because *Jabotinsky* triangles):

$$G2(k;x) \;=\; (k-1)\, g2(k; \frac{x}{k-1}) \;,$$

with $g2(k;y) \;=\; \left(1 \;-\; (1 \;-\; (1-k)^2\, y\,)^{\frac{1}{1-k}}\right) / (1-k)\;.$

$g2(k;y)$ is the $o.g.f.$ of the first column of the associated triangles

$$s2(k;n,m) \;:=\; (k-1)^{n-m}\, \frac{m!}{n!}\, S2(k;n,m)\;.$$

Problem 1

with recurrence

$$s2(k;n,m) = \frac{k-1}{n}\left[(k-1)(n-1)+m\right] s2(k;n-1,m) + \frac{m}{n}\, s2(k;n-1,m-1)\ ,$$

where $s2(k;n,m) = 0,\ n < m$, $s2(k;n,0) = \delta_{n,0}$, $s2(k;1,1) = 1$.

$$\boxed{c2(l;y) := \frac{1-(1-l^2\,y)^{\frac{1}{l}}}{l\,y}}\ ,$$

which appears here as $g2(k;y) = y\,c2(1-k;y)$ is, for $l \neq 0$ the $o.g.f.$ for generalized *Catalan*-numbers ($l = 2$ case).

Before commenting on generalized *Stirling* numbers of the first kind, $S1(k;n,m)$, an interlude on the *Sheffer* group and its *Jabotinsky* subgroup. This is similar to the case of ordinary convolution polynomials where the corresponding group has been called *Riordan* group with its associated subgroup.

L. W. Shapiro, Seyoum Getu, Wen-Jin Woan and L. C. Woodson, DAM 34 (1991) 229-239

Interlude: The Sheffer group $\mathcal{S}$ S. Roman, Umbral Calculus p.43

Elements are (g, f), with e.g.f.s $g(y) := 1 + \sum_{k=1}^{\infty} g_k\, y^k / k!$ and $f(y) = y + \sum_{n=2}^{\infty} f_n\, y^n / n!$, standing for the infinite dimensional, lower triangular matrix $\mathbf{S}$

$$S(n, m) := \left[\frac{y^n}{n!}\right] g_m(y), \text{ with } g_m(y) := g(y)\, \frac{(f(y))^m}{m!} \text{ for } n \geq m \geq 0, \text{ and } 0 \text{ if } n < m .$$

Multiplication is matrix multiplication $S^{(1)} \cdot S^{(2)} = S^{(3)}$ which produces the law

$$(g^{(1)}, f^{(1)}) \cdot (g^{(2)}, f^{(2)}) = (g^{(3)}, f^{(3)}) \text{ , with}$$
$$g^{(3)} = g^{(1)}\, (\, g^{(2)} \circ f^{(1)}\,) \text{ and } f^{(3)} = f^{(2)} \circ f^{(1)}$$

- Unit element: matrix $\mathbf{1}_{\infty}$, corresponding to $(1, y)$.

- Inverse element to (g, f) is $(g, f)^{-1} := (1/(g \circ \overline{f}) , \overline{f})$ with the compositional inverse $\overline{f} \equiv f^{-1}$ of f.

Sheffer polynomials $s_n(x) = \sum_{m=0}^{n} S(n, m)\, x^m$ have *e.g.f.* $\boxed{\text{g(y) exp(x f(y))}}$.

Interlude: The Sheffer group $\mathcal{S}$

- $(1, f)$ are the elements of the *Jabotinsky* subgroup $\mathcal{I}$: $(1, f^{(1)}) \cdot (1, f^{(2)}) = (1, f^{(2)} \circ f^{(1)})$.

  $(1, f)^{-1} = (1, \overline{f})$.

- Exponential convolution polynomials: $\{s_n(x)\}$ together with the associated *Jabotinsky* polynomials $\{p_n(x)\}$ with $(1, f)$:

$$s_n(x + y) = \sum_{k=0}^{n} \binom{n}{k} s_k(x)\, p_{n-k}(y) = \sum_{k=0}^{n} \binom{n}{k} s_k(y)\, p_{n-k}(x) .$$

- Multinomial $M3$ expression for *Jabotinsky* matrix elements:

$$J(n, m) = n! \sum_{\vec{\alpha} \in Pa(n,m)} \prod_{k=1}^{n} f_k^{\alpha_k} / (\alpha_k!\, (k!)^{\alpha_k}) .$$

Example: $(1, exp(y) - 1)$ for $S2(n, m)$ and its inverse $(1, ln(1 + y))$ for $S1(n, m)$.

$$\mathbf{S2} \cdot \mathbf{S1} = \mathbf{1}_\infty = \mathbf{S1} \cdot \mathbf{S2} .$$

Also for the general $k \in \mathbb{Z}$ case: $\mathbf{S2}(k) \cdot \mathbf{S1}(k) = \mathbf{1}_\infty = \mathbf{S1}(k) \cdot \mathbf{S2}(k)$.

Interlude: The Sheffer group $\mathcal{S}$

- Observation by *E. Neuwirth* for $k \neq 1$: $\mathbf{S2}(k) = \mathbf{kS1} \cdot \mathbf{S2}$, and $\mathbf{S1}(k) = \mathbf{S1} \cdot \mathbf{kS2}$,

  with $kS1(n,m) := (1-k)^{n-m}\, S1(n,m)$, and $kS2(n,m) := (1-k)^{n-m}\, S2(n,m)$.

From his work: Recursively defined combinatorial functions: extending Galton's board, DM 239(2001)33-51.

- E.g.f.for $\mathbf{S}$ row sums $r_n := \sum_{m=0}^{n} S(n,m)$ is $g(x)\, exp(f(x))$.

- Recurrence: $s_n(x) = [x + (ln(g(\bar{f}(t))))' / \bar{f}'(t)]\big|_{t^k = d_x^k}\, s_{n-1}(x)$, $n \geq 1$, $s_0(x) = 1$.

S. Roman, Umbral Calculus p.50.

- OPS of *Sheffer* type: *J. Meixner*'s 1934 paper. **London Math. Soc. 9, 6-13.**

Problem 2

- Rightsided normal ordering in thermal quantum field theory for harmonic Bose oscillators.

H. Umezawa, H. Matsumoto and M. Tachiki: Thermo Field Dynamics and Condensed States, North-Holland, 1982 , HHW 1967, TT 1967

$$exp(\theta\,(\mathbf{A}^{+}\;-\;\mathbf{A}))\,|\,0)\;=\;\frac{1}{cosh\,\theta}\,exp(tanh(\theta)\,\mathbf{A}^{+})\,|\,0)\;,$$

which is the thermo-vacuum $|0;\beta)$; inv. temperature $\beta\;=\;1/(k\,T)$, $tanh(\theta)\;=\;exp(-\beta\,\hbar\,\omega/2)$

$$\mathbf{A}^{+}\;:=\;\mathbf{a}^{+}\otimes\tilde{\mathbf{a}}^{+}\,,\quad\mathbf{A}\;:=\;\mathbf{a}\otimes\tilde{\mathbf{a}}\,,$$

$$|\,0)\;:=\;|\,0>\otimes|\,\tilde{0}>\;\;.$$

- Tilde-system $\tilde{\mathbf{a}}$, $\tilde{\mathbf{a}}^{+}$ with $|\,\tilde{0}>$ twin version of harmonic *Bose* oscillators.
- *Lie* algebra $su(1,1)$ with $\mathbf{J}_{-}\;:=\;\mathbf{A}\,,\;\mathbf{J}_{+}\;:=\;\mathbf{A}^{+}\;=\;(\mathbf{J}_{-})^{+}$, $\mathbf{J}_{3}\;:=\;(\mathbb{1}\;+\;\mathcal{N})/2\;=\;(\mathbf{J_3})^{+}$.

$$[\mathbf{A}\,,\;\mathbf{A}^{+}]\;=\;\mathbb{1}\;+\;\mathcal{N}\,,\quad\text{with }\;\mathcal{N}\;:=\;\mathbf{N}\otimes\tilde{\mathbf{1}}\;+\;\mathbf{1}\otimes\tilde{\mathbf{N}}\,,\;\text{where }\mathbf{N}\;:=\;\mathbf{a}^{+}\,\mathbf{a}\,,\;\tilde{\mathbf{N}}\;:=\;\tilde{\mathbf{a}}^{+}\,\tilde{\mathbf{a}}$$

$$[\,\mathcal{N}\,,\;\mathbf{A}\,]\;=\;-2\,\mathbf{A}\,,\quad[\,\mathcal{N}\,,\;\mathbf{A}^{+}\,]\;=\;+2\,\mathbf{A}^{+}\,,\quad\mathbb{1}\;:=\;\mathbf{1}\otimes\mathbf{1}\;.$$

- Holom. repr.: $\mathbf{A}^{+}\;\doteq\;(1/2)\,\partial_z^2\,,\;\;\mathbf{A}\;\doteq\;(1/2)\,z^2\,,\;(\mathbb{1}\;+\;\mathcal{N})/\mathbf{2}\;\doteq\;-(\mathbf{1/2})\,(\mathbf{z}\,\partial_\mathbf{z}\;+\;\mathbf{1/2})\,.$

Compute above l.h.s. with $\mathcal{N}\,|\,0)\;=\;0$ and $\mathbf{A}\,|\,0)\;=\;0$, keep $\mathbb{1}\,|\,\mathbf{0})$ and $\mathbf{A}^{+}\,|\,0)\,.$ Rightsided normal ordering: all $\mathbf{A}$ and $\mathcal{N}$ to the right.

Problem 2

- $\mathcal{O}(\mathbf{A},\mathbf{A}^+) = \mathcal{U}(\mathcal{O}) = Nr(\mathcal{O}) + R(\mathcal{O})$ with $Nr(\mathcal{O})|0) = 0$.
- Example: $\mathcal{U}\left((\mathbf{A}^+ - \mathbf{A})^2\right) = \mathbf{A}^{+2} - \mathbf{A}^+\mathbf{A} - (\mathbf{A}^+\mathbf{A} + \mathbb{1} + \mathcal{N}) + \mathbf{A^2}$
  $i.e.\ Nr((\mathbf{A}^+ - \mathbf{A})^2) = -2\,\mathbf{A}^+\mathbf{A} + \mathbf{A}^2 - \mathcal{N}$ and $R((\mathbf{A}^+ - \mathbf{A})^2) = \mathbf{A}^{+\mathbf{2}} - \mathbb{1}$.
- Interested in $R(\mathcal{O})$. Use x for $\mathbf{A}^+$, 1 for $\mathbb{1}$: polynomials $R_n(x)$. $R_2(x) = x^2 - 1$.
- Finds integer coefficient triangle for $R(n,m) := [x^m]\,R_n(x)$. OEIS A060081 (WL).
- $R_n(x) = \sum_{k=0}^{\lfloor n/2\rfloor} (-1)^k\, a(n-(2\,k-1),k)\,x^{n-2\,k}$, where
  $a(n,k) = \sum_{j=1}^{n} a(j+1,k-1)\,j^2$, with input $a(n,0) = 1$.
  rectangular array:

$$\boxed{a(n,k) = a(n-1,k) + n^2\,a(n+1,k-1)\,,}$$

  with input $a(n,-1) = 0$, $a(0,k) = \delta_{0,k}$.
  Example: $R_6(x) = x^6 - a(5,1)\,x^4 + a(3,2)\,x^2 - a(1,3)\,1 = x^6 - 55\,x^4 + 331\,x^2 - 61$.
- $R_n(x)$ polynomials *Sheffer* for $(\mathbf{1/\cosh y}, \mathbf{\tanh y})$, $i.e.$
  $\sum_{n=0}^{\infty} R_n(x)\,y^n/n! = (1/cosh\,y)\,exp(x\,tanh\,y)\,;\ x \to \mathbf{A}^+\,,\ y \to \theta$.
- *Euler* numbers $\bar{E}_n$ (signed, aerated) in first (and second) column.
  Symbolically $(\bar{E} + 1)^k + (\bar{E} - 1)^k = 0,\ k \in \mathbb{N},\ \bar{E}_0 = 1$.

Problem 2

New(?) representation for *Euler* numbers $E_n = (-1)^n \bar{E}_{2\,n}$, $n \in \mathbb{N}_0$ and generalizations, *via* iterated sums of squares:

$$a(n,m) = \sum_{j_m=1}^{n} j_m^2 \sum_{j_{m-1}=1}^{j_m+1} j_{m-1}^2 \cdots \sum_{j_1=1}^{j_2+1} j_1^2\,,\ a(n,0) := 1,\ a(0,m) = \delta_{m,0}\,.$$

$E_{m+1} = a(2,m)$: last sum only up to $n = 2$.

Trigonometric version: *Sheffer* $(\mathbf{1/\cos y, \tan y})$ used in *Moyal* star product for harmonic *Bose* oscillator.

Th. Spernat, Clifford-Algebren in der Quantenmechanik, Diplomarbeit 2004, Dortmund (A. C. Hirshfeld).

$f = f(\bar{a}, a)$, $g = g(\bar{a}, a)$; $f * g := f\, exp(i\,\hbar/2\, \overleftrightarrow{P}_{a,\bar{a}})\, g$ with
$\overleftrightarrow{P}_{a,\bar{a}} := -i\,(\overleftarrow{\partial}_a \overrightarrow{\partial}_{\bar{a}} - \overleftarrow{\partial}_{\bar{a}} \overrightarrow{\partial}_a)$ *Poisson* bidifferential. $[a, \bar{a}]_* := a * \bar{a} - \bar{a} * a = 1$.
$U(t) := exp_*(-i\,H\,t/\hbar)$ with $H = \omega\,\bar{a}\,a$, with $H^{*n} := \underbrace{H * H * \ldots * H}_{n \text{ times}}$.

$$U(t) = \frac{1}{cosh\,y}\, exp(x\,tanh\,y) = \frac{1}{cos(\omega\,t/2)}\, exp(-i\,(2\,\bar{a}\,a/\hbar)\,tan(\omega\,t/2))\,,$$

where $x \equiv 2\,\bar{a}\,a/\hbar$ and $y \equiv -i\,\omega\,t/2$.

From: $x^{*n} = R_n(x) = \sum_{m=0}^{n} R(n,m)\,x^m$.

Problem 2

TAB. 1: R(n,m) Sheffer triangle OEIS A060081

n/m	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	0	1									
2	-1	0	1								
3	0	-5	0	1							
4	5	0	-14	0	1						
5	0	61	0	-30	0	1					
6	-61	0	331	0	-55	0	1				
7	0	-1385	0	1211	0	-91	0	1			
8	1385	0	-12284	0	3486	0	-140	0	1		
9	0	50521	0	-68060	0	8526	0	-204	0	1	
10	-50521	0	663061	0	-281210	0	18522	0	-285	0	1
⋮											

Problem 2

TAB. 2: a(n,m) array (as triangle OEIS A060074)

n/m	0	1	2	3	4	5	6	7
0	1	0	0	0	0	0	0	0
1	1	1	5	61	1385	50521	2702765	199360981
2	**1**	**5**	**61**	**1385**	**50521**	**2702765**	**199360981**	**19391512145**
3	1	14	331	12284	663061	49164554	4798037791	596372040824
4	1	30	1211	68060	5162421	510964090	64108947631	9954077496120
5	1	55	3486	281210	28862471	3706931865	584856590956	111432850130020
6	1	91	8526	948002	127838711	20829905733	4059150905356	935210483855284
7	1	140	18522	2749340	475638163	96508175400	22882712047924	6296554692590120
8	1	204	36762	7097948	1544454483	384154309032	109415187933364	35575114290521256
⋮								

Problem 2

TAB.: associated Sheffer triangle (1,tanh y)

n/m	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	0	1									
2	0	0	1								
3	0	-2	0	1							
4	0	0	-8	0	1						
5	0	16	0	-20	0	1					
6	0	0	136	0	-40	0	1				
7	0	-272	0	616	0	-70	0	1			
8	0	0	-3968	0	2016	0	-112	0	1		
9	0	7936	0	-28160	0	5376	0	-168	0	1	
10	0	0	176896	0	-135680	0	12432	0	-240	0	1
⋮											

Problem 2, different approach

- Define (with *Umezawa* et al.) $f_n(\theta) := (0 \,|\, \mathbf{A}^n \, exp(-\theta\,(\mathbf{A}^+ - \mathbf{A}))\,|\,0) \equiv (0\,|\,\mathbf{A}^n\,\mathbf{U}(-\theta)\,|\,0)$.
  Consider
  $$f_0'(\theta) = -(0\,|\,(\mathbf{A}^+ - \mathbf{A})\,\mathbf{U}(-\theta)|\,0) = -(0\,|\,\mathbf{U}(-\theta)(\mathbf{A}^+ - \mathbf{A})|\,0)$$
  and derive, using *Bogoliubov* transformations like
  $$\mathbf{U}(\theta)\,\mathbf{a}\,\mathbf{U}(-\theta) = cosh\,\theta\,\mathbf{a} - sinh\,\theta\,\tilde{\mathbf{a}}^+.\ \text{etc.}$$
  the differential-difference eq.
  $$\boxed{f_{n+1}(\theta) = f_n'(\theta) + n^2\,f_{n-1}(\theta)\,,}$$
  with inputs $f_0(\theta) = 1/cosh\,\theta$ and $f_1(\theta) = f_0'(\theta)$.
  For general input $f_0(\theta)$ use
  $f_n(\theta) = \sum_{m=0}^{n} f(n,m)\,\frac{d^m}{d\theta^m}\,f_0(\theta) = s_n(\frac{d}{d\theta})\,f_0(\theta)$, with
  $$\boxed{s_n(\theta) = \theta\,s_{n-1}(\theta) + (n-1)^2\,s_{n-2}(\theta)\,, \quad s_0(\theta) = 1\,,\ s_{-1}(\theta) = 0\,.}$$
  $\{s_n(\theta \to x)\}$ *Sheffer* polynomials for $(\frac{1}{\sqrt{1-\mathbf{y}^2}}, \mathbf{Artanh\,y})$.
  $f(n,m)$ triangle is inverse of $R(n,m)$ triangle! OEIS A060524 V. Jovovic

Problem 2, different approach

- Combinatorial interpretation: $f(n,m) = \sum_{\vec{\alpha}\in Pao(n,m)} M2(\vec{\alpha})$.
  $Pao(n,m)$: partitions of n with m odd parts (and possibly even ones).
  $M2$ multinomial numbers *Abramowitz-Stegun*: $n!/\prod_{j=1}^{n} j^{a_j}\, a_j!$.
  Example: $5 = f(3,1) = M2([3]) + M2([1,2]) = 2 + 3$.

  Reformulation of exercise 3.3.13. on p.189 of I. P. Goulden, D. M. Jackson: Combinatorial Enumeration, Wiley, 1983.
- Here: Input $f_0(\theta) = 1/cosh\,\theta \;\Rightarrow\; f_n(\theta) = n!(1/cosh\,\theta)\,(-tanh\,\theta)^n = (n!)^2\, G_n^R(-\theta)$.
  Like matrix elements $(0\,|\,\mathbf{A}^n\,\mathbf{U}(-\theta)\,|\,0)$ with $\mathbf{U}(-\theta)|\,0) = (1/cosh\,\theta)\, exp(-tanh(\theta)\,\mathbf{A}^+)\,|\,0)$ due to $(0\,|\mathbf{A}^n\,(\mathbf{A}^+)^m\,|\,0) = (n!)^2\,\delta_{n,m}$.

Problem 2, different approach

TAB.: f(n,m) Sheffer triangle OEIS A060524

n/m	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	0	1									
2	1	0	1								
3	0	5	0	1							
4	9	0	14	0	1						
5	0	89	0	30	0	1					
6	225	0	439	0	55	0	1				
7	0	3429	0	1519	0	91	0	1			
8	11025	0	24940	0	4214	0	140	0	1		
9	0	230481	0	122156	0	10038	0	204	0	1	
10	893025	0	2250621	0	463490	0	21378	0	285	0	1
⋮											

Problem 2, different approach

TAB.: associated Sheffer triangle (1,Artanh y)

n/m	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	0	1									
2	0	0	1								
3	0	2	0	1							
4	0	0	8	0	1						
5	0	24	0	20	0	1					
6	0	0	184	0	40	0	1				
7	0	720	0	784	0	70	0	1			
8	0	0	8448	0	2464	0	112	0	1		
9	0	40320	0	52352	0	6384	0	168	0	1	
10	0	0	648576	0	229760	0	14448	0	240	0	1
⋮											

Conclusion

- ★ Two simple harmonic quantum oscillator problems featuring some nice elements of the *Sheffer* group.
- ★ **Problem 1**: Sometimes it is rewarding not to take the *diretissima*.
- ★ **Problem 2**: *Ditto* to take different routes to the same summit.

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